Cyclic descents of standard Young tableaux

Sergi Elizalde

Dartmouth College

Joint work with Ron Adin and Yuval Roichman

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Descents and cyclic descents of permutations

Let $\pi = \pi_1 \ldots \pi_n \in S_n$ be a permutation.

The descent set of a $\pi$ is

$$\text{Des}(\pi) = \{ i \in [n - 1] : \pi_i > \pi_{i+1} \},$$

where $[m] := \{1, 2, \ldots, m\}$. 
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where $[m] := \{1, 2, \ldots, m\}$.

The cyclic descent set of $\pi$ is

$$\text{cDes}(\pi) := \begin{cases} 
\text{Des}(\pi) \cup \{n\}, & \text{if } \pi_n > \pi_1, \\
\text{Des}(\pi), & \text{otherwise}.
\end{cases}$$
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Introduced by Cellini ’95; further studied by Dilks, Petersen and Stembridge ’09 among others.
Examples

\[ \pi = 23154 : \quad \text{Des}(\pi) = \{2, 4\} , \]

\[ \pi = 3415 : \quad \text{Des}(\pi) = \{2, 4\} , \]

\[ \text{cDes}(\pi) = \{2, 4, 5\} . \]
Descents and cyclic descents of permutations

Examples

$$\pi = 23154 : \quad \text{Des}(\pi) = \{2, 4\}, \quad \text{cDes}(\pi) = \{2, 4, 5\}.$$
Descents and cyclic descents of permutations

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\[ \pi = 34152 : \ Des(\pi) = \{2, 4\} , \]
Descents and cyclic descents of permutations

Examples

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Properties of cDes

For $D \subseteq [n]$, let $D + 1$ be the subset of $[n]$ is obtained from $D$ by adding 1 mod $n$ to each element.
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The map $c\text{Des} : S_n \to 2^n$ has two properties:

(a) $c\text{Des}(\pi) \cap [n - 1] = \text{Des}(\pi) \quad \forall \pi \in S_n$, 
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(b) there exists a bijection $\phi : S_n \to S_n$ such that

$$c\text{Des}(\phi(\pi)) = c\text{Des}(\pi) + 1.$$
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(b) there exists a bijection \( \phi : S_n \to S_n \) such that

\[
\text{cDes}(\phi(\pi)) = \text{cDes}(\pi) + 1.
\]

Indeed, we can just define \( \phi \) by

\[
\pi_1 \pi_2 \ldots \pi_{n-1} \pi_n \quad \xrightarrow{\phi} \quad \pi_n \pi_1 \pi_2 \ldots \pi_{n-1}
\]
A partition of $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lambda_1 + \lambda_2 + \cdots = n$. We write $\lambda \vdash n$.

$\lambda$ can be represented as a Young diagram.
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Example: \( \lambda = (4, 3, 1) \)
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$\lambda$ can be represented as a Young diagram.

Example: $\lambda = (4, 3, 1)$

If the diagram of $\mu$ is contained in the diagram of $\lambda$, then the difference of these diagrams is a diagram of skew shape $\lambda/\mu$.

Example: $\lambda/\mu = (5, 3, 3, 1)/(2, 1)$

When $\mu$ is the empty partition, $\lambda/\mu$ is simply $\lambda$. 
A standard Young tableau (SYT) of shape $\lambda/\mu$ is a filling of the diagram of $\lambda/\mu$ with the numbers $1, \ldots, n$ (where $n = \#\text{boxes}$) so that entries increase along rows and along columns.

Examples:

$$\lambda = (4, 3, 1)$$

\[
\begin{array}{ccc}
1 & 2 & 4 & 8 \\
3 & 5 & 7 \\
6
\end{array}
\]
Standard Young Tableaux

A standard Young tableau (SYT) of shape $\lambda/\mu$ is a filling of the diagram of $\lambda/\mu$ with the numbers $1, \ldots, n$ (where $n = \#\text{boxes}$) so that entries increase along rows and along columns.

Examples:

$\lambda = (4, 3, 1)$

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 7 \\
6 & & \\
\end{array}
\]

$\lambda/\mu = (5, 3, 3, 1)/(2, 1)$

\[
\begin{array}{ccc}
2 & 3 & 9 \\
1 & 5 & \\
4 & 7 & 8 \\
6 & & \\
\end{array}
\]
A standard Young tableau (SYT) of shape $\lambda/\mu$ is a filling of the diagram of $\lambda/\mu$ with the numbers $1, \ldots, n$ (where $n = \#\text{boxes}$) so that entries increase along rows and along columns.

