Multiple pattern avoidance with respect to fixed points and excedances

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Definitions

$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_k$

π *avoids* σ if there do not exist $i_1 < \ldots < i_k$ s.t. $\pi_{i_1}\cdots\pi_{i_k}$ has type σ (i.e., $\pi_{i_a} < \pi_{i_b} \iff \sigma_a < \sigma_b$).

$\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$

$$\mathcal{S}_n(\sigma_1, \ldots, \sigma_m) := \bigcap_{i=1}^{m} \mathcal{S}_n(\sigma_i)$$

Patterns of length 3

By trivial bijections,

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$$

$$|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$$

Proposition (Knuth '73).

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(132)| = \mathrm{C}_n = \frac{1}{n+1}\binom{2n}{n}$$

Permutation statistics

i is a *fixed point* of π if $\pi_i = i$.
$\mathrm{fp}(\pi) :=$ number of fixed points of π.

i is an *excedance* of π if $\pi_i > i$.
$\mathrm{exc}(\pi) :=$ number of excedances of π.

i is a *descent* of π if $\pi_i > \pi_{i+1}$
(otherwise i is an *ascent*).
$\mathrm{des}(\pi) :=$ number of descents of π.

$\mathrm{lis}(\pi) :=$ length of longest increasing subseq. of π.

$\mathrm{lds}(\pi) :=$ length of longest decreasing subseq. of π.

Ex: if $\pi = 4{\cdot}2{\cdot}17{\cdot}5{\cdot}36$, then
$\mathrm{fp}(\pi) = 2$, $\mathrm{exc}(\pi) = 2$, $\mathrm{des}(\pi) = 4$,
$\mathrm{lis}(\pi) = 3$, $\mathrm{lds}(\pi) = 3$.

Our goal is to study the distribution of statistics on pattern-avoiding permutations.

Motivation:

Thm (Robertson, Saracino, Zeilberger '02).
For any k, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathrm{fp}(\pi) = k\}| \\ = |\{\pi \in \mathcal{S}_n(132) : \mathrm{fp}(\pi) = k\}|$$

Their proof is not bijective.

Questions:

I. Is there a simple bijective proof?

II. Are there similar results for other permutation statistics?

III. What about other patterns?

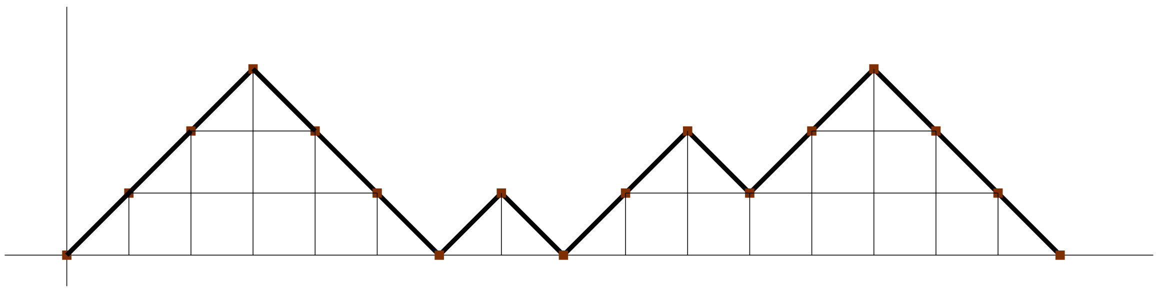
I. Bijective proof

We need some definitions:

Dyck paths

Starts and ends at the x-axis; never goes below it.

Dyck path: steps $\mathbf{u} = (1,1)$ and $\mathbf{d} = (1,-1)$



$\mathcal{D}_n :=$ set of Dyck paths of length $2n$.
$|\mathcal{D}_n| = \mathbf{C}_n$ (Catalan number).
$\mathcal{D} := \bigcup_{n \geq 0} \mathcal{D}_n$ class of all Dyck paths.
$|D| := n$ if $D \in \mathcal{D}_n$.

For a Dyck path D, define:

double rise: $\mathbf{uu}$ (two consecutive up-steps).

peak: $\mathbf{ud}$ (up-step followed by down-step).

hill: peak at lowest height.

$h(D) :=$ number of hills of D

tunnel: horizontal segment between two lattice points of D that stays always below D.

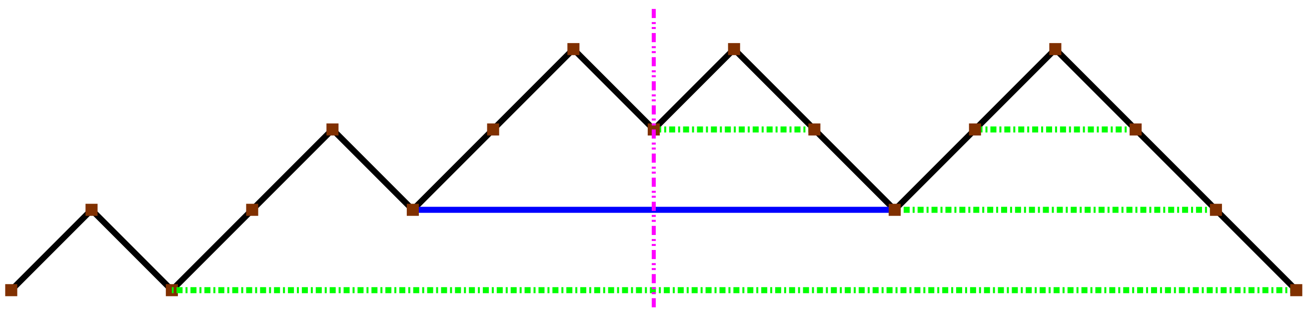
centered tunnel: x-coordinate of midpoint is at the middle of D.

ct$(D) :=$ number of centered tunnels of D.

right tunnel: x-coordinate of midpoint is in the right half of D.

