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This is not a very practical way to compute $\chi(G;k)$, since there are too many partitions to consider.

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Note: We can disregard multiple copies of the same edge, since they don't affect the number of colorings.

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The chromatic recurrence can be used to compute $\chi(G; k)$ for any graph, since both $G - e$ and $G \cdot e$ have fewer edges than G, and we know how to compute it for graphs with no edges: $\chi(\overline{K_n}; k) = k^n$.

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This also gives another proof of the fact that $\chi(G; k)$ is always a polynomial.

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Exercise: Compute $\chi(K_n - e; k)$.

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If G has a simplicial elimination ordering, then

$$\chi(G; k) = (k - a_1)(k - a_2) \cdots (k - a_n),$$

where $a_i = |N(v_i) \cap \{v_1, \ldots, v_{i-1}\}|$.

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Theorem

A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Acyclic orientations

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Example: $\chi(C_4; k) = k(k-1)(k^2 - 3k + 3)$, so

$$(-1)^4\chi(C_4; -1) = 14.$$