A greedy sorting algorithm

Sergi Elizalde Peter Winkler

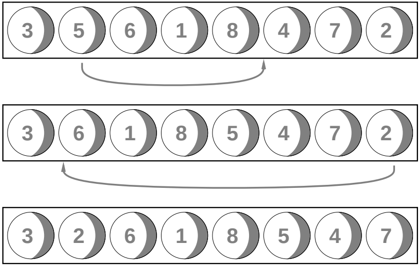
Dartmouth College

Rutgers Experimental Mathematics Seminar

The *homing* algorithm

Given a permutation π, repeat the following *placement* step:

- Choose an entry $\pi(i)$ such that $\pi(i) \neq i$.
- Place $\pi(i)$ in the correct position.
- Shift the other entries as necessary.



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- How many steps does it take in the worst case...
  - with a good choice of placements?
  - with a random choice of placements?
  - with a bad choice of placements?

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- In hand-sorting files, it is common to take the first file and move it to the front, then the second, and so on. This is a (fast) special case of homing.
- It is fun to analyze this algorithm.
- If you have to sort a list and you are paid by the hour, this is a great algorithm to use.

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  $7132568\underline{4} \to \underline{7}1348652 \to 5684317\underline{2} \to 5271\underline{3}486 \to$
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- Loren Larson misunderstood the definition of the algorithm, and thought the intervening numbers were shifted.

History (cont'd)

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- Let k be the largest number which is placed *upward* in the cycle.
- Once k is placed, it can be dislodged upward and placed again downward, but nothing can ever push it below position k.
- Hence it can never again be placed upward, a contradiction.

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Theorem

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- *An algorithm that always places the smallest or largest available number will terminate in at most* $n-1$ *steps.*
- *Let* k *be the length of the longest increasing subsequence in* π. *Then no sequence of fewer than* $n-k$ *placements can sort* π.
- *The permutation* $n \dots 21$ *is the only one requiring* $n-1$ *steps.*

Random placements

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- Suppose that we have a permutation where k of the extremal numbers are home:

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- With probability $\geq \frac{2}{n-k}$, the next step will place an additional extremal number.
- Total expected number of steps is $\leq \sum_{k=0}^{n-2} \frac{n-k}{2}$.

$\square$

Slow Homing: Example

Starting from

$$2\,3\,4\,5\,6\,7\ldots n\,1$$

place always the leftmost possible entry:

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It takes $2^{n-1}-1$ steps to sort this permutation.

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To prove this, consider the reverse algorithm.
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$$2^{n-1}-1 = \underbrace{2^{n-2}}_{\text{until 1 and } n \text{ are displaced}} + \underbrace{2^{n-2}-1}_{\text{after displacing 1 and } n}$$

Lemma

After 2^{n-2} displacements, both 1 and n have been displaced and will never be displaced again.

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- Note that 1 and n can each be displaced only once.
- If after 2^{n-2} displacements one of these values hasn't been displaced, then it played no role in the process.
- Hence the remaining $n-1$ numbers allowed more than $2^{n-2}-1$ steps, contradicting the induction hypothesis.

□

The code of a permutation

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Assign to each permutation π a code $\alpha(\pi) = \alpha_2\alpha_3\ldots\alpha_{n-1}$, where

$$\alpha_i = \left\{ \begin{array}{c} 0 \\ + \\ - \end{array} \right\} \text{ if entry } i \text{ is } \left\{ \begin{array}{c} \text{exactly} \\ \text{to the right of} \\ \text{to the left of} \end{array} \right\} \text{ home.}$$

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Example

$$\pi = 3\,5\,6\,1\,8\,4\,7\,2 \quad \longrightarrow \quad \alpha(\pi) = + - + - - 0$$

The weight of a code

$$\alpha = \quad + \quad - \quad + \quad - \quad - \quad 0$$

Define the weight of a code α recursively:

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$$\alpha = \begin{array}{cccccc} + & - & + & - & - & 0 \\ 5 & 1 & 3 & 3 & 4 & \end{array}$$

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- For each $-$, count the number of symbols to its left, and for each $+$, count the number of symbols to its right.

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- Let d be the largest of these numbers, and let $\hat{\alpha}$ be the code obtained by deleting the corresponding symbol.

