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Let $G \Box H$ denote the graph with vertex set $V(G) \times V(H)$, where (u, v) is adjacent to (u', v') if either

$$\begin{cases} u = u' \text{ and } vv' \in E(H), \text{ or} \\ \text{or } v = v' \text{ and } uu' \in E(G). \end{cases}$$

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Proposition

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Upper bounds

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Proof: Use a **greedy coloring**: order the vertices and then color them in that order, assigning to each vertex the smallest color not used by the neighbors colored so far.

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Proposition

If G is an **interval graph***, then*

$$\chi(G) = \omega(G).$$

5.3 Enumerative aspects

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[Example for C_4]

This is not a very practical way to compute $\chi(G;k)$, since there are too many partitions to consider.