Examples:

$$\lambda = (4, 3, 1)$$

$$\lambda/\mu = (5, 3, 3, 1)/(2, 1)$$

Denote the set of all SYT of shape $\lambda/\mu$ by $\text{SYT}(\lambda/\mu)$. 
The descent set of a standard Young tableau $T$ is

$$\text{Des}(T) = \{ i : i + 1 \text{ is in a lower row than } i \}.$$
Descents of SYT

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Examples:

$$T = \begin{array}{cccc}
1 & 2 & 4 & 8 \\
3 & 5 & 7 & \\
6 & \\
\end{array} \in \text{SYT}((4, 3, 1)) \quad \text{Des}(T) = \{2, 4, 5\}$$
Descents of SYT

The descent set of a standard Young tableau $T$ is
\[ \text{Des}(T) = \{i : i + 1 \text{ is in a lower row than } i\}. \]

Examples:

1. \[
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3 & 5 & 7 & \\
6 & & & \\
\end{array}
\in SYT((4, 3, 1)) \quad \text{Des}(T) = \{2, 4, 5\}
\]

2. \[
T = \begin{array}{ccc}
2 & 3 & 9 \\
1 & 5 & \\
4 & 7 & 8 \\
6 & & & \\
\end{array}
\in SYT((5, 3, 3, 1)/(2, 1)) \quad \text{Des}(T) = \{3, 5\}
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The descent set of a standard Young tableau $T$ is

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Examples:

$$T = \begin{array}{cccc}
1 & 2 & 4 & 8 \\
3 & 5 & 7 \\
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\end{array} \in \text{SYT}((4, 3, 1)) \quad \text{Des}(T) = \{2, 4, 5\}$$

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1 & 5 \\
4 & 7 & 8 \\
6
\end{array} \in \text{SYT}((5, 3, 3, 1)/(2, 1)) \quad \text{Des}(T) = \{3, 5\}$$

Motivating Problem:
Define a cyclic descent set for SYT of any shape $\lambda/\mu$. 
For $r \mid n$, let $\lambda = (r, \ldots, r) \vdash n$ be a **rectangular** shape.
SYT of rectangular shapes

For \( r \mid n \), let \( \lambda = (r, \ldots, r) \vdash n \) be a rectangular shape.

**Theorem (Rhoades ’10)**

For \( \lambda = (r, \ldots, r) \), there exists a cyclic descent map \( \text{cDes} : \text{SYT}(\lambda) \to 2^n \) satisfying

\( \text{cDes}(T) \cap [n-1] = \text{Des}(T) \quad \forall T \in \text{SYT}(\lambda), \)
For $r | n$, let $\lambda = (r, \ldots, r) \vdash n$ be a rectangular shape.

**Theorem (Rhoades ’10)**

For $\lambda = (r, \ldots, r)$, there exists a cyclic descent map \( c\text{Des} : \text{SYT}(\lambda) \to 2^n \) satisfying

(a) \( c\text{Des}(T) \cap [n - 1] = \text{Des}(T) \quad \forall T \in \text{SYT}(\lambda), \)

(b) *there is a bijection* \( \phi : \text{SYT}(\lambda) \to \text{SYT}(\lambda) *such that* \( c\text{Des}(\phi(T)) = c\text{Des}(T) + 1. \)
For \( r \mid n \), let \( \lambda = (r, \ldots, r) \vdash n \) be a rectangular shape.

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For \( \lambda = (r, \ldots, r) \), there exists a cyclic descent map
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 satisfying

(a) \( \text{cDes}(T) \cap [n - 1] = \text{Des}(T) \quad \forall T \in \text{SYT}(\lambda) \),

(b) there is a bijection \( \phi : \text{SYT}(\lambda) \to \text{SYT}(\lambda) \) such that
\[ \text{cDes}(\phi(T)) = \text{cDes}(T) + 1. \]

Here, \( \phi \) is Schützenberger’s *jeu-de-taquin* promotion operator \( p \).
SYT of rectangular shapes

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & 6 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & \ \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 & 5 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 4 & \ \\
2 & 3 & 5 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 4 & \ \\
2 & 3 & 5 \\
\end{array} \rightarrow \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & 6 \\
\end{array}
\]

\[p\]

\[p^{-1}\]
Rhoades’ definition of cDes for $T \in \text{SYT}(r, \ldots, r)$ declares that 

$$n \in \text{cDes}(T) \text{ iff } n - 1 \in \text{Des}(p^{-1}(T)).$$
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Rhoades’ definition of cDes for \( T \in \text{SYT}(r, \ldots, r) \) declares that
\[ n \in \text{cDes}(T) \iff n - 1 \in \text{Des}(p^{-1}(T)). \]

In fact, \( p \) determines a \( \mathbb{Z}_n \)-action. Here it is for \( \lambda = (3, 3) \):

\[ \begin{array}{ccc}
T & \xrightarrow{p} & T' \\
1 & 3 & 4 \\
\downarrow & & \downarrow \\
2 & 5 & 6
\end{array} \quad \begin{array}{ccc}
1 & 2 & 5 \\
\downarrow & & \downarrow \\
3 & 4 & 6
\end{array} \quad \begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & & \downarrow \\
4 & 5 & 6
\end{array} \quad \begin{array}{ccc}
1 & 3 & 5 \\
\downarrow & & \downarrow \\
2 & 4 & 6
\end{array} \quad \begin{array}{ccc}
1 & 2 & 4 \\
\downarrow & & \downarrow \\
3 & 5 & 6
\end{array} \]

\[ \text{cDes}(T) \quad \{1, 4\} \quad \{2, 5\} \quad \{3, 6\} \quad \{1, 3, 5\} \quad \{2, 4, 6\} \]
Definition

