Penney's game for permutations

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Dartmouth

AMS Fall Eastern Sectional Meeting
Special Session on Permutation Patterns
Albany, NY, October 2024

Penney's game: original version

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It is known that, for any word picked by Player A, Player B can always pick a word that will be more likely to appear first.

A game with this property is called a *non-transitive* game.

Conway's formula

Given two binary words v and w, let

$$b(v, w) = \sum_{i} 2^{i-1},$$

where the sum is over those i such that $w_1 \dots w_i = v_{\ell-i+1} \dots v_k$.

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Theorem (Conway '74)

The probability that w appears before v is

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Felix'16 characterized the words w that maximize this probability for any given v.

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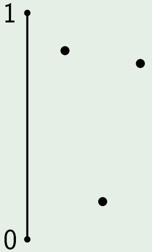

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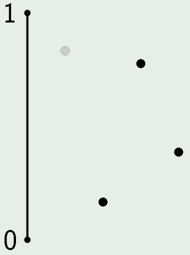

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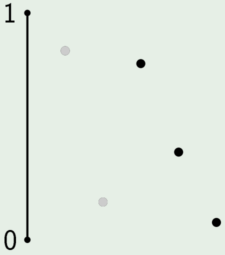

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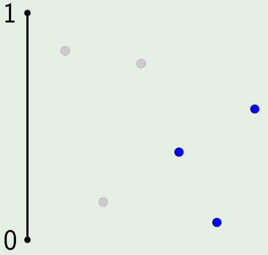

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213 $\rightarrow$ Player B wins

Questions about Penney's game for permutations

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Let us start with an easier question:

- How long do we have to wait until σ appears for the first time?

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$$\mathbb{E}T_\sigma = P_\sigma(1).$$

Examples:

$$\mathbb{E}T_{123} = \frac{\sqrt{3e}}{2\cos(\frac{\sqrt{3}}{2} + \frac{\pi}{6})} \approx 7.924, \quad \mathbb{E}T_{132} = \frac{1}{1 - \int_0^1 e^{-t^2/2} dt} \approx 6.926.$$

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Side note: The theorem $\mathbb{E}T_\sigma = P_\sigma(1)$ extends to vincular patterns. For example,

$$\mathbb{E}T_{1\text{-}23} = \sum_{n\geq 0} \frac{\mathrm{Bell}_n}{n!} = e^{e-1} \approx 5.575.$$

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And if σ is any classical pattern of length 3,

$$\mathbb{E}T_\sigma = \sum_{n\geq 0} \frac{\mathrm{Cat}_n}{n!} \approx 5.091.$$

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For any vincular pattern σ, the value $\mathbb{E}T_\sigma$ gives a measure of how easy it is to avoid σ in a random permutation.

The probability of seeing one pattern before another

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Here are the values of $\Pr(\sigma \prec \tau)$ for patterns of length 3:

σ \ τ	123	132	213	231	312	321
123	–	**0.5**	**0.412**	**0.550**	**0.342**	0.5
132	0.5	–	**0.461**	**0.476**	0.5	0.658
213	0.588	0.539	–	0.5	0.524	0.450
231	0.450	0.524	0.5	–	0.539	0.588
312	0.658	0.5	0.476	0.461	–	0.5
321	0.5	0.342	0.550	0.412	0.5	–

Because of trivial symmetries, it is enough to compute the **orange** entries.

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For $\sigma, \tau \in \mathcal{S}_k$, let

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Lemma

$$\Pr(\sigma \prec \tau) = \sum_{n \geq k} \frac{|\mathrm{Av}_n^\sigma(\sigma,\tau)|}{n!}.$$

Theorem

$$\Pr(123 \prec 132) = \frac{1}{2},$$

$$\Pr(132 \prec 231) = \frac{e^2 - 2e - 1}{2} \approx 0.476,$$

$$\Pr(123 \prec 213) = e^{\frac{3}{2}} \left(2 - 3 \int_0^1 e^{-t-\frac{t^2}{2}} dt \right) - 1 \approx 0.412.$$

Non-transitivity

Penney's game for permutations is non-transitive:

$$\Pr(123 \prec 231) > \tfrac{1}{2}\ , \quad \Pr(312 \prec 123) > \tfrac{1}{2}, \quad \Pr(231 \prec 312) > \tfrac{1}{2}.$$

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$\Pr(123 \prec 231) > \frac{1}{2}$, $\Pr(312 \prec 123) > \frac{1}{2}$, $\Pr(231 \prec 312) > \frac{1}{2}$.

Compare this to the fact that $\mathbb{E}T_{123} > \mathbb{E}T_{231}$, i.e.,
123 is more likely to appear before 231, but
on average we have to wait longer to see 123.

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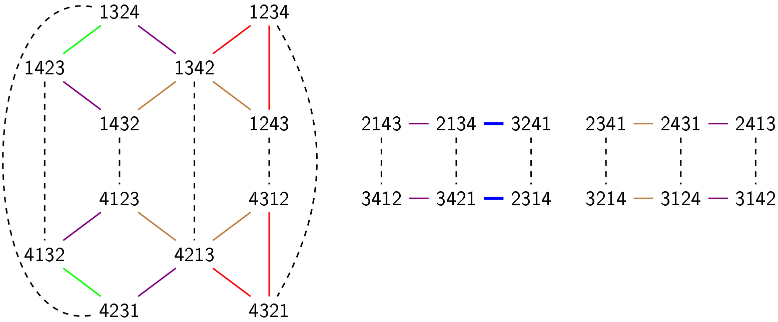
Theorem

Let $2 \le i, i' \le k-1$. *Then,*

$$\begin{aligned} 12\dots k &\equiv 12\dots(i-1)(i+1)\dots ki, \\ 12\dots(i-1)(i+1)\cdots ki &\equiv 12\cdots(i'-1)(i'+1)\dots ki' \quad , \\ 134\dots(k-2)k2(k-1) &\equiv 134\dots(k-2)(k-1)2k \\ 1k\alpha(k-2)(k-1) &\equiv 1(k-1)\beta(k-2)k, \end{aligned}$$

where α *and* β *are permutations of* $\{2,3,\dots,k-3\}$.