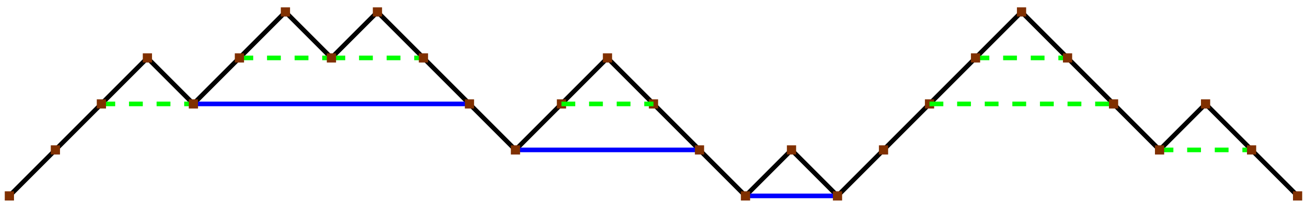
rt$(D) :=$ number of right tunnels of D

lt$(D) :=$ number of left tunnels of D



depth of a tunnel:

$$\text{depth}(T) := \frac{1}{2}\text{length}(T) - \text{height}(T) - 1$$



$\text{td}_0(D)$:= number of **tunnels of depth 0** of D

$\text{td}_{<0}(D)$:= number of **tunnels of negative depth** of D

A simple bijective proof of

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k\}|$$

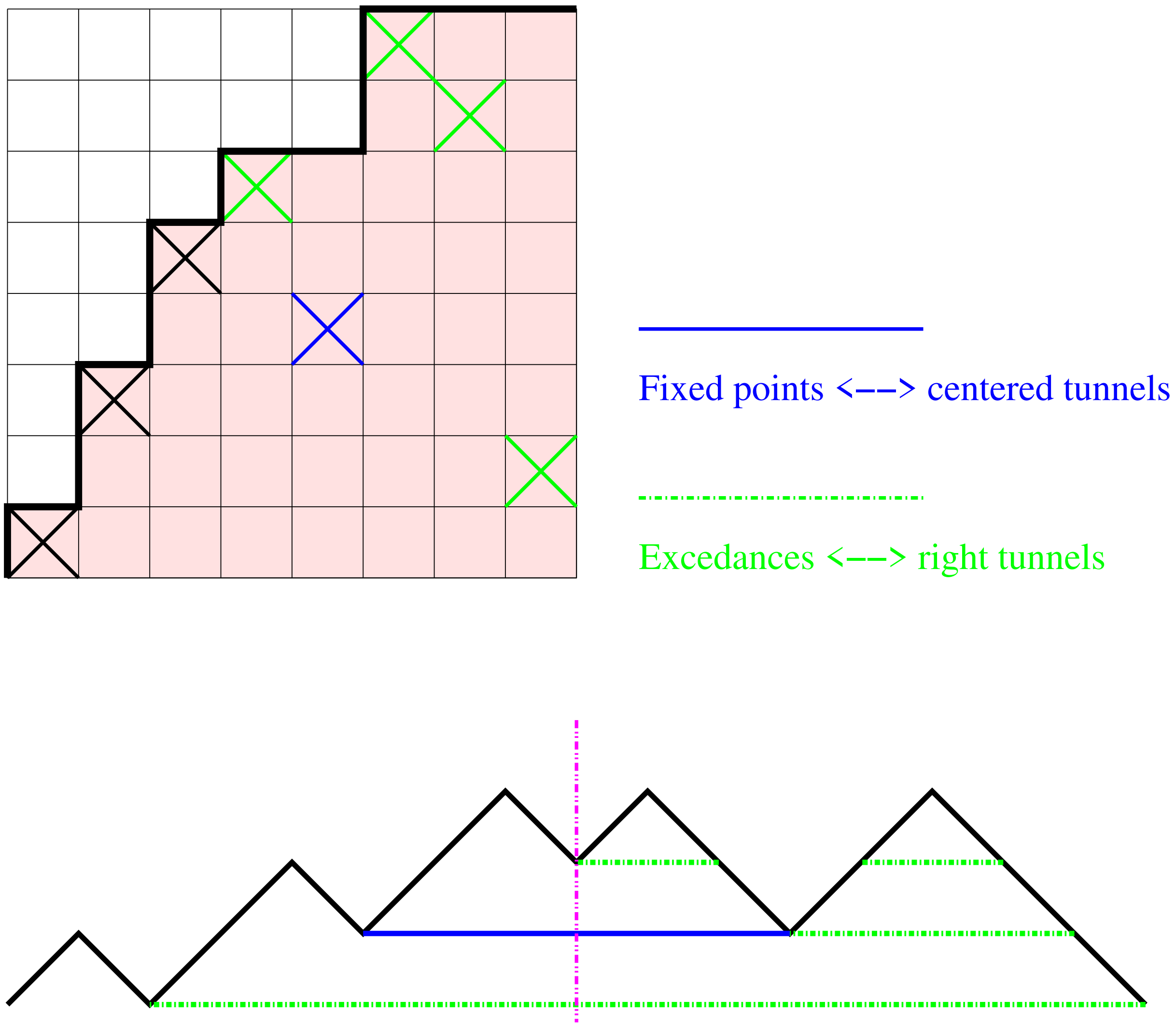
Composition of bijections:

$$\mathcal{S}_n(132) \overset{\varphi}{\longleftrightarrow} \mathcal{D}_n \overset{\Phi}{\longleftrightarrow} \mathcal{D}_n \overset{\psi_{\llcorner}}{\longleftrightarrow} S_n(321)$$

fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills	$\leftrightarrow$	fixed points