Given a set $\mathcal{T}$ and map $\text{Des} : \mathcal{T} \rightarrow 2^{[n-1]}$, a cyclic descent extension is a pair $(\text{cDes}, \phi)$, where

- $\text{cDes} : \mathcal{T} \rightarrow 2^{[n]}$,
- $\phi : \mathcal{T} \rightarrow \mathcal{T}$ is a bijection,
Reformulation

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(a) $\text{cDes}(T) \cap [n-1] = \text{Des}(T)$,
Reformulation

Definition
Given a set \( \mathcal{T} \) and map \( \text{Des} : \mathcal{T} \to 2^{[n-1]} \), a cyclic descent extension is a pair \((\text{cDes}, \phi)\), where 
\[ \text{cDes} : \mathcal{T} \to 2^{[n]}, \]
\[ \phi : \mathcal{T} \to \mathcal{T} \] is a bijection, 
satisfying the following conditions for all \( T \in \mathcal{T} \):

(a) \( \text{cDes}(T) \cap [n - 1] = \text{Des}(T) \),
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Definition
Given a set $\mathcal{T}$ and map $\text{Des} : \mathcal{T} \to 2^{[n-1]}$, a \textbf{cyclic descent extension} is a pair $(c\text{Des}, \phi)$, where $c\text{Des} : \mathcal{T} \to 2^{[n]}$, $\phi : \mathcal{T} \to \mathcal{T}$ is a bijection, satisfying the following conditions for all $T \in \mathcal{T}$:

(a) $c\text{Des}(T) \cap [n-1] = \text{Des}(T)$,
(b) $c\text{Des}(\phi(T)) = c\text{Des}(T) + 1$.

Examples
- $\mathcal{T} = S_n$, with Cellini’s cDes and $\phi = \text{cyclic rotation}$.
- $\mathcal{T} = \text{SYT}(r, \ldots, r)$, with Rhoades’ cDes and $\phi = \text{promotion}$.
Reformulation

Motivating Problem:

Is there a cyclic descent extension on $\text{SYT}(\lambda/\mu)$?
Cyclic descents on \( \text{SYT}(\lambda \Box) \)

For a partition \( \lambda \vdash n - 1 \), let \( \lambda \Box \) be the skew shape obtained from \( \lambda \) by placing a disconnected box at its upper right corner.

Example

\[
(3, 3, 1) \Box = \begin{array}{ccc}
\square & & \\
 & \square & \\
\end{array}
\]

Theorem (E.-Roichman '16)

For every \( \lambda \vdash n - 1 \), there exists a cyclic descent extension on \( \text{SYT}(\lambda \Box) \). What is the definition of \( \text{cDes} \) and \( \phi \) in this case?
Cyclic descents on $\text{SYT}(\lambda \Box)$

For a partition $\lambda \vdash n - 1$, let $\lambda \Box$ be the skew shape obtained from $\lambda$ by placing a disconnected box at its upper right corner.

Example

$$(3, 3, 1) \Box = \begin{array}{ccc}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}$$

Theorem (E.-Roichman '16)

For every $\lambda \vdash n - 1$, there exists a cyclic descent extension on $\text{SYT}(\lambda \Box)$. 
Cyclic descents on $\text{SYT}(\lambda \Box)$

For a partition $\lambda \vdash n - 1$, let $\lambda \Box$ be the skew shape obtained from $\lambda$ by placing a disconnected box at its upper right corner.

Example

$$(3, 3, 1) \Box = \begin{array}{c}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \\
\cdot & & \\
\end{array}
\end{array}$$

Theorem (E.-Roichman '16)

For every $\lambda \vdash n - 1$, there exists a cyclic descent extension on $\text{SYT}(\lambda \Box)$.

What is the definition of cDes and $\phi$ in this case?
Definition of cDes on SYT(λ□)

Example:

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 3 & 4 \\
\end{array} & \begin{array}{ccc}
2 & 4 & 1 \\
3 & 4 & 2 \\
\end{array} & \begin{array}{ccc}
1 & 3 & 2 \\
4 & 3 & 4 \\
\end{array} & \begin{array}{ccc}
1 & 2 & 3 \\
4 & 2 & 3 \\
\end{array}
\]

\{1, 4\} \quad \{1, 2\} \quad \{2, 3\} \quad \{3, 4\}
Definition of cDes on SYT(\(\lambda\Box\))

Example:

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 3 & \\
\end{array}
\quad
\begin{array}{ccc}
2 & 4 & 1 \\
4 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 2 \\
4 & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & \\
\end{array}
\]

\{1, 4\} \quad \{1, 2\} \quad \{2, 3\} \quad \{3, 4\}

For \( T \in \text{SYT}(\lambda\Box) \), let \( n \in \text{cDes}(T) \) iff

- \( n \) is strictly north of 1, or
- \( n - d \in \text{Des}(\text{jdt}(T - d)) \), where \( d \) is the letter in the disconnected cell of \( T \).
Definition of \textit{cDes} on SYT(\(\lambda\square\))

Example:

\[
\begin{array}{ccc}
1 & 3 & 4 \\
2 & 4 & \\
3 & & \\
\end{array}
\quad
\begin{array}{ccc}
2 & 4 & 1 \\
3 & 4 & \\
& & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 2 \\
4 & & \\
& & \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & & \\
& & \\
\end{array}
\]

