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Caveat: If the capacities are irrational, we could get augmenting paths forever!
There is a way to fix the algorithm so that this never happens.

Integrality Theorem

Corollary

If all capacities are integers, then there exists a maximum flow assigning integer values to all edges.

Chapter 5
Coloring of Graphs

5.1 Vertex coloring

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Graph Theory (GT), Combinatorics (C), Linear Algebra (LA), Analysis (A), Geometry (G), and Topology (T).

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Martin:	LA, C	Ethan:	T, LA, G	Abby:	T, G, LA
Jillian:	G, LA, A	Jessie:	A, LA, C	Albert:	G, A
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Use graph theory to determine the minimum number of class periods needed to offer these courses so that no student has a conflict with their courses.

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Computing $\chi(G)$ for a general graph G is a hard problem.

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Exercise: Find a graph G for which $\chi(G) > \omega(G)$.

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Let $G \Box H$ denote the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') if either

$$\begin{cases} u = u' \text{ and } vv' \in E(H), \text{ or} \\ \text{or } v = v' \text{ and } uu' \in E(G). \end{cases}$$

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Proposition

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