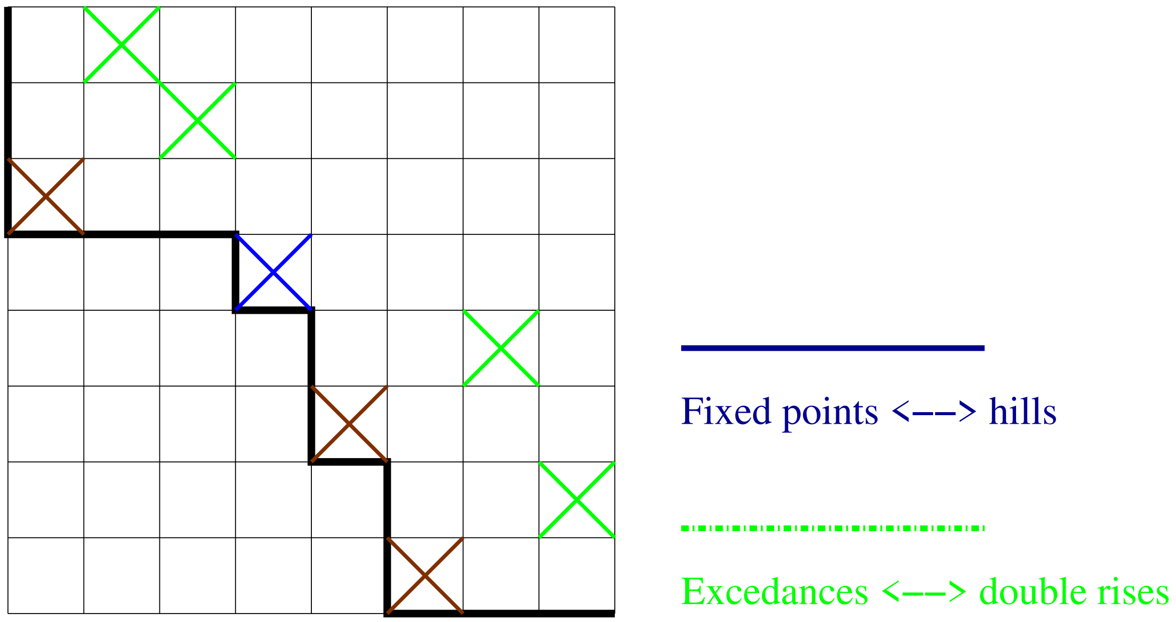
$\varphi : \mathcal{S}_n(132) \xrightarrow{\sim} \mathcal{D}_n$ (Krattenthaler '01)

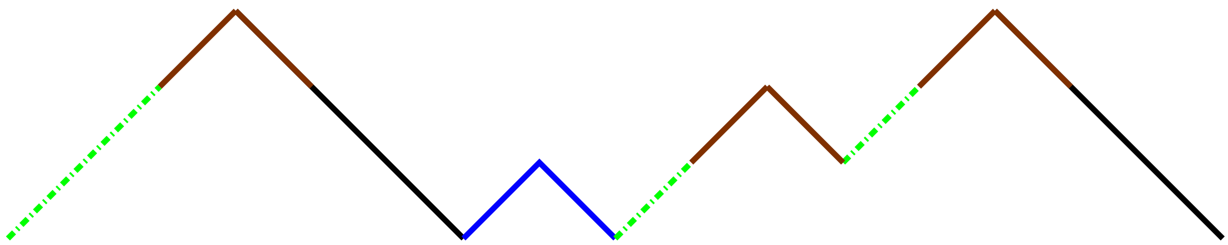
Example: $\pi = 67435281 \in \mathcal{S}_8(132)$



$\psi_{\llcorner} : \mathcal{S}_n(321) \xrightarrow{\sim} \mathcal{D}_n$ (Stanley '03)

