Bijections for lattice paths between two boundaries

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Joint work with Martin Rubey
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Dyck paths

For \( P \in \mathcal{D}_n \) (Dyck paths with \( 2n \) steps), let

\[
\begin{align*}
  t(P) &= \# \text{ of } E \text{ steps in common with } T \\
        &= \text{“height” of the last “peak”} \\
  b(P) &= \# \text{ of } E \text{ steps in common with } B \\
        &= \text{number of returns}
\end{align*}
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$= \text{number of returns}$

**Theorem (Deutsch ’98)**

*The joint distribution of the pair $(t, b)$ over $\mathcal{D}_n$ is symmetric, i.e.,*

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\sum_{P \in \mathcal{D}_n} x^{t(P)} y^{b(P)} = \sum_{P \in \mathcal{D}_n} x^{b(P)} y^{t(P)}.
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**Proof 1 (Deutsch):** Recursive bijection. **Proof 2:** Generating fcts.
Both proofs rely on the recursive structure of Dyck paths.
A generalization to arbitrary boundaries

$T$ and $B$ paths from $O$ to $F$ with steps $N$ and $E$, with $T$ weakly above $B$

$P \in \mathcal{P}(T, B) = \text{set of paths from } O \text{ to } F$ weakly between $T$ and $B$

$t(P) = \# \text{ of } E \text{ steps in common with } T$

(top contacts of $P$)

$b(P) = \# \text{ of } E \text{ steps in common with } B$

(bottom contacts of $P$)
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**Theorem**

The joint distribution of $(t, b)$ over $\mathcal{P}(T, B)$ is symmetric, i.e.,

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\sum_{P \in \mathcal{P}(T, B)} x^{t(P)} y^{b(P)} = \sum_{P \in \mathcal{P}(T, B)} x^{b(P)} y^{t(P)}.
$$
Example

\[ \sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{b(P)} = x^3 + x^2 y + xy^2 + y^3 + 2x^2 + 2xy + 2y^2 + 2x + 2y + 1 \]
The known proofs for Dyck paths do not seem to generalize to arbitrary boundaries.

Proof

We give an involution \( \Phi : P(T, B) \to P(T, B) \) with the property \( t(\Phi(P)) = b(P) \) and \( b(\Phi(P)) = t(P) \).

Idea: Given \( P \in P(T, B) \) with \( t(P) > b(P) \), turn some of its top contacts into bottom contacts, one at a time.
Proof

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We give an involution

$$\Phi : \mathcal{P}(T, B) \rightarrow \mathcal{P}(T, B)$$

with the property $t(\Phi(P)) = b(P)$ and $b(\Phi(P)) = t(P)$.

Idea: Given $P \in \mathcal{P}(T, B)$ with $t(P) > b(P)$, turn some of its top contacts into bottom contacts, one at a time.
We define the involution $\Phi$ by iterating a map $\phi$, which turns one top contact into one bottom contact.

\[(t, b) = (4, 2) \quad \mapsto \quad (t, b) = (3, 3) \quad \mapsto \quad (t, b) = (2, 4)\]
To define $\phi(P)$, we first find the top contact that will be changed into a bottom contact.
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1. Record top and bottom contacts of $P$ as a word $w$ over $\{t, b\}$:

$$w = bttbtt$$
2. Having built $w$, select a top contact as follows:

$$w = bttbtbbbtbttbttbtt$$
From paths to words

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   - Draw a path with a step $(1, 1)$ for each $t$, and a step $(1, -1)$ for each $b$.

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2. Having built $w$, select a top contact as follows:
   - Draw a path with a step $(1, 1)$ for each $t$, and a step $(1, -1)$ for each $b$.
   - Match $t$’s and $b$’s that “face” each other in the path.

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2. Having built $w$, select a top contact as follows:

- Draw a path with a step $(1, 1)$ for each $t$, and a step $(1, -1)$ for each $b$.
- Match $t$’s and $b$’s that “face” each other in the path.
- Select the leftmost unmatched $t$ as the top contact that will be changed.

$$w = bttbtbbbttbttbttbtt$$
The map $\phi$

Given $P \in \mathcal{P}(T,B)$, define $\phi(P)$ as follows:

- Record top and bottom contacts of $P$ as a word $w$ over $\{t, b\}$. 

$$w = bttbtt$$
The map $\phi$

Given $P \in \mathcal{P}(T, B)$, define $\phi(P)$ as follows:

- Record top and bottom contacts of $P$ as a word $w$ over $\{t, b\}$.
- Find leftmost unmatched $t$; let $E$ be the corresponding step.

