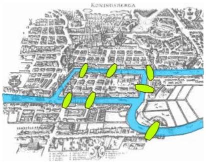
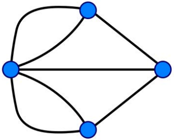
Characterization of Eulerian graphs

Recall the Königsberg bridge problem.





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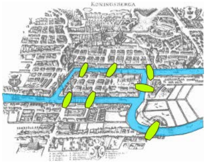

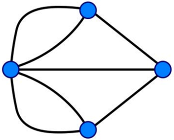

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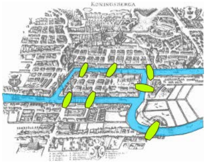

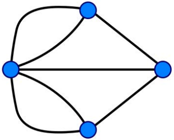

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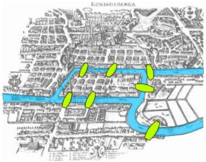

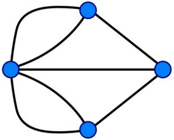

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Theorem (Characterization of Eulerian graphs)

A connected graph is Eulerian if and only if all its vertices have even degree.

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The **degree** of a vertex v is the number of edges incident to v.

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The **neighborhood** of v, denoted by $N_G(v)$ or $N(v)$, is the set of vertices that are adjacent to v.

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  In particular, if n is odd, then k must be even.

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Proposition

A k-regular bipartite graph has the same number of vertices in each partite set.

The Reconstruction Conjecture

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