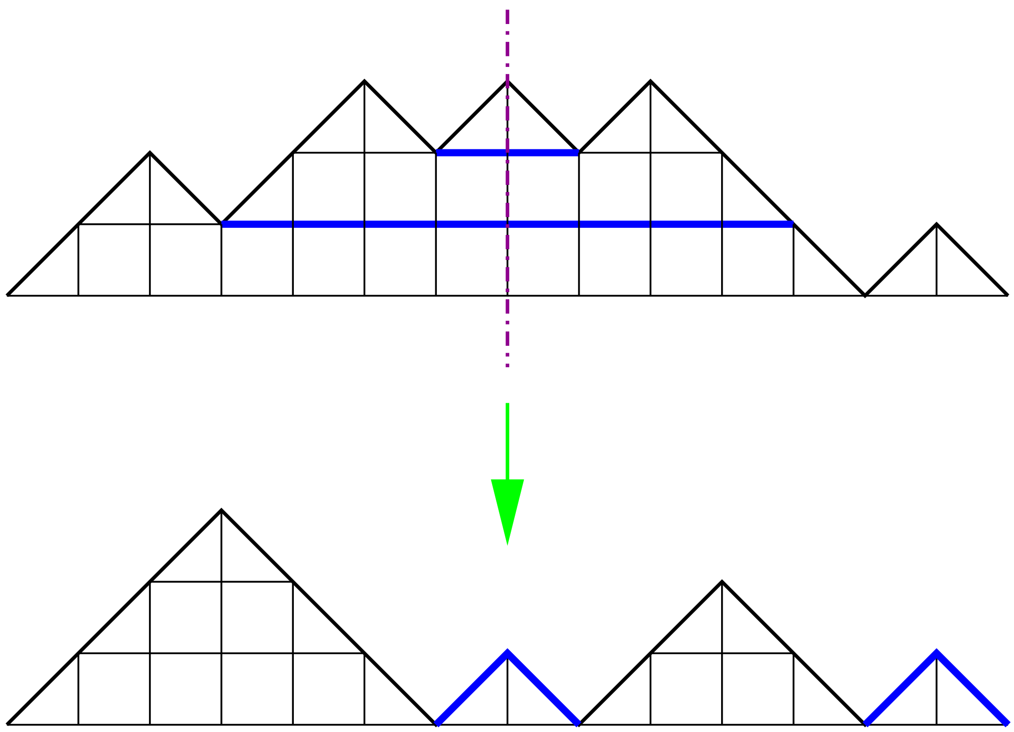
Example: $\pi = 23147586 \in \mathcal{S}_8(321)$





$\Phi : \mathcal{D}_n \xrightarrow{\sim} \mathcal{D}_n$ (Deutsch, E.)

1. Each $\mathbf{u}$ in D has a matching $\mathbf{d}$ (together they determine a tunnel)
2. Read the steps of D in zigzag: $1, 2n, 2, 2n-1, \ldots$
3. For each step, if its corresponding matching step has not yet been read, draw an up-step in $\Phi(D)$. Otherwise, draw a down-step



Φ maps centered tunnels to hills

$\square$

More generally:

Theorem (Deutsch, E.). Let

$$\alpha_r(\pi) := |\{i : \pi_i = i + r\}|$$

$$\beta_r(\pi) := |\{i : i > r, \pi_i = i\}|$$

Then, for any k, r, n,

$$|\{\pi \in \mathcal{S}_n(132) : \alpha_r(\pi) = k\}| \\ = |\{\pi \in \mathcal{S}_n(321) : \beta_r(\pi) = k\}|$$

Proof uses a generalization of Φ

Another application of Φ

Theorem (Deutsch, E.).

$$1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} p^{\mathsf{des}(\pi)+1} z^n$$
$$= \frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{\spadesuit}}$$

where

$$\spadesuit = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2 \\ -2(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$$

Proof:

$\mathcal{S}_n(132)$	$\overset{\varphi}{\longleftrightarrow}$	$\mathcal{D}_n$	$\overset{\Phi}{\longleftrightarrow}$	$\mathcal{D}_n$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	even up-steps
descents+1	$\leftrightarrow$	peaks	$\leftrightarrow$	$\underbrace{\text{“certain statistic”}}_{\text{easier to enumerate}}$

II. Generalization to excedances

Theorem. For any k, l, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$

Equivalently,

$$\sum_{\pi \in \mathcal{S}_n(321)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} = \sum_{\pi \in \mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)}$$

Example (n=4):

$\mathcal{S}_4(321)$

π	fp	exc
1234	4	0
1243	2	1
1324	2	1
1342	1	2
1423	1	1
2134	2	1
2143	0	2
2314	1	2
2341	0	3
2413	0	2
3124	1	1
3142	0	2
3412	0	2
4123	0	1

$\mathcal{S}_4(132)$

π	fp	exc
1234	4	0
2134	2	1
2314	1	2
2341	0	3
3124	1	1
3214	2	1
3241	1	2
3412	0	2
3421	0	2
4123	0	1
4213	1	1
4231	2	1
4312	0	2
4321	0	2

$$x^4 + 3x^2q + 2xq^2 + 2xq + q^3 + 4q^2 + q$$

The original proof is analytical (not bijective) and uses nonstandard techniques of generating functions.

Here we will see a bijective proof (with Igor Pak).

Composition of bijections:

$$\begin{array}{ccccc}
\mathcal{S}_n(132) & \xrightarrow{\varphi} & \mathcal{D}_n & \xleftarrow{\Psi} & \mathcal{S}_n(321) \\
\text{fixed points} & \leftrightarrow & \text{centered tunnels} & \leftrightarrow & \text{fixed points} \\
\text{excedances} & \leftrightarrow & \text{right tunnels} & \leftrightarrow & \text{excedances}
\end{array}$$

$$\Psi : \mathcal{S}_n(321) \xrightarrow{\sim} \mathcal{D}_n$$

Step 1: RSK correspondence

$$\pi \mapsto (P, Q)$$

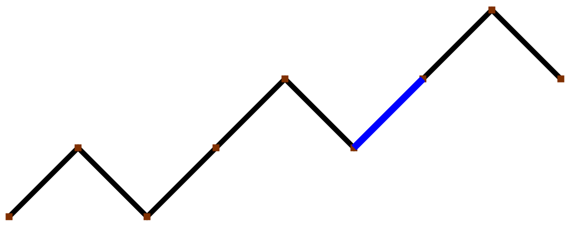
Example: $\pi = 23514687$

$\begin{array}{|c|}\hline 2 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$ $\begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline\end{array}$ $\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$ $\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array}$

