Descent sets of permutations with only even or only odd cycle lengths

Ron Adin, Sergi Elizalde, Pál Hegedűs and Yuval Roichman

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$$(\mathtt{a},\mathtt{b})(\mathtt{a},\mathtt{b})(\mathtt{a},\mathtt{a},\mathtt{b},\mathtt{c}) \quad \longleftrightarrow \quad (\mathtt{b})(\mathtt{a})(\mathtt{a},\mathtt{b},\mathtt{a})(\mathtt{a},\mathtt{c},\mathtt{b})$$

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$|O_4| = |C_{(1,1,1,1)}| + |C_{(3,1)}| = 1 + 8 = 9$.

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For every even n and $J \subseteq [n-1]$,

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it follows that our main result is equivalent to the claim

$$\sum_{\pi \in O_n} \mathcal{F}_{n,\mathrm{Des}(\pi)} = \omega\left(\sum_{\pi \in E_n} \mathcal{F}_{n,\mathrm{Des}(\pi)}\right).$$

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Theorem [Gessel-Reutenauer '93] For every $\lambda \vdash n$, $L_\lambda = \mathcal{Q}_{C_\lambda}$ is symmetric and Schur-positive.

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Examples

$$L_{(4)} = L_{(2,2)}$$
$$L_{(4,1)} = L_{(3,2)} + L_{(2,1,1,1)}$$
$$L_{(6)} + L_{(3,3)} = L_{(3,2,1)} + L_{(3,1,1,1)}$$

Higher Lie characters

Let Z_λ be the centralizer in $\mathfrak{S}_n$ of a permutation of cycle type λ.

Observation If k_i denotes the number of parts of λ equal to i, then

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For each i, let ω_i be the linear character on $\mathbb{Z}_i \wr S_{k_i}$ which is trivial on $\mathfrak{S}_{k_i}$ and primitive on $\mathbb{Z}_i$. Let $\omega^\lambda := \bigotimes_i \omega_i$, a linear character on Z_λ. The higher Lie character indexed by λ is

$$\psi^\lambda := \omega^\lambda \uparrow_{Z_\lambda}^{\mathfrak{S}_n} .$$

Theorem [Garsia, Gessel, Reutenauer]

$$\mathrm{ch}(\psi^\lambda) = L_\lambda \quad (\forall \lambda \vdash n).$$

Main theorem restated (plus)

Theorem (Adin-Hegedűs-R '25)

For every $n = 2m > 0$*,*

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where $\psi^{(\mu^+,\mu^-)}$ is the type B higher Lie character corr. to (μ^+,μ^-).

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Note that $|\{\pi \in B_m : \text{ all cycles of } \pi \text{ are positive}\}| = (2m-1)!!$, and also the index of B_m in $\mathfrak{S}_{2m}$ is $(2m-1)!!$. Thus the theorem implies that $|E_{2m}| = |O_{2m}| = (2m-1)!!^2$.

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Combining the theorem with the Gessel-Reutenauer theorem implies

$$|\{\pi \in O_n : \mathrm{Des}(\pi) = J\}| = |\{\pi \in E_n : \mathrm{Des}(\pi) = [n-1] \setminus J\}|.$$

Explicit generating functions

Let $\underline{x} = (x_1, x_2, \ldots)$ be a countable set of indeterminates. For a permutation $\pi \in \mathfrak{S}_n$, with n_i cycles of size i for each $i \geq 1$, denote

$$\underline{x}^{c(\pi)} := \prod_i x_i^{n_i}.$$

Let OP_n (EP_n) be the set of all partitions of n with only odd (respectively, even) parts. Then

$$\sum_{n\geq 0} \sum_{\lambda \in OP_n} \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} \psi^{\lambda}_{S_n}(\pi)\, \underline{x}^{c(\pi)}$$
$$= \sum_{n\geq 0} \sum_{\lambda \in EP_n} \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} sign(\pi)\, \psi^{\lambda}_{S_n}(\pi)\, \underline{x}^{c(\pi)}$$
$$= \prod_{p\geq 0} \left(\frac{1 + x_{2^p}}{1 - x_{2^p}} \right)^{1/2^{p+1}}.$$

Note: The equality of the first and last expressions was proved already in [Calderbank-Hanlon-Sundaram '94].

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Corollary For any positive integer n,

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Theorem [Adin-Hegedűs-R '25] For any integers $k \geq 0$ and $n > 0$

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Corollary For every classical Weyl group W and any $k \geq 0$, the k-th root enumerator ρ_k^W is a proper character (i.e., positively spanned by the irreducible characters).

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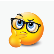
We will construct an explicit bijection

$$\{\pi \in O_n : \mathrm{Asc}(\pi) \subseteq S\} \leftrightsquigarrow \{\pi \in E_n : \mathrm{Des}(\pi) \subseteq S\}.$$

Structure of the bijection



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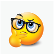

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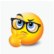

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$\cong$		$\cong$
Words whose Lyndon factors are odd and distinct	$\longleftrightarrow$	Words whose Lyndon factors are even

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A word $u \in \mathcal{W}$ is primitive if it is not of the form $u = r^j$ for $j \geq 2$.

A necklace is an equivalence class of primitive words under cyclic rotation, e.g. $(\mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{b}, \mathtt{a}) = (\mathtt{b}, \mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{a}) = (\mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{a}, \mathtt{b}) = \dots$

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Ex: $M = (\mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{b}, \mathtt{a})(\mathtt{a}, \mathtt{c})(\mathtt{a}, \mathtt{c})(\mathtt{b}) \in \mathcal{M}_{10}$, $\mathrm{wt}(M) = \mathtt{a}^5\mathtt{b}^3\mathtt{c}^2$.