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- Define

$$w(\alpha) = 2^d + w(\hat{\alpha}).$$

The weight of a code: example

$$w(\ + \ - \ + \ - \ - \ 0\ )$$

The weight of a code: example

$$w\begin{pmatrix} + & - & + & - & - & 0 \\ \mathbf{5} & 1 & 3 & 3 & 4 & \end{pmatrix}$$

The weight of a code: example

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$$

$$= 2^5 + w(\ -\ \ +\ \ -\ \ -\ \ 0\ )$$

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&= 2^5 + 2^3 + 2^2 + 2^1 + 2^0 + w(\,0\,) \\
&= 2^5 + 2^3 + 2^2 + 2^1 + 2^0 = 47
\end{aligned}
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Bound on the weight

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The maximum of $w(\alpha)$ over codes α of length k is $2^k - 1$, for codes of the form $++\cdots+--\cdots-$.

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Proof.

In the recursion,

$$w(\alpha) \leq 2^{k-1} + w(\hat{\alpha}),$$

with equality when a $-$ is deleted from the right or a $+$ from the left. $\square$

The weight increases at each displacement

Lemma

Let $\pi \in S_n$ with $\pi(1) \neq 1$ and $\pi(n) \neq n$, and let π' be the result of applying some displacement to π. Let $\alpha = \alpha(\pi)$ and $\alpha' = \alpha(\pi')$. Then

$$w(\alpha') > w(\alpha).$$

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Proof sketch.

- A number i can be displaced iff $\alpha_i = 0$ in the code.

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$$w(\alpha') > w(\alpha).$$

Proof sketch.

- A number i can be displaced iff $\alpha_i = 0$ in the code.
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- It can be shown that this increases the weight of the code.

Finishing the proof

Combining these lemmas, the maximum number of displacements is

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So at most $2^{n-1}-1$ in total.

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B_n grows super-exponentially:

$$B_n \sim \frac{1}{\sqrt{n}} \lambda(n)^{n+1/2} e^{\lambda(n)-n-1},$$

where $\lambda(n) = \frac{n}{W(n)}$, and $W(n)e^{W(n)} = n$.

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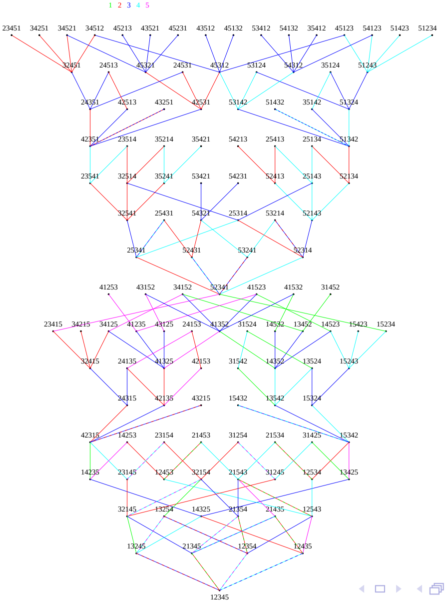
Theorem

$$F(u,v) = uv + uv\frac{\partial}{\partial u}F(u,v) + uv\frac{\partial}{\partial v}F(u,v) - u^2v^2\frac{\partial^2}{\partial u \partial v}F(u,v)$$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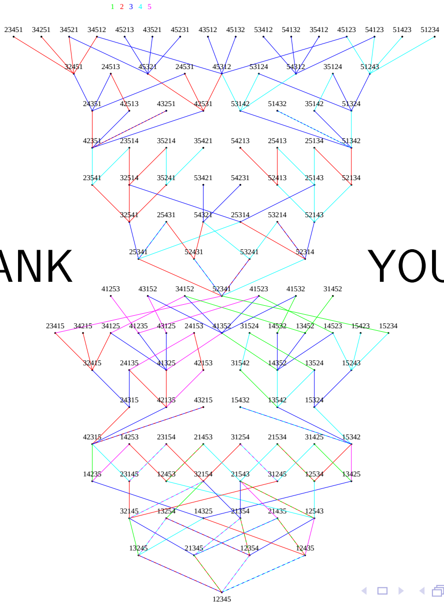
THANK YOU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