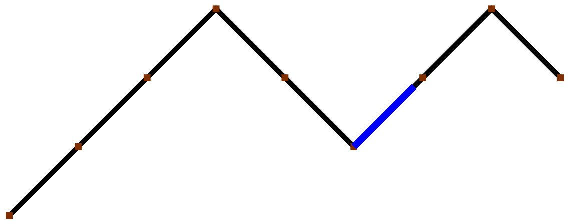
$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array}$$

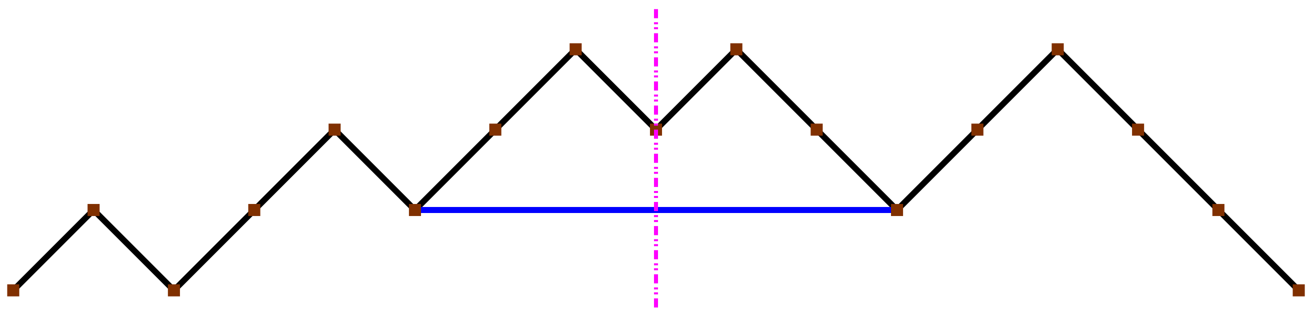
Step 2: $(P, Q) \mapsto \Psi(\pi)$

$$P = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3} \end{array} \longrightarrow$$



$$Q = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3} \end{array} \longrightarrow$$

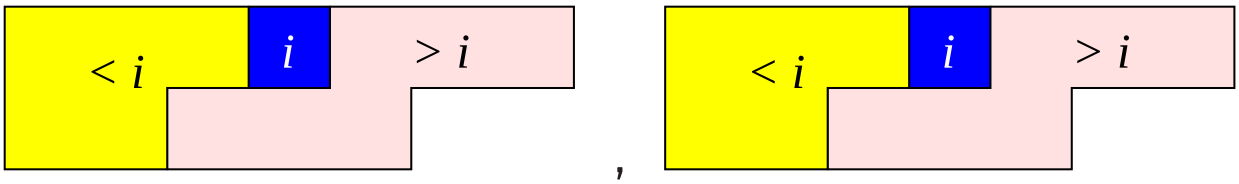




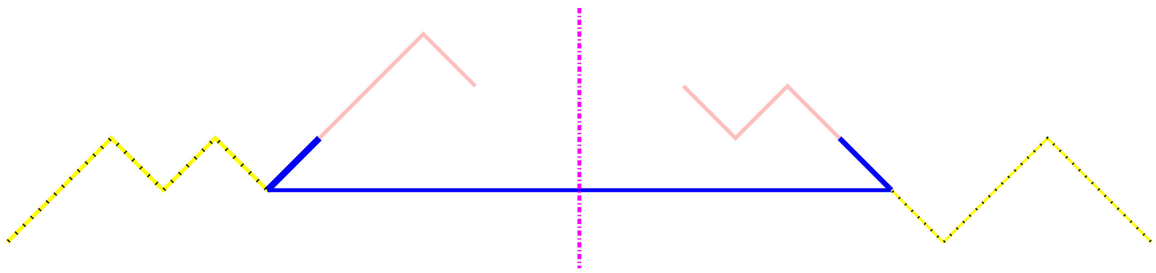
Ψ maps fixed points to centered tunnels:

$$\left.\begin{array}{c}\pi \text{ 321-avoiding}\\ \pi_i = i\end{array}\right\} \Rightarrow \pi = \underbrace{\cdots\cdots}_{<i}\, i\, \underbrace{\cdots\cdots\cdots}_{>i}$$

Then, both P and Q have shape



and i produces a centered tunnel



It can also be checked that Ψ maps excedances to right tunnels

$\square$

III. Other patterns

For a set of patterns $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$, let

$$F_{\Sigma}(x, q, z) := \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} z^n$$

Notation:

$$\Sigma_1 \approx \Sigma_2 \quad \text{if} \quad F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(x, q, z)$$

$$\Sigma_1 \sim \Sigma_2 \quad \text{if} \quad F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(\frac{x}{q}, \frac{1}{q}, qz)$$

Permutations avoiding one pattern of length 3

a) 123
b) $132 \approx 213 \approx 321$
c) 231 $\sim$ **c')** 312

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b) $132 \approx 213 \approx 321$
It's the case already studied.

