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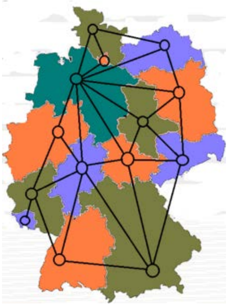
Theorem (The Four Color Theorem (Appel, Haken, Koch '77))

Every planar graph is 4-colorable.



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- 1890: Percy Heawood publishes the paper *Map colouring theorem*, in which he points out a problem with Kempe's proof, and produces a counter-example to Kempe's technique. However, he shows that one can use Kempe's ideas to prove a "5-color theorem".

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- 1976: Appel, Haken and Koch announce that they have constructed an unavoidable set of 1936 configurations, which they verified using 1200 hours of computer time.
- 1997: A simpler solution using an unavoidable set of 633 configurations is announced by Robertson, Sanders, Seymour and Thomas (`http://people.math.gatech.edu/~thomas/FC/fourcolor.html`). It requires a relatively short computation.

Six- and Five-Color Theorems

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Theorem (Five-color Theorem)

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Embedding graphs on surfaces

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The **genus** of a surface is the number of “holes” (or “handles”):

Genus 0

Genus 1

Genus 2

Genus 3

Generalized Euler's Formula

Theorem

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Examples:

For graphs embedded in a sphere, $n - e + f = 2$.

For graphs embedded in a torus, $n - e + f = 0$.

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For the torus, there are more than 17,000 minimal obstructions known, and the list is probably not complete.