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Since then, smaller examples have been found. The current smallest one has 509 vertices.

Edge coloring

Definition

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Theorem (Vizing 1964)

For any simple graph G,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

An application of planar graphs: regular polyhedra

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Substituting $n = \frac{2e}{k}$ and $f = \frac{2e}{\ell}$ into Euler's formula, we deduce, after some manipulations, that

$$(k-2)(\ell-2) < 4.$$

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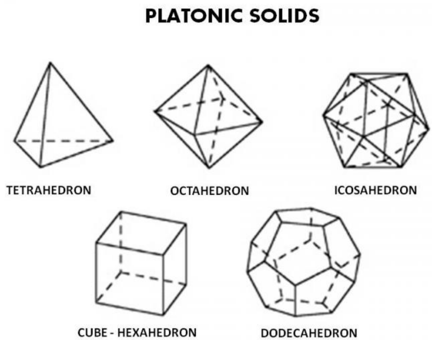
Also,

$$e = \frac{2k\ell}{4-(k-2)(\ell-2)}.$$

An application of planar graphs: regular polyhedra

k	ℓ	$(k-2)(\ell-2)$	e	n	f	name of polyhedron
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	icosahedron



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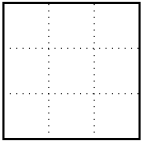
One can interpret perfect matchings of $P_2 \square P_n$ as domino tilings of a $2 \times n$ rectangle.

Domino tilings of rectangles

In how many ways can we tile an 3×3 rectangle with dominoes?



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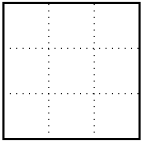
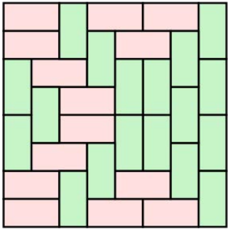


Answer: 0, because the number of unit squares that need to be covered is odd.

Domino tilings of rectangles

In how many ways can we tile an 8×8 rectangle with dominoes?



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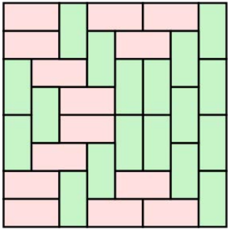

Answer: $12,988,816$.

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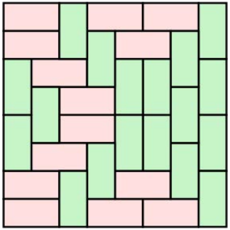

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In general, a complicated formula is known for the number of domino tilings of an $m \times n$ rectangle, for any given m and n.

Domino tilings of rectangles

Theorem (Fisher, Temperley, Kasteleyn, 1961)

The number of tilings of a $2m \times 2n$ rectangle with dominoes is

$$4^{mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left(\cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+1} \right).$$

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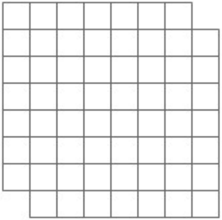
For example, for a chessboard with $m = n = 4$, and we get

$$4^{16} \prod_{j=1}^{4} \prod_{k=1}^{4} \left(\cos^2 \frac{j\pi}{9} + \cos^2 \frac{k\pi}{9} \right).$$

Note that $\cos^2 \frac{\pi}{9} = 0.8830222216\ldots.$

A small variation

If we remove two opposite corners of the 8×8 board, in how many ways can we tile it now with dominoes?



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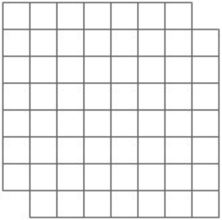

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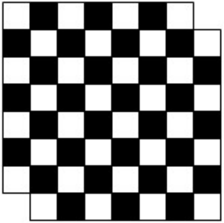
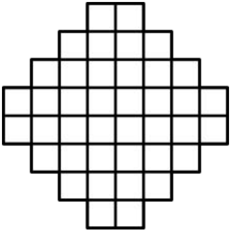


Answer: 0.

Coloring it as in a chessboard, each domino covers one unit square of each color, but there are more white squares in total.

Domino tilings of an Aztec diamond

In how many ways can we tile the following figure using dominoes?



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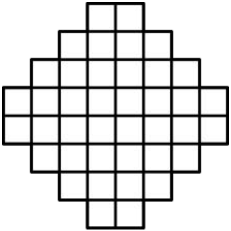

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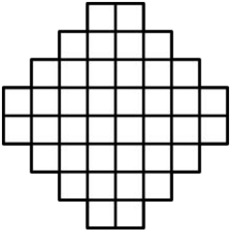

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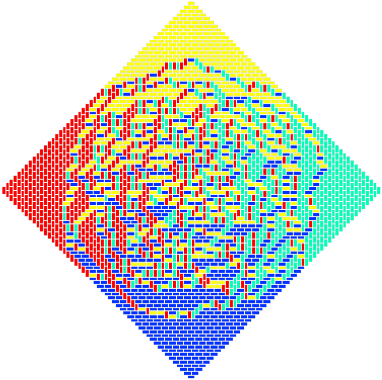
In general, for a similar diamond having n corners on each side, the number of tilings is

$$2^{n(n+1)/2}.$$

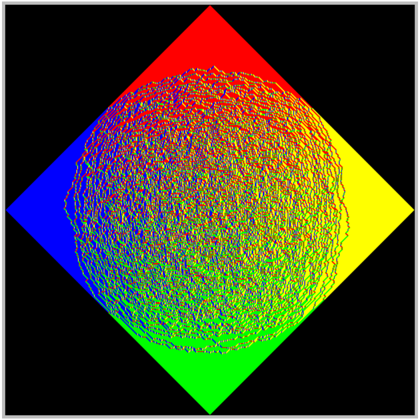
Tilings of an Aztec diamond

This is how a typical tiling of a large Aztec diamond looks like:



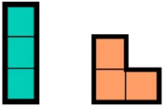
Tilings of an Aztec diamond

Here's an even larger one:



Polyominoes

Instead of dominoes, we can consider larger tiles. For example, trominoes are tiles consisting of 3 little squares:



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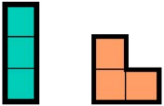
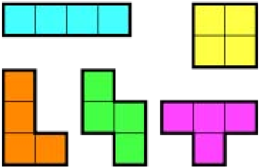


Tetrominoes are tiles consisting of 4 squares:



Polyominoes

How many different polynominoes can we form with n squares?

# of squares	1	2	3	4	5	6	$\dots$	n	$\dots$
# of polyominoes	1	1	2	5	12	35	$\dots$		$\dots$

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No formula is known!