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The underlying graph of a digraph D is the graph obtained by treating the edges of D as unordered pairs.

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The **incidence matrix** of a diagraph D of order n and size e is the $n \times e$ matrix with entries

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of } e_j, \\ -1 & \text{if } v_i \text{ is the head of } e_j, \\ 0 & \textit{otherwise}. \end{cases}$$

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A diagraph is **weakly connected** if its underlying graph is connected.
A diagraph is **strongly connected** if for every ordered pair of vertices u and v there is a path from u to v.

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Proposition

In a diagraph D,

$$\sum_{v \in V(D)} d^+(v) = \sum_{v \in V(D)} d^-(v) = e(D).$$

Eulerian digraphs

Theorem

A weakly connected digraph is Eulerian if and only if $d^+(v) = d^-(v)$ for each vertex v.

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Theorem

The digraph D_n is Eulerian, and the edge labels of any Eulerian circuit form a de Bruijn sequence.

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