\{1, 4\} \quad \{1, 2\} \quad \{2, 3\} \quad \{3, 4\}

For \(T \in \text{SYT}(\lambda\square)\), let \(n \in \text{cDes}(T)\) iff

\begin{itemize}
  \item \(n\) is strictly north of 1, or
  \item \(n - d \in \text{Des}(\text{jdt}(T - d))\), where \(d\) is the letter in the disconnected cell of \(T\).
\end{itemize}

What is \(\text{jdt}(T - d)\)?
A *jeu-de-taquin* straightening algorithm

Given an SYT $T$ with $n$ boxes, let $T + k$ be obtained by adding $k \mod n$ to each entry. 

$T = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & \end{array} \quad T + 3 = \begin{array}{ccc}
4 & 6 & 2 \\
5 & 1 & \end{array}$
A *jeu-de-taquin* straightening algorithm

Given an SYT $T$ with $n$ boxes, let $T + k$ be obtained by adding $k \mod n$ to each entry.

Let $\text{jdt}(T + k)$ be the SYT obtained from $T + k$ by repeatedly applying the following step:

Let $i$ be the minimal entry for which the entry immediately above or to its left is $> i$.

Switch $i$ with the larger of these two entries.
A *jeu-de-taquin* straightening algorithm

Given an SYT $T$ with $n$ boxes, let $T + k$ be obtained by adding $k \mod n$ to each entry.

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1 & 3 & 5 \\
2 & 4 \\
\end{array}$

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2 & 4 & \\
\end{array} \quad T + 3 = \begin{array}{ccc}
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5 & 1 & \\
\end{array}$$

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\end{array}$$

Let $jdt(T + k)$ be the SYT obtained from $T + k$ by repeatedly applying the following step:

- Let $i$ be the minimal entry for which the entry immediately above or to its left is $> i$.
- Switch $i$ with the larger of these two entries.
A *jeu-de-taquin* straightening algorithm

Given an SYT $T$ with $n$ boxes, let $T + k$ be obtained by adding $k \mod n$ to each entry.

Let $jdt(T + k)$ be the SYT obtained from $T + k$ by repeatedly applying the following step:

- Let $i$ be the minimal entry for which the entry immediately above or to its left is $> i$.
- Switch $i$ with the larger of these two entries.

\[
\begin{array}{c}
4 & 6 & 2 \\
5 & 1 \\
\end{array} \quad \mapsto \quad \begin{array}{c}
4 & 1 & 2 \\
5 & 6 \\
\end{array} \quad \mapsto \quad \begin{array}{c}
1 & 4 & 2 \\
5 & 6 \\
\end{array}
\]

\[
T = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
\end{array} \quad T + 3 = \begin{array}{ccc}
4 & 6 & 2 \\
5 & 1 \\
\end{array}
\]

Note: promotion is just $p(T) = jdt(T + 1)$, $p^{-1}(T) = jdt(T - 1)$. 
A jeu-de-taquin straightening algorithm

Given an SYT $T$ with $n$ boxes, let $T + k$ be obtained by adding $k \mod n$ to each entry.

Let $\text{jdt}(T + k)$ be the SYT obtained from $T + k$ by repeatedly applying the following step:

1. Let $i$ be the minimal entry for which the entry immediately above or to its left is $> i$.
2. Switch $i$ with the larger of these two entries.

\[
\begin{array}{cccc}
4 & 6 & 2 & \rightarrow 4 & 1 & 2 \\
5 & 1 & & \rightarrow 1 & 4 & 2 \\
& & & \rightarrow 1 & 2 & 4 \\
\end{array}
\]

$= \text{jdt}(T + 3)$
A *jeu-de-taquin* straightening algorithm

Given an SYT $T$ with $n$ boxes, let $T + k$ be obtained by adding $k \mod n$ to each entry.

Let $\text{jdt}(T + k)$ be the SYT obtained from $T + k$ by repeatedly applying the following step:

- Let $i$ be the minimal entry for which the entry immediately above or to its left is $> i$.
- Switch $i$ with the larger of these two entries.

\[
\begin{align*}
4 & \quad 6 & \quad 2 \\
5 & \quad 1 \\
\end{align*} \quad \mapsto \quad \begin{align*}
4 & \quad 1 & \quad 2 \\
5 & \quad 6 \\
\end{align*} \quad \mapsto \quad \begin{align*}
1 & \quad 4 & \quad 2 \\
5 & \quad 6 \\
\end{align*} \quad \mapsto \quad \begin{align*}
1 & \quad 2 & \quad 4 \\
5 & \quad 6 \\
\end{align*} = \text{jdt}(T + 3)
\]

Note: promotion is just $p(T) = \text{jdt}(T + 1)$, $p^{-1}(T) = \text{jdt}(T - 1)$. 
Definition of cDes on SYT(λ□)

For $T \in \text{SYT}(\lambda □)$, define $n \in \text{cDes}(T)$ iff

$\triangleright$ $n$ is strictly north of 1, or

$\triangleright$ $n - d \in \text{Des}(\text{jdt}(T - d))$, where $d$ is the letter in the disconnected cell of $T$.