All tied pairs of patterns of length 4



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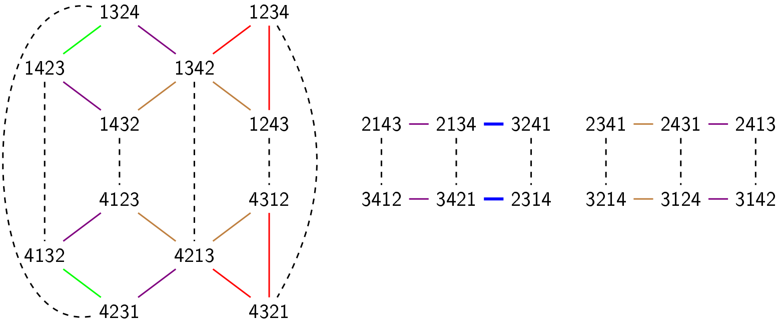

All tied pairs follow from the previous theorem, except for the two blue edges.

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We do not have a direct bijection between these sets.
But for any set I of positions, we can construct a bijection between $\{\pi \in \mathcal{S}_n$ having occurrences of 2134 and 3241 in positions $I\}$ and $\{\pi \in \mathcal{S}_n$ having occurrences of 2314 and 3421 in positions $I\}$ (there may be occurrences in other positions as well).

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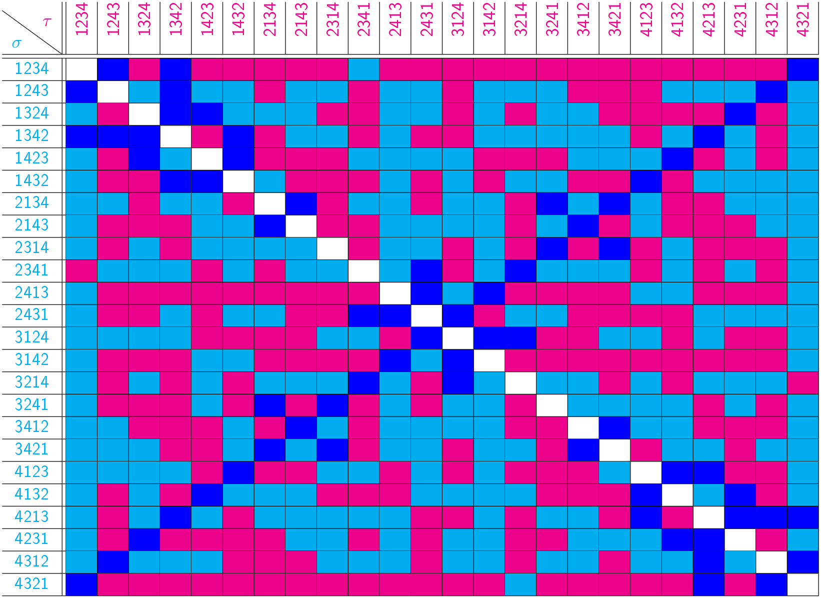
Finally, we apply inclusion-exclusion to get the above equality.

The approximate probabilities $\Pr(\sigma \prec \tau)$ for $\sigma, \tau \in \mathcal{S}_4$



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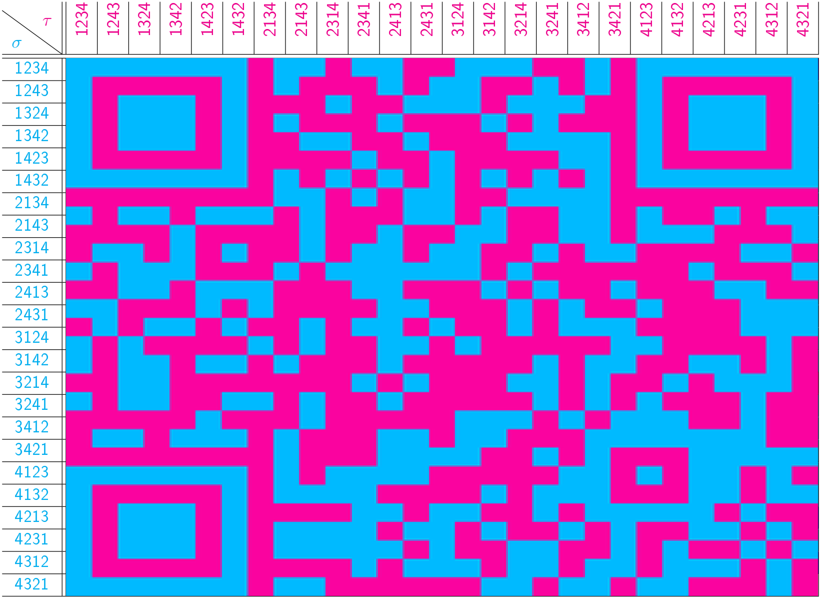

Open questions and conjectures

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- Consider the analogous questions for classical patterns.