Using the bijection $\psi_\llcorner$,

$$F_{321}(x,q,z) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n$$
$$= \sum_{D\in\mathcal{D}} x^{h(D)} q^{\mathrm{dr}(D)} z^{|D|}$$

Hills and double rises in Dyck paths are easy to enumerate. We get

$$F_{321}(x,q,z) = \frac{2}{1+z\,(1+q-2x)+\sqrt{1-2z(1+q)+z^2(1-q)^2}}$$

We can also consider descents:

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathsf{fp}(\pi)}q^{\mathsf{exc}(\pi)}p^{\mathsf{des}(\pi)}z^n$$

$$= \frac{2}{1+z(1+q-2x)+\sqrt{1-2z(1+q)+z^2((1+q)^2-4qp)}}.$$

$$1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)}q^{\mathsf{exc}(\pi)}p^{\mathsf{des}(\pi)+1}z^n$$
$$= 1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(213)} x^{\mathsf{fp}(\pi)}q^{\mathsf{exc}(\pi)}p^{\mathsf{des}(\pi)+1}z^n$$
$$= \frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{\spadesuit}},$$

where

$$\spadesuit = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2 - 2(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$$

a) 123

We only have long summation formulas for

$$s_n^k(123) := |\{\pi \in \mathcal{S}_n(123) : \mathrm{fp}(\pi) = k\}|$$

Theorem.

$s_n^k(123) = 0$ for $k \geq 3$.

$s_n^2(123) =$

$$= \sum_{i=1}^{n-1} \sum_{r,s=1}^{i} \Bigg[\left(\binom{2i-r-1}{i-1} - \binom{2i-r-1}{i} \right) \cdot \left(\binom{2i-s-1}{i-1} - \binom{2i-s-1}{i} \right) \cdot \sum_{\substack{h=1 \\ n-h \text{ even}}}^{n} \sum_{k=0}^{n-2i} f(k,r,h,n-2i+r) f(n-2i-k,s,h,n-2i+s) \Bigg],$$

where

$$f(k,r,h,\ell) = \begin{cases} \binom{\frac{\ell+h-r}{2}-1}{k}\binom{\frac{\ell-h+r}{2}-1}{k-1} - \binom{\frac{\ell-h-r}{2}-1}{k}\binom{\frac{\ell+h+r}{2}-1}{k-1} & \text{if } k \geq 1, \\ 1 & \text{if } k = 0 \text{ and } \ell = h-r, \\ 0 & \text{otherwise,} \end{cases}$$

with $\binom{a}{b} := 0$ if $a < 0$.

$$s_n^1(123) = \begin{cases} 2(\mathrm{C}_{n-1} - s_n^2(123)) & \text{if } n \text{ even}, \\ 2(\mathrm{C}_{n-1} - s_n^2(123)) - \mathrm{C}^2_{\frac{n-1}{2}} & \text{if } n \text{ odd}. \end{cases}$$

$$s_n^0(123) = \begin{cases} \mathrm{C}_n - 2\mathrm{C}_{n-1} + s_n^2(123) & \text{if } n \text{ even}, \\ \mathrm{C}_n - 2\mathrm{C}_{n-1} + s_n^2(123) + \mathrm{C}^2_{\frac{n-1}{2}} & \text{if } n \text{ odd}. \end{cases}$$

Corollary. For all $n \geq 4$, $s_n^0(132) < s_n^0(123)$.

This had been conjectured by Bóna and Guibert.

c,c') $231 \sim 312$

Theorem.

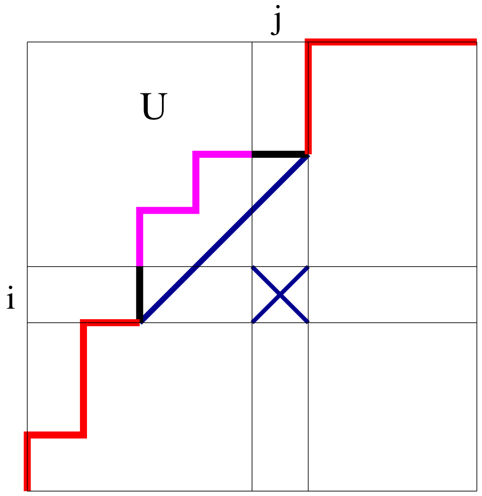
$$F_{312}(x,q,z) = \cfrac{1}{1 - zK_0 + \cfrac{z}{1 - zK_1 + \cfrac{z}{1 - zK_2 + \cfrac{z}{1 - zK_3 + \cfrac{z}{\ddots}}}}},$$

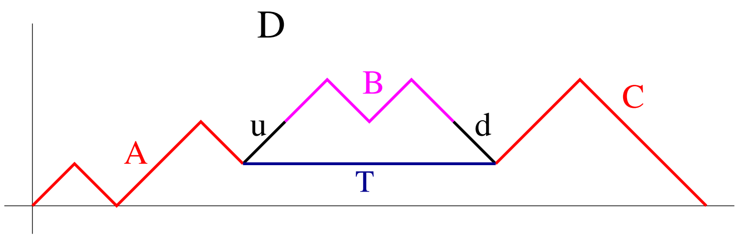
where

$$K_n = (x-1)\,\mathbf{C}_n\, q^n z^n + (q-1)\sum_{j=0}^{n-1} \mathbf{C}_j\, q^j z^j$$

Proof idea: bijection to Dyck paths mapping fixed points to tunnels of depth 0 and excedances to tunnels of negative depth

$$\begin{array}{ccccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{S}_n(132) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \bar{\pi} & \mapsto & \varphi(\bar{\pi}) \end{array}$$