$T = \begin{array}{ccc}
1 & 2 & 3 \\
& 4 & \\
& & 3
\end{array}$

$\{1, 4\}$ $\{1, 2\}$ $\{2, 3\}$ $\{3, 4\}$
Definition of cDes on SYT($\lambda \square$)

For $T \in \text{SYT}(\lambda \square)$, define $n \in \text{cDes}(T)$ iff

- $n$ is strictly north of 1, or
- $n - d \in \text{Des}(\text{jdt}(T - d))$, where $d$ is the letter in the disconnected cell of $T$.
Definition of $c\text{Des}$ on $\text{SYT}(\lambda \square)$

For $T \in \text{SYT}(\lambda \square)$, define $n \in c\text{Des}(T)$ iff

- $n$ is strictly north of 1, or
- $n - d \in \text{Des}(\text{jdt}(T - d))$, where $d$ is the letter in the disconnected cell of $T$.

$T = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 2 & 3 \end{array}$

$T - 3 = \begin{array}{ccc} 2 & 3 & 4 \\ 1 & 2 & 4 \end{array} \rightarrow \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & 1 \end{array} = \text{jdt}(T - 3)$
Definition of cDes on $\text{SYT}(\lambda \square)$

For $T \in \text{SYT}(\lambda \square)$, define $n \in \text{cDes}(T)$ iff

- $n$ is strictly north of 1, or
- $n - d \in \text{Des}(\text{jdt}(T - d))$, where $d$ is the letter in the disconnected cell of $T$.

For $T = \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 2 & 3 \end{array}$, $T - 3 = \begin{array}{ccc} 2 & 3 & 4 \\ 1 & 2 & 4 \end{array}$, $\text{jdt}(T - 3) = \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 4 & 3 \end{array}$

$4 \in \text{cDes}$

$4 - 3 = 1 \in \text{Des}$
The bijection $\phi$ that rotates $c\text{Des}$ on $\text{SYT}(\lambda \Box)$

The map $\phi : \text{SYT}(\lambda \Box) \to \text{SYT}(\lambda \Box)$ given by

$$\phi(T) = jdt \left( jdt(T - d) + d + 1 \right),$$

where $d$ is the letter in the disconnected cell of $T$, is a bijection such that $c\text{Des}(\phi(T)) = c\text{Des}(T) + 1$ for all $T$. 
The bijection $\phi$ that rotates $\text{cDes}$ on $\text{SYT} (\lambda \square)$

The map $\phi : \text{SYT} (\lambda \square) \to \text{SYT} (\lambda \square)$ given by

$$\phi(T) = \text{jdt} \left( \text{jdt} (T - d) + d + 1 \right),$$

where $d$ is the letter in the disconnected cell of $T$,

is a bijection such that $\text{cDes}(\phi(T)) = \text{cDes}(T) + 1$ for all $T$.

In fact, $\phi$ determines a $\mathbb{Z}_n$-action on $\text{SYT} (\lambda \square)$. 
The bijection $\phi$ that rotates $c\text{Des}$ on $\text{SYT}(\lambda\Box)$

The map $\phi : \text{SYT}(\lambda\Box) \rightarrow \text{SYT}(\lambda\Box)$ given by

$$\phi(T) = \text{jdt} \left( \text{jdt}(T - d) + d + 1 \right),$$

where $d$ is the letter in the disconnected cell of $T$, is a bijection such that $c\text{Des}(\phi(T)) = c\text{Des}(T) + 1$ for all $T$.

In fact, $\phi$ determines a $\mathbb{Z}_n$-action on $\text{SYT}(\lambda\Box)$.

Example:

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
&6& & & & & & \\
1&3&5& & & & & \\
2&4& & & & & & \\
3&5& & & & & & \\
1&3&5& & & & & \\
2&4& & & & & & \\
3&5& & & & & & \\
1&2&4& & & & & \\
3&4&6& & & & & \\
1&3&5& & & & & \\
1&2&4& & & & & \\
1&2&4& & & & & \\
\end{array}
\]

$c\text{Des}$

\{1, 3, 6\} \rightarrow \{1, 2, 4\} \rightarrow \{2, 3, 5\} \rightarrow \{3, 4, 6\} \rightarrow \{1, 4, 5\} \rightarrow \{2, 5, 6\}
Cyclic descent extensions for other shapes

Theorem (Adin-E.-Roichman ’17)

There exists a cyclic descent extension on $\text{SYT}(\lambda/\mu)$ for $\lambda/\mu$ of each of these shapes:
Theorem (Adin-E.-Roichman '17)

There exists a cyclic descent extension on $\text{SYT}(\lambda/\mu)$ for $\lambda/\mu$ of each of these shapes:

(strip)
Cyclic descent extensions for other shapes

Theorem (Adin-E.-Roichman '17)

There exists a cyclic descent extension on $\text{SYT}(\lambda/\mu)$ for $\lambda/\mu$ of each of these shapes:

- (strip)
- (hook plus a box)
Theorem (Adin-E.-Roichman ’17)