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Define the weight of a set $S = \{s_1, s_2, \dots, s_k\} \subseteq [n-1]$ to be the monomial with exponents $s_1, s_2 - s_1, \dots, n - s_k$.

Ex: If $n = 6$ and $S = \{2, 3\} \subseteq [5]$, then $\mathrm{wt}(S) = \mathtt{a}^2\mathtt{b}\mathtt{c}^3$.

From permutations to multisets of necklaces: $\mathrm{Des} \subseteq S$

In 1993, Gessel and Reutenauer described a bijection

$$\{\pi \in \mathfrak{S}_n : \mathrm{Des}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n : \mathrm{wt}(M) = \mathrm{wt}(S)\}$$

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Example

Let $n = 8$ and $S = \{4, 7\}$, so $\mathrm{wt}(S) = \mathrm{a}^4\mathrm{b}^3\mathrm{c}$.

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From permutations to multisets of necklaces: Des $\subseteq S$

$\mathcal{M}_n^e$ = multisets of necklaces of even length.

$$\{\pi \in O_n : \mathrm{Asc}(\pi) \subseteq S\} \quad \leftrightsquigarrow \quad \{\pi \in E_n : \mathrm{Des}(\pi) \subseteq S\}$$

$$\updownarrow$$

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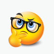

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To deal with ascent sets, we use a related bijection due to Gessel–Restivo–Reutenauer '12 and Steinhardt '10.

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However, those of *odd length* are primitive and **distinct**.

$\mathcal{M}_n^o =$ multisets of distinct necklaces of odd length.

We get a bijection

$$\{\pi \in O_n : \mathrm{Asc}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathrm{wt}(S)\}.$$

The bijections so far

$\mathcal{M}_n^e =$ multisets of necklaces of even length.

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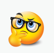

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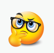

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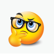

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We can interpret multisets of necklaces as words.

Lyndon words

A Lyndon word is a primitive word that is lexicographically smaller than all of its cyclic rotations. Let $\mathcal{L}$ be the set of Lyndon words.

Example: $\mathtt{aabab} \in \mathcal{L}$, but $\mathtt{ababa} \notin \mathcal{L}$.

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We identify necklaces with Lyndon words: $(\texttt{a}, \texttt{b}, \texttt{a}, \texttt{b}, \texttt{a}) \leftrightarrow \texttt{aabab}$.

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Theorem (Lyndon '58)

Every $w \in \mathcal{W}$ has a unique Lyndon factorization $w = \ell_1|\ell_2|\dots|\ell_m$ where $\ell_i \in \mathcal{L}$ for all i, and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m$ lexicographically.

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Example

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$$\leftrightarrow (\mathtt{d}, \mathtt{e})(\mathtt{d})(\mathtt{c}, \mathtt{c}, \mathtt{e}, \mathtt{d}, \mathtt{c}, \mathtt{d})(\mathtt{b}, \mathtt{d})(\mathtt{b}, \mathtt{d})(\mathtt{a}, \mathtt{a}, \mathtt{b}, \mathtt{d}).$$

We identify multisets of necklaces with words.

The bijections so far

Define the following sets of length-n words:

$\mathcal{W}_n^e =$ words all of whose Lyndon factors have even length.

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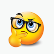

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$(\mathrm{b})(\mathrm{a})(\mathrm{a,b,a})(\mathrm{a,c,b}) \equiv \mathrm{b|acb|aab|a}$ $\qquad$ $(\mathrm{a,b})(\mathrm{a,b})(\mathrm{a,a,b,c}) \equiv \mathrm{ab|ab|aabc}$

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We want a weight-preserving bijection between $\mathcal{W}_n^o$ and $\mathcal{W}_n^e$.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Given $w \in \mathcal{W}_n^o$,

Example

$$w = \texttt{dadcdebccc}$$

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Given $w \in \mathcal{W}_n^o$, initially set $\alpha = w$ and $\beta = -$.

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Given $w \in \mathcal{W}_n^o$, initially set $\alpha = w$ and $\beta = -$.

Repeat until α is empty:

- Let $\alpha = o_1|o_2|\ldots|o_m$ be the Lyndon factorization of α.

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Example

	α	β
$w =$	d|adcdebccc	$-$
	d|ad¦cde	bccc

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Given $w \in \mathcal{W}_n^o$, initially set $\alpha = w$ and $\beta = -$.

Repeat until α is empty:

- Let $\alpha = o_1|o_2|\dots|o_m$ be the Lyndon factorization of α.
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Example

	α	β
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	d|adcde	bccc
	d|c ¦ de	adbccc
	$-$	cdedadbccc

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Set $\Psi(w) = \beta$.

Example

	α	β	
$w =$	d|adcdebccc	$-$	
	d|adcde	bccc	
	d|cde	adbccc	
	$-$	cdedadbccc	$= \Psi(w)$

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

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Example

	α	β	
$w =$	d|adcdebccc	$-$	
	d|adcde	bccc	
	d|cde	adbccc	
	$-$	cded|adbccc	$= \Psi(w)$

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ is a weight-preserving bijection.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ is a weight-preserving bijection.

Composing the maps, we obtain the desired bijection:

$$
\begin{array}{ccc}
\{\pi \in O_n : \mathrm{Asc}(\pi) \subseteq S\} & \rightsquigarrow & \{\pi \in E_n : \mathrm{Des}(\pi) \subseteq S\} \\
\updownarrow & & \updownarrow \\
\{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathrm{wt}(S)\} & & \{M \in \mathcal{M}_n^e : \mathrm{wt}(M) = \mathrm{wt}(S)\} \\
||| & & ||| \\
\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = \mathrm{wt}(S)\} & \xrightarrow{\Psi} & \{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = \mathrm{wt}(S)\}
\end{array}
$$

Thank you