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Theorem

A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.

Acyclic orientations

Does it make any sense to compute $\chi(G; -1)$?

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Example: $\chi(C_4; k) = k(k-1)(k^2 - 3k + 3)$, so

$$(-1)^4\chi(C_4; -1) = 14.$$

Chapter 6
Planar Graphs

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Example: Three enemies living in different houses want to have access to three utilities (gas, water and electricity). Can we build paths from each house to each utility so that the paths don't cross?

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A plane graph divides the plane into connected pieces called **regions** or **faces**.

The dual of a plane graph

Definition

The **dual graph** G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. Edges of G^* correspond to edges of G, so that if $e \in E(G)$ bounds two faces, then the endpoints of the corresponding edge $e^* \in E(G^*)$ are the vertices that correspond to those two faces.

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Note: Different embeddings of the same graph can yield different dual graphs.

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Proof: Apply the handshaking lemma to the dual graph.

Translating between a graph and its dual

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- *K_5 is not planar.*

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- *K_5 is not planar.*
- *$K_{3,3}$ is not planar.*