There exists a cyclic descent extension on \( \text{SYT}(\lambda/\mu) \) for \( \lambda/\mu \) of each of these shapes:

- (strip)
- (hook plus a box)
- (two-row straight)
Theorem (Adin-E.-Roichman '17)

There exists a cyclic descent extension on $\text{SYT}(\lambda/\mu)$ for $\lambda/\mu$ of each of these shapes:

- **(strip)**
- **(hook plus a box)**
- **(two-row straight)**
- **(two-row skew)**
Theorem (Adin-E.-Roichman '17)

There exists a cyclic descent extension on $\text{SYT}(\lambda/\mu)$ for $\lambda/\mu$ of each of these shapes:

- (strip)
- (hook plus a box)
- (two-row straight)
- (two-row skew)

In each case we have an explicit combinatorial definition of cDes.
Definition of cDes on strips

Let $\lambda/\mu$ be a strip of size $n$, i.e., a shape whose components are one-row or one-column shapes.
Definition of $c\text{Des}$ on strips

Let $\lambda/\mu$ be a **strip** of size $n$, i.e., a shape whose components are one-row or one-column shapes.

For $T \in \text{SYT}(\lambda/\mu)$, let $n \in c\text{Des}(T)$ iff
- $n$ is strictly north of 1, or
- 1 and $n$ are in the same vertical component.
Definition of cDes on strips

Let $\lambda/\mu$ be a strip of size $n$, i.e., a shape whose components are one-row or one-column shapes.

For $T \in \text{SYT}(\lambda/\mu)$, let $n \in \text{cDes}(T)$ iff

- $n$ is strictly north of 1, or
- 1 and $n$ are in the same vertical component.

Equivalently, $n \in \text{cDes}(T)$ iff $n - 1 \in \text{Des}(p^{-1}(T))$. 
Definition of $\phi$ on strips

Let $\lambda/\mu$ be a strip of size $n$, i.e., a shape whose components are one-row or one-column shapes.

As in the case of rectangles, the promotion operator $p : T \mapsto \text{jdt}(T + 1)$ shifts $\text{cDes}$.

Let $\lambda/\mu$ be a strip of size $n$, i.e., a shape whose components are one-row or one-column shapes.

As in the case of rectangles, the promotion operator $p : T \mapsto \text{jdt}(T + 1)$ shifts $\text{cDes}$.
Definition of cDes on hooks plus a box

Let \( \lambda = (n - k - 2, 2, 1^k) \), where \( 0 \leq k \leq n - 4 \).
Definition of $c\text{Des}$ on hooks plus a box

Let $\lambda = (n - k - 2, 2, 1^k)$, where $0 \leq k \leq n - 4$.

For $T \in \text{SYT}(\lambda)$, let $n \in c\text{Des}(T)$ iff

- $T_{2,2} - 1$ is in the first column of $T$. 
Definition of cDes on hooks plus a box

Let $\lambda = (n - k - 2, 2, 1^k)$, where $0 \leq k \leq n - 4$.

For $T \in \text{SYT}(\lambda)$, let $n \in \text{cDes}(T)$ iff

- $T_{2,2} - 1$ is in the first column of $T$.

For this shape, this definition of cDes is unique.
Definition of cDes on hooks plus a box

Let $\lambda = (n - k - 2, 2, 1^k)$, where $0 \leq k \leq n - 4$.

For $T \in \text{SYT}(\lambda)$, let $n \in c\text{Des}(T)$ iff

$T_{2,2} - 1$ is in the first column of $T$.

For this shape, this definition of cDes is unique.

We have a complicated explicit definition of a bijection $\phi$ that shifts cDes. It determines a $\mathbb{Z}$-action, but not a $\mathbb{Z}_n$-action.
Non-uniqueness of cDes

For many shapes, cyclic descent completions are not unique.

Example: Let $\lambda = (4, 2)/(2)$.

\[
\begin{array}{ccccccccc}
1 & 4 & 2 & 3 & 3 & 4 & 1 & 4 & 1 & 3 & 2 & 4 \\
2 & 3 & 1 & 2 & 1 & 2 & 3 & 4 & 2 & 4 & 1 & 3
\end{array}
\]
Non-uniqueness of cDes

For many shapes, cyclic descent completions are not unique.

Example: Let $\lambda = (4, 2)/(2)$.

\[
\begin{array}{cccccc}
1 & 4 & 1 & 2 & 2 & 3 \\
2 & 3 & 3 & 4 & 1 & 4 \\
\end{array}
\begin{array}{cccccc}
2 & 3 & 1 & 4 & 1 & 2 \\
3 & 4 & 2 & 4 & 2 & 4 \\
\end{array}
\begin{array}{cccccc}
3 & 4 & 1 & 3 & 1 & 3 \\
2 & 4 & 1 & 3 & 2 & 4 \\
\end{array}
\]

Our definition of cDes:

\[
\{1\} \quad \{2\} \quad \{3\} \quad \{4\} \quad \{1, 3\} \quad \{2, 4\}
\]
Non-uniqueness of cDes

For many shapes, cyclic descent completions are not unique.

Example: Let $\lambda = (4, 2)/(2)$.