$P \quad \quad E$  

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- Record top and bottom contacts of $P$ as a word $w$ over $\{t, b\}$.
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- Write $P = X Y E Z$, where $Y$ touches $B$ only at its left endpoint.

![Diagram illustrating the bijection]

$w = \text{bttbtt}$
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- Find leftmost unmatched $t$; let $E$ be the corresponding step.
- Write $P = XYEZ$, where $Y$ touches $B$ only at its left endpoint.
- Let $\phi(P) = XEYZ$.

$\phi(P)$

$w = bttbtt$

$bbtbtt$
The involution $\Phi$

For $P \in \mathcal{P}(T, B)$ with $t(P) = e$ and $b(P) = f$, define

$$\Phi(P) = \phi^{e-f}(P).$$
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**Theorem**

$\Phi$ is an involution on $\mathcal{P}(T, B)$ that satisfies $t(\Phi(P)) = b(P)$ and $b(\Phi(P)) = t(P)$. 
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For $P \in \mathcal{P}(T, B)$ with $t(P) = e$ and $b(P) = f$, define

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**Theorem**

$\Phi$ is an involution on $\mathcal{P}(T, B)$ that satisfies $t(\Phi(P)) = b(P)$ and $b(\Phi(P)) = t(P)$.
A generalization to paths with $S$ steps

$\tilde{\mathcal{P}}(T, B) =$ set of paths from $O$ to $F$
with steps $N$, $E$ and $S$
weakly between $T$ and $B$.

For $P \in \tilde{\mathcal{P}}(T, B)$, define $t(P)$ and $b(P)$ as before.
The descent set of $P$ is the set of $x$-coordinates where $S$ steps occur.
A generalization to paths with $S$ steps

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For $P \in \tilde{\mathcal{P}}(T, B)$, define $t(P)$ and $b(P)$ as before.

The descent set of $P$ is the set of $x$-coordinates where $S$ steps occur.

**Theorem**

*There is an involution $\tilde{\mathcal{P}}(T, B) \rightarrow \tilde{\mathcal{P}}(T, B)$ that switches the statistics $(t, b)$ and preserves the descent set.*
A generalization: examples

The map $\phi$ for paths with $S$ steps:
A generalization: examples

The involution \( \Phi \) for paths with \( S \) steps:

\[
\phi \mapsto \Phi(\phi) \mapsto \phi \mapsto \Phi(\phi) \mapsto \phi
\]
A related theorem

For $P \in \mathcal{P}(T, B)$, let

\[ \ell(P) = \# \text{ of } N \text{ steps in common with } T \]
\[ r(P) = \# \text{ of } N \text{ steps in common with } B \]

Example: $t(P) = 4$, $b(P) = 3$, $\ell(P) = 2$, $r(P) = 1$. 
A related theorem

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Example: $t(P) = 4$, $b(P) = 3$, $\ell(P) = 2$, $r(P) = 1$.

Theorem

The pairs $(b, \ell)$ and $(t, r)$ have the same joint distribution over $\mathcal{P}(T, B)$, i.e.,

$$\sum_{P \in \mathcal{P}(T, B)} x^{b(P)} y^{\ell(P)} = \sum_{P \in \mathcal{P}(T, B)} x^{t(P)} y^{r(P)}.$$
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We do not know of a bijective proof similar to the previous one.
Proof idea

Both

$$\sum_{P \in \mathcal{P}(T,B)} x^{b(P)} y^{\ell(P)} \quad \text{and} \quad \sum_{P \in \mathcal{P}(T,B)} x^{t(P)} y^{r(P)}$$

equal the Tutte polynomial of a lattice path matroid, as defined by Bonin–De Mier–Noy '03.

The statistics $b$ and $\ell$ ($t$ and $r$) are internal and external activities with respect to different linear orderings of the ground set.
$P_0 = T$  
$P_1, P_2, \ldots, P_k \in \mathcal{P}(T, B)$, 
$P_i$ weakly above $P_{i+1}$ for all $i$. 
Let $P_0 = T$, $P_{k+1} = B$. 
For $0 \leq i \leq k$, let 

$$h_i = \# \text{ of } E \text{ steps where } P_i \text{ and } P_{i+1} \text{ coincide}$$
$P_0 = T$

$P_1, P_2, \ldots, P_k \in P(T, B),$

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**Theorem**

The distribution of $(h_0, h_1, \ldots, h_k)$ over $k$-fans of paths as above is symmetric.
Connection to flagged SSYT

Let $T = NN \ldots NEE \ldots E$. 

$h_i = \# E \text{ steps in } P_i \cap P_{i+1}$

$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$
Connection to flagged SSYT

Let $T = NN \ldots NEE \ldots E$.

