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Example: The maximum number of edges in a simple graph of order n is $\binom{n}{2}$.

Example: edges and minimum degrees in connected graphs

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Definition

The sum of two graphs G and H is the graph obtained by taking the union of disjoint copies of G and H.

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Question: What is the maximum number of edges in a triangle-free simple graph with n vertices?

Theorem (Mantel 1907)

The maximum number of edges in a triangle-free simple graph with n vertices is $\lfloor n^2/4 \rfloor$.