Our definition of cDes:

\[
\begin{align*}
\{1\} & \quad \{2\} & \quad \{3\} & \quad \{4\} & \quad \{1, 3\} & \quad \{2, 4\}
\end{align*}
\]

Another possible definition of cDes:

\[
\begin{align*}
\{1\} & \quad \{2, 4\} & \quad \{3\} & \quad \{4\} & \quad \{1, 3\} & \quad \{2\}
\end{align*}
\]
Non-uniqueness of $\phi$

Even for shapes where cDes in unique, different definitions of $\phi$ may give different orbit lengths:
Non-uniqueness of $\phi$

Even for shapes where $c\text{Des}$ is unique, different definitions of $\phi$ may give different orbit lengths:

(cDes in red)
Non-uniqueness of $\phi$

Even for shapes where cDes in unique, different definitions of $\phi$ may give different orbit lengths:

(cDes in red)
Definition of cDes on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$. 

![Diagram of a two-row shape with labels](image)

Examples:
- $9 \in \text{cDes}(\begin{array}{cccc} 1 & 2 & 3 & 5 \\ 4 & 6 & 7 & 8 \end{array})$ because $8 = 7 + 1$, $4 > 2$ and $6 > 3$.
- $9 / \in \text{cDes}(\begin{array}{cccc} 1 & 3 & 4 & 6 \\ 2 & 5 & 7 & 8 \end{array})$ because $2 < 3$. 
Definition of cDes on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$.

For $T \in \text{SYT}(\lambda)$, let $n \in \text{cDes}(T)$ iff

- the last two entries in the second row of $T$ are consecutive, that is, $T_{2,k} = T_{2,k-1} + 1$;
Definition of cDes on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$.

For $T \in \text{SYT}(\lambda)$, let $n \in \text{cDes}(T)$ iff

- the last two entries in the second row of $T$ are consecutive, that is, $T_{2,k} = T_{2,k-1} + 1$; and
- $T_{2,i-1} > T_{1,i}$ for every $1 < i < k$. 

Examples:

- $9 \in \text{cDes}(123594678)$ because $8 = 7 + 1$, $4 > 2$ and $6 > 3$.
- $9 \notin \text{cDes}(134692578)$ because $2 < 3$. 

Sergi Elizalde
Cyclic descents of standard Young tableaux
Definition of cDes on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$.

For $T \in \text{SYT}(\lambda)$, let $n \in \text{cDes}(T)$ iff

- the last two entries in the second row of $T$ are consecutive, that is, $T_{2,k} = T_{2,k-1} + 1$; and
- $T_{2,i-1} > T_{1,i}$ for every $1 < i < k$.

Examples:

$9 \in \text{cDes} \begin{pmatrix} 1 & 2 & 3 & 5 & 9 \\ 4 & 6 & 7 & 8 \end{pmatrix}$ because $8 = 7 + 1$, $4 > 2$ and $6 > 3$. 

Definition of cDes on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$.

For $T \in \text{SYT}(\lambda)$, let $n \in \text{cDes}(T)$ iff

- the last two entries in the second row of $T$ are consecutive, that is, $T_{2,k} = T_{2,k-1} + 1$; and
- $T_{2,i-1} > T_{1,i}$ for every $1 < i < k$.

Examples:

- $9 \in \text{cDes} \begin{pmatrix} 1 & 2 & 3 & 5 & 9 \\ 4 & 6 & 7 & 8 \end{pmatrix}$ because $8 = 7 + 1$, $4 > 2$ and $6 > 3$.
- $9 \notin \text{cDes} \begin{pmatrix} 1 & 3 & 4 & 6 & 9 \\ 2 & 5 & 7 & 8 \end{pmatrix}$ because $2 < 3$. 
Definition of cDes on two-row straight shapes

Remarks

- When $\lambda = (n - 2, 2)$, the definition of cDes viewed as a two-row shape coincides with the definition viewed as a hook plus a box.

\[
\begin{array}{cccc}
\text{□} & & & \\
\text{□} & & & \\
\end{array}
\]
Definition of $c\text{Des}$ on two-row straight shapes

Remarks

- When $\lambda = (n - 2, 2)$, the definition of $c\text{Des}$ viewed as a two-row shape coincides with the definition viewed as a hook plus a box.

- For $\lambda = (r, r)$, the definition of $c\text{Des}$ viewed as a two-row shape coincides with Rhoades’ definition viewed as a rectangular shape.
Definition of $\phi$ on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$. 

\[
\begin{array}{|c|c|c|}
\hline
\ & \ & \\
\hline
\ & \ & \\
\hline
\end{array}
\]
Definition of $\phi$ on two-row straight shapes

Let $\lambda = (n - k, k)$, where $2 \leq k \leq n/2$.

We have a complicated explicit definition of a map $\phi$ that shifts cDes, which determines a $\mathbb{Z}$-action (but not a $\mathbb{Z}_n$-action).
Definition of \( \phi \) on two-row straight shapes

Let \( \lambda = (n - k, k) \), where \( 2 \leq k \leq n/2 \).

We have a complicated explicit definition of a map \( \phi \) that shifts cDes, which determines a \( \mathbb{Z} \)-action (but not a \( \mathbb{Z}_n \)-action).