$h_i = \# \ E \ steps \ in \ P_i \cap \mathcal{P}_{i+1}$

$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$

$u_j = \# \ of \ unused \ E \ steps \ at \ level \ j$
Connection to flagged SSYT

Let $T = NN \ldots NEE \ldots E$. $u_1 = 2$ $u_2 = 2$ $u_3 = 1$ $u_4 = 1$

$h_i = \# \text{ E steps in } P_i \cap P_{i+1}$ $h_0 = 4$ $h_1 = 3$ $h_2 = 3$ $h_3 = 3$

$u_j = \# \text{ of unused E steps at level } j$

$\lambda = (6, 4, 3, 3, 1)$

$T$ and $B$ form the shape of a Young diagram of a partition $\lambda$. 
Connection to flagged SSYT

Let $T = NN \ldots NEE \ldots E$.

$u_1 = 2$
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$u_3 = 1$
$u_4 = 1$

$h_i = \# E$ steps in $P_i \cap P_{i+1}$
$h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3$
$u_j = \# of unused $E$ steps at level $j$

$\lambda = (6, 4, 3, 3, 1)$

$T$ and $B$ form the shape of a Young diagram of a partition $\lambda$.

**Def:** A SSYT of shape $\lambda$ is called $k$-flagged if the entries in row $r$ are $\leq k + r$ for each $r$.

<table>
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<th>1</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>\leq 8</td>
</tr>
</tbody>
</table>
Connection to flagged SSYTB

Let \( T = NN \ldots NEE \ldots E \).

\[
\begin{align*}
u_1 &= 2 \\
u_2 &= 2 \\
u_3 &= 1 \\
u_4 &= 1
\end{align*}
\]

\[
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 & 4 \\
4 & 5 & 6 \\
5 & 6 & 7 \\
8
\end{array}
\]

\( h_i = \# \text{ of } E \text{ steps in } P_i \cap P_{i+1} \)

\( h_0 = 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3 \)

\( u_j = \# \text{ of unused } E \text{ steps at level } j \)

\( \lambda = (6, 4, 3, 3, 1) \)

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\[
\begin{align*}
1 & \leq 4 \\
2 & \leq 5 \\
4 & \leq 6 \\
5 & \leq 7 \\
8 & \leq 8
\end{align*}
\]

weight = \((\#1s, \#2s, \ldots) = (2, 3, 3, 3, 2, 2, 1, 1)\)
Connection to flagged SSYT

**Theorem**

There is an explicit bijection between

- $k$-fans of paths in $\mathcal{P}(T, B)$ with statistics $h_i$ and $u_j$, and
- $k$-flagged SSYT of shape $\lambda$ and weight
  $$(\lambda_1 - h_0, \lambda_1 - h_1, \ldots, \lambda_1 - h_k, u_1, u_2, \ldots, u_r).$$

\[\begin{align*}
u_1 &= 2 \\
u_2 &= 2 \\
u_3 &= 1 \\
u_4 &= 1 \\
h_0 &= 4 \quad h_1 = 3 \quad h_2 = 3 \quad h_3 = 3
\end{align*}\]

\[\begin{array}{cccccc}
1 & 1 & 2 & 2 & 3 & 4 \\
2 & 3 & 3 & 4 & \lesssim & 5 \\
4 & 5 & 6 & \lesssim & 6 \\
5 & 6 & 7 & \lesssim & 7 \\
8 & \lesssim & 8 \\
\end{array}\]

$\lambda_1 = 6$

weight $= (2, 3, 3, 3, 2, 2, 1, 1)$
Theorem

There is an explicit bijection between

- $k$-fans of paths in $\mathcal{P}(T, B)$ with statistics $h_i$ and $u_j$, and
- $k$-flagged SSYT of shape $\lambda$ and weight $(\lambda_1 - h_0, \lambda_1 - h_1, \ldots, \lambda_1 - h_k, u_1, u_2, \ldots, u_r)$.

The bijection uses a variation of jeu de taquin.
Connection to $k$-triangulations

Theorem (conjectured by C. Nicolás ’09)

The joint distribution of the degrees of $k+1$ consecutive vertices in a $k$-triangulation of a convex $n$-gon equals the distribution of $(h_0, h_1, \ldots, h_k)$ over $k$-fans of Dyck paths of semilength $n - 2k$. 

The proof uses the previous theorem in the special case of Dyck paths, together with a bijection of SerranoStump between $k$-triangulations and flagged SSTY.
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The proof uses the previous theorem in the special case of Dyck paths, together with a bijection of Serrano–Stump between $k$-triangulations and $k$-flagged SSYT.