Simultaneous avoidance of patterns of length 3

Double restrictions

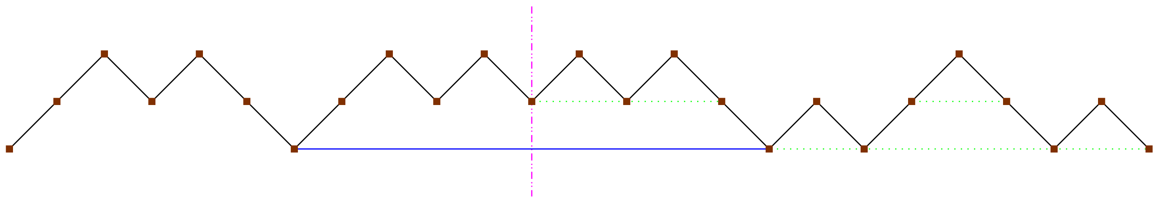
$$
\begin{array}{c}
\textbf{a)}\ \{123, 132\} \approx \{123, 213\} \\
\textbf{b)}\ \{231, 321\} \quad \sim \quad \textbf{b')}\ \{312, 321\} \\
\textbf{c)}\ \{132, 213\} \\
\textbf{d)}\ \{231, 312\} \\
\textbf{e)}\ \{132, 231\} \approx \{213, 231\} \sim \textbf{e')}\ \{132, 312\} \approx \{213, 312\} \\
\textbf{f)}\ \{132, 321\} \approx \{213, 321\} \\
\textbf{g)}\ \{123, 231\} \quad \sim \quad \textbf{g')}\ \{123, 312\} \\
\textbf{h)}\ \{123, 321\}
\end{array}
$$

Theorem. Explicit expressions for $F_{\Sigma}(x, q, z)$ for every $\Sigma \subset \mathcal{S}_3$, $|\Sigma| \geq 2$.

Idea of the proof: bijections between pattern-avoiding permutations and Dyck paths with certain restrictions, so that fp and exc correspond to statistics that are easier to enumerate.

a) $\{123, 132\} \approx \{123, 213\}$

$$\pi \in \mathcal{S}_n(132) \text{ avoids } 123$$
$$\Updownarrow$$
$$\varphi(\pi) \in \mathcal{D}_n \text{ has height } \leq 2$$



This gives a bijection between $\mathcal{S}_n(123, 132)$ and Dyck paths of height ≤ 2.

$$F_{\{123,132\}}(x, q, z) = \sum_{D \in \mathcal{D}^{\leq 2}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}$$

$$= \frac{1 + xz + (x^2 - 4q)z^2 + (-3xq + q + q^2)z^3 + (xq + xq^2 - 3x^2q + 3q^2)z^4}{(1 - qz^2)(1 - 4qz^2)}.$$

b, b') $\{231, 321\} \sim \{312, 321\}$

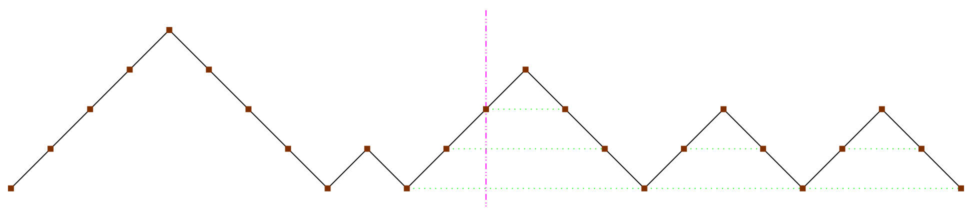
By a similar reasoning,

$$F_{\{312,321\}}(x, q, z) = \sum_{D \in \mathcal{D}^{\leq 2}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}$$

$$= \frac{1 - qz}{1 - (x + q)z + (x - 1)qz^2}.$$

c) $\{132, 213\}$

$$\pi \in \mathcal{S}_n(132) \text{ avoids } 213$$
$$\Updownarrow$$
all the valleys of $\varphi(\pi) \in \mathcal{D}_n$ are at height 0
(sequence of *pyramids*)



We get a bijection between $\mathcal{S}_n(132, 213)$ and Dyck paths that are sequences of pyramids.

$$F_{\{132,213\}}(x, q, z) = \sum_{D \text{ seq. of pyr.}} x^{\mathsf{ct}(D)} q^{\mathsf{rt}(D)} z^{|D|}$$

$$= \frac{1-(1+q)z-2qz^2+4q(1+q)z^3-(xq^2+xq+5q^2)z^4+2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}.$$

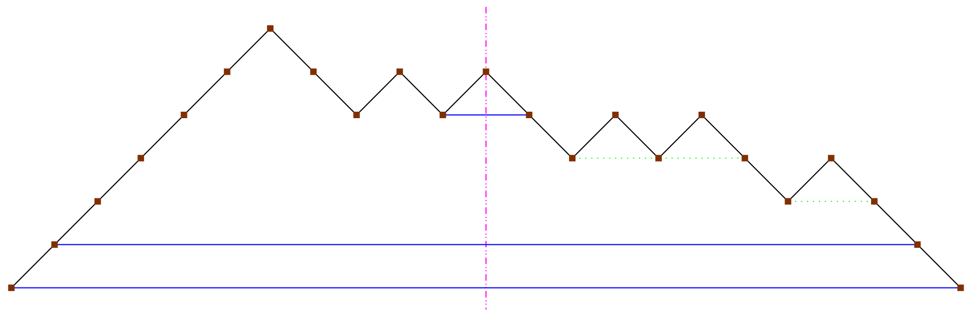
d) Similar.

