The dual of a plane graph

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The **dual graph** G^* of a plane graph G is a plane graph whose vertices correspond to the faces of G. Edges of G^* correspond to edges of G, so that if $e \in E(G)$ bounds two faces, then the endpoints of the corresponding edge $e^* \in E(G^*)$ are the vertices that correspond to those two faces.

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Note: Different embeddings of the same graph can yield different dual graphs.

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Proof: Apply the handshaking lemma to the dual graph.

Translating between a graph and its dual

$$G \quad \longleftrightarrow \quad G^*$$

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faces		

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Theorem (Euler's formula)

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If, additionally, G has no triangles, then

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Corollary

- *K_5 is not planar.*

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Corollary

- *K_5 is not planar.*
- *$K_{3,3}$ is not planar.*

A characterization of planar graphs

Definition

A **subdivision** of a graph is obtained from it by replacing edges with pairwise internally disjoint paths.
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Theorem (Kuratowski's Theorem)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

Easy direction: if G contains a subdivision of K_5 or $K_{3,3}$, then G is not planar.

Another similar characterization

Definition

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Theorem (Wagner '37)

A graph is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a minor.