Example:

\[
\begin{array}{cccc}
1 & 3 & 5 & 6 \\
2 & 4 & 8 & 9
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 2 & 4 & 7 \\
3 & 5 & 6 & 9
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 2 & 3 & 5 \\
4 & 6 & 7 & 8
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 3 & 4 & 6 \\
2 & 5 & 7 & 8
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 2 & 5 & 7 \\
3 & 4 & 6 & 8
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 6 \\
4 & 5 & 7 & 9
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 8 & 9
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 3 & 4 & 5 \\
2 & 6 & 7 & 9
\end{array}
\xrightarrow{\phi}
\begin{array}{cccc}
1 & 2 & 4 & 5 \\
3 & 7 & 8 & 9
\end{array}
\]

(cDes in red)
Definition of cDes on two-row skew shapes

Let $\lambda/\mu = (n-k+m, k)/(m)$ with $k \neq m+1$. 

\[
\begin{array}{cccc}
& & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]
Definition of cDes on two-row skew shapes

Let $\lambda/\mu = (n - k + m, k)/(m)$ with $k \neq m + 1$. 

\[
\begin{array}{c}
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\end{array}
\end{array}
Definition of cDes on two-row skew shapes

Let $\lambda/\mu = (n - k + m, k)/(m)$ with $k \neq m + 1$.

We have two different definitions of cDes on $\lambda/\mu$ that work, but both are complicated.
Let $\lambda/\mu = (n - k + m, k)/(m)$ with $k \neq m + 1$.

We have two different definitions of cDes on $\lambda/\mu$ that work, but both are complicated.

We do not have an explicit description of $\phi$ in this case.
How about other shapes?

For which shapes $\lambda/\mu$ is there a cyclic descent extension for $\text{SYT}(\lambda/\mu)$?
Definition
A connected skew shape $\lambda/\mu$ is a ribbon if it does not contain a $2 \times 2$ rectangle.

Examples:
Connected ribbons

Definition
A connected skew shape $\lambda/\mu$ is a ribbon if it does not contain a $2 \times 2$ rectangle.

Examples:

```
  A connected skew shape
  B
C
```

Proposition
If $\lambda/\mu$ is a connected ribbon, then there is no cyclic descent extension on $\text{SYT}(\lambda/\mu)$.
After running computations for all partitions of size $n < 16$...

**Conjecture (Adin-E.-Roichman '16)**

*For every $\lambda$ that is not a hook, there is a cyclic descent extension on SYT($\lambda$).*
After running computations for all partitions of size $n < 16$...

**Conjecture (Adin-E.-Roichman ’16)**

_for every $\lambda$ that is not a hook, there is a cyclic descent extension on $\text{SYT}(\lambda)$._

**Theorem (Adin-Reiner-Roichman ’17)**

_for every skew shape $\lambda/\mu$ that is not a connected ribbon, there is a cyclic descent extension on $\text{SYT}(\lambda/\mu)$._
After running computations for all partitions of size $n < 16$...

**Conjecture (Adin-E.-Roichman '16)**

*For every $\lambda$ that is not a hook, there is a cyclic descent extension on $\text{SYT}(\lambda)$.***

**Theorem (Adin-Reiner-Roichman '17)**

*For every skew shape $\lambda/\mu$ that is not a connected ribbon, there is a cyclic descent extension on $\text{SYT}(\lambda/\mu)$.***

The proof uses affine symmetric functions, Gromov-Witten invariants, and nonnegativity properties of Postnikov’s toric Schur polynomials.
After running computations for all partitions of size $n < 16$...

**Conjecture (Adin-E.-Roichman ’16)**

*For every $\lambda$ that is not a hook, there is a cyclic descent extension on $\text{SYT}(\lambda)$.*

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*For every skew shape $\lambda/\mu$ that is not a connected ribbon, there is a cyclic descent extension on $\text{SYT}(\lambda/\mu)$.***

The proof uses affine symmetric functions, Gromov-Witten invariants, and nonnegativity properties of Postnikov’s toric Schur polynomials.

Unfortunately, it *does not provide an explicit description* of $\text{cDes}$ on a given $\text{SYT}$. 
Problem: For each non-ribbon shape $\lambda/\mu$:

- Find an explicit combinatorial description of $c\text{Des}$ on $\text{SYT}(\lambda/\mu)$. 
Future work

**Problem:** For each non-ribbon shape $\lambda/\mu$:

- Find an explicit combinatorial description of $c\text{Des}$ on $\text{SYT}(\lambda/\mu)$.
- Describe an explicit bijection $\phi$ that shifts $c\text{Des}$ cyclically and, ideally, generates a $\mathbb{Z}_n$-action.
Problem: For each non-ribbon shape $\lambda/\mu$:

- Find an explicit combinatorial description of $c$Des on $\text{SYT}(\lambda/\mu)$.
- Describe an explicit bijection $\phi$ that shifts $c$Des cyclically and, ideally, generates a $\mathbb{Z}_n$-action.

Thanks!
Deadline for submissions: November 14, 2017
Deadline for submissions:
November 14, 2017

Also:
Permutation Patterns
Dartmouth College
July 9-14, 2018