e, e') $\{132, 231\} \approx \{213, 231\} \sim \{132, 312\} \approx \{213, 312\}$

$\pi \in \mathcal{S}_n(132)$ avoids 231

$\Updownarrow$

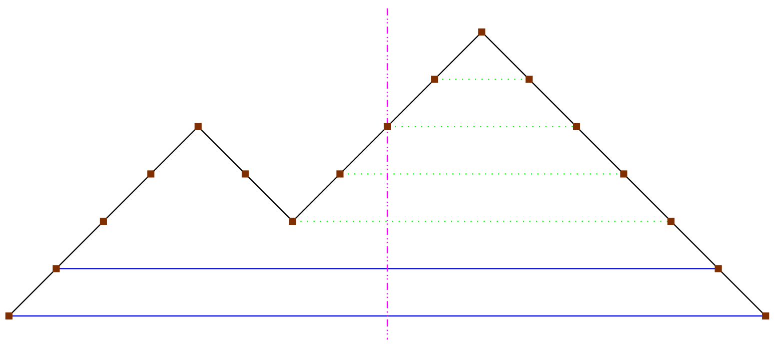
$\varphi(\pi) \in \mathcal{D}_n$ does not have double rises after the first down-step



$$F_{\{132,231\}}(x, q, z) = \frac{1 - z - qz^2 + xqz^3}{(1 - xz)(1 - z - 2qz^2)}$$

f) $\{132, 321\} \approx \{213, 321\}$

$$\pi \in \mathcal{S}_n(132) \text{ avoids } 321$$
$$\Updownarrow$$
$$\varphi(\pi) \in \mathcal{D}_n \text{ has at most two peaks}$$



$$F_{\{132,321\}}(x, q, z) = \frac{1 - (1+q)z + 2qz^2}{(1-z)(1-xz)(1-qz)}.$$

g, g') Similar.

h) $\{123, 321\}$ Trivial.

We can generalize some cases to $\Sigma = \{\sigma_1, \sigma_2\}$ where $\sigma_1 \in \mathcal{S}_3$ and $\sigma_2 \in \mathcal{S}_k$.

Theorem. Expressions for

$$F_{132,12\cdots(k+1)}(x,1,z)$$

$$F_{312,(k+1)k\cdots 1}(x,q,z)$$

$$F_{132,(k+1)k\cdots 1}(x,q,z)$$

for any k.

Triple restrictions

a) $\{123, 132, 213\}$
b) $\{231, 312, 321\}$
c) $\{123, 132, 231\} \approx \{123, 213, 231\} \sim$
c') $\{123, 132, 312\} \approx \{123, 213, 312\}$
d) $\{132, 231, 321\} \approx \{213, 231, 321\} \sim$
d') $\{132, 312, 321\} \approx \{213, 312, 321\}$
e) $\{132, 213, 231\}$ $\sim$ **e')** $\{132, 213, 312\}$
f) $\{132, 231, 312\} \approx \{213, 231, 312\}$
g) $\{123, 231, 312\}$
h) $\{132, 213, 321\}$
i) $\{123, 132, 321\} \approx \{123, 213, 321\}$
j) $\{123, 231, 321\}$ $\sim$ **j')** $\{123, 312, 321\}$

We get all the generating functions using the same methods as before.

Example:

$$F_{123,132,213} = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

Pattern-avoiding involutions

Let $\mathcal{I}_n(\Sigma) := \{\pi \in \mathcal{S}_n(\Sigma) : \pi = \pi^{-1}\}$

Theorem. Expressions for

$$G_{\Sigma}(x,q,z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{I}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n$$

for every $\Sigma \subset \mathcal{S}_3$

Note: The same bijection we had maps involutions to involutions.

$$\begin{array}{ccccc}
\mathcal{I}_n(132) & \xrightarrow{\varphi} & \mathcal{E}_n & \xleftarrow{\Psi} & \mathcal{I}_n(321) \\
\text{fixed points} & \leftrightarrow & \text{centered tunnels} & \leftrightarrow & \text{fixed points} \\
\text{excedances} & \leftrightarrow & \text{right tunnels} & \leftrightarrow & \text{excedances}
\end{array}$$

where $\mathcal{E}_n := \{D \in \mathcal{D}_n : D \text{ symmetric}\}$

Expected number of fixed points

X_n^{σ} random variable for # of fixed points in $\mathcal{S}_n(\sigma)$

Proposition.

$$E[X_n^{123}] = \begin{cases} \frac{n+1}{2n-1} & \text{if } n \text{ is even} \\ \frac{2\,\mathbf{C}_{n-1} - \mathbf{C}^2_{(n-1)/2}}{\mathbf{C}_n} & \text{if } n \text{ is odd} \end{cases}$$

$$E[X_n^{132}] = E[X_n^{213}] = E[X_n^{321}] = 1$$

$$\sum_{n\geq 1} E[X_n^{231}]\mathbf{C}_n z^n = \sum_{n\geq 1} E[X_n^{312}]\mathbf{C}_n z^n = \frac{1-\sqrt{-1+4z+2\sqrt{1-4z}}}{2z}$$

Open questions

- Are there other statistics in restricted permutations that have the same distribution for different patterns?

- Find distribution of statistics in permutations avoiding longer patterns (e.g., of length 4)