What pattern-avoidance shouldn't have avoided

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Basic definitions

$\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n$, $\sigma \in \mathcal{S}_k$

π *contains* σ if there exist $i_1 < \ldots < i_k$ s.t.
$\rho(\pi_{i_1} \cdots \pi_{i_k}) = \sigma$
where ρ is the relabeling with $\{1, \ldots, k\}$ keeping the same order relationships

Otherwise, we say that π *avoids* σ

Example:
$\underline{2}4\underline{53}1$ contains 132
42351 avoids 132

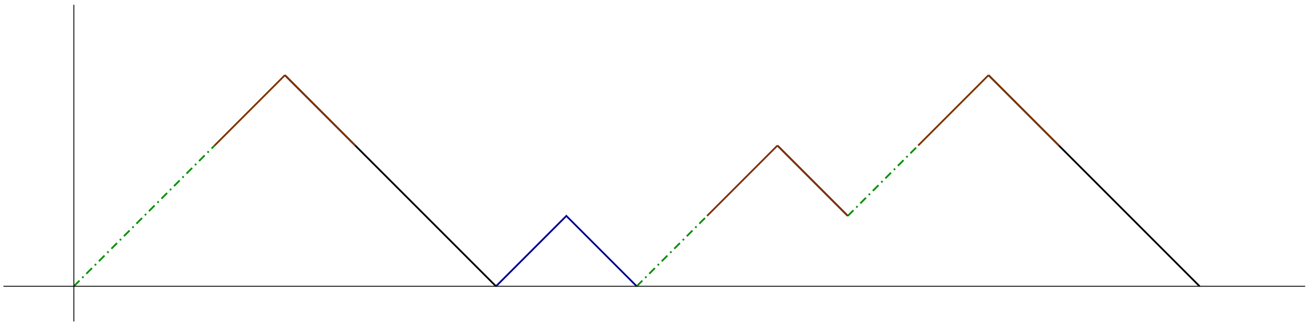
Def. $\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n |\ \pi \text{ avoids } \sigma\}$

Def. i is a *fixed point* of π if $\pi_i = i$
$fp(\pi) :=$ number of fixed points of π

Def. i is an *excedance* of π if $\pi_i > i$
$exc(\pi) :=$ number of excedances of π

Ex: if $\pi = \overline{4}\mathbf{2}\mathbf{3}\overline{5}1$, then $fp(\pi) = 2$, $exc(\pi) = 2$

Def. A *Dyck path* is a path between $(0,0)$ and $(2n,0)$ with steps $(1,1)$ (*up-steps*) and $(1,-1)$ (*down-steps*) that never goes below the x-axis



Let $\mathcal{D}_n$ be the set of Dyck paths of length $2n$.

$|\mathcal{D}_n| = c_n = \frac{1}{n+1}\binom{2n}{n}$ (Catalan number)

Let $D \in \mathcal{D}_n$. Define:

double rise $=$ two consecutive up-steps
$dr(D) :=$ number of double rises of D
peak $=$ up-step followed by down-step
$pk(D) :=$ number of peaks of D
hill $=$ peak of height 1
$h(D) :=$ number of hills of D

Example: In the above Dyck path, $dr(D) = 4$, $pk(D) = 4$, and $h(D) = 1$

Patterns of length 3

$$\#\mathcal{S}_n(123) = \#\mathcal{S}_n(321)$$

$\pi_1\pi_2\cdots\pi_n$ avoids 123 $\Leftrightarrow$ $\pi_n\cdots\pi_2\pi_1$ avoids 321

$$\#\mathcal{S}_n(132) = \#\mathcal{S}_n(231) = \#\mathcal{S}_n(312) = \#\mathcal{S}_n(213)$$

$$\begin{array}{ccc}
\pi_1\pi_2\cdots\pi_n \text{ avoids } 132 & \Leftrightarrow & \pi_n\cdots\pi_2\pi_1 \text{ avoids } 231 \\
\Updownarrow & & \Updownarrow \\
\begin{array}{c}(n+1-\pi_1)\cdots(n+1-\pi_n) \\ \text{avoids } 312\end{array} & \Leftrightarrow & \begin{array}{c}(n+1-\pi_n)\cdots(n+1-\pi_1) \\ \text{avoids } 213\end{array}
\end{array}$$

Theorem.

$$\#\mathcal{S}_n(123) = \#\mathcal{S}_n(132) = c_n$$

There are several proofs.

Some of them give bijections between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$.

Theorem (Robertson-Saracino-Zeilberger '02)
For any $i \leq n$

$$\#\{\pi \in \mathcal{S}_n(321)|\ fp(\pi) = i\} = \\ \#\{\pi \in \mathcal{S}_n(132)|\ fp(\pi) = i\}$$

No bijective proof is known

None of the known bijections between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ preserves fixed points

Question: Can this theorem be generalized, considering other statistics in permutations?

Theorem (October '02)
For any $i, j \leq n$

$$\#\{\pi \in \mathcal{S}_n(321) |\ fp(\pi) = i,\ exc(\pi) = j\} = \\ \#\{\pi \in \mathcal{S}_n(132) |\ fp(\pi) = i,\ exc(\pi) = j\}$$

Equivalently,

$$\sum_{\pi \in \mathcal{S}_n(321)} x^{fp(\pi)} q^{exc(\pi)} = \sum_{\pi \in \mathcal{S}_n(132)} x^{fp(\pi)} q^{exc(\pi)}$$

Example (n=4):

$\mathcal{S}_4(321)$

π	fp	exc
1234	4	0
1243	2	1
1324	2	1
1342	1	2
1423	1	1
2134	2	1
2143	0	2
2314	1	2
2341	0	3
2413	0	2
3124	1	1
3142	0	2
3412	0	2
4123	0	1

$\mathcal{S}_4(132)$

π	fp	exc
1234	4	0
2134	2	1
2314	1	2
2341	0	3
3124	1	1
3214	2	1
3241	1	2
3412	0	2
3421	0	2
4123	0	1
4213	1	1
4231	2	1
4312	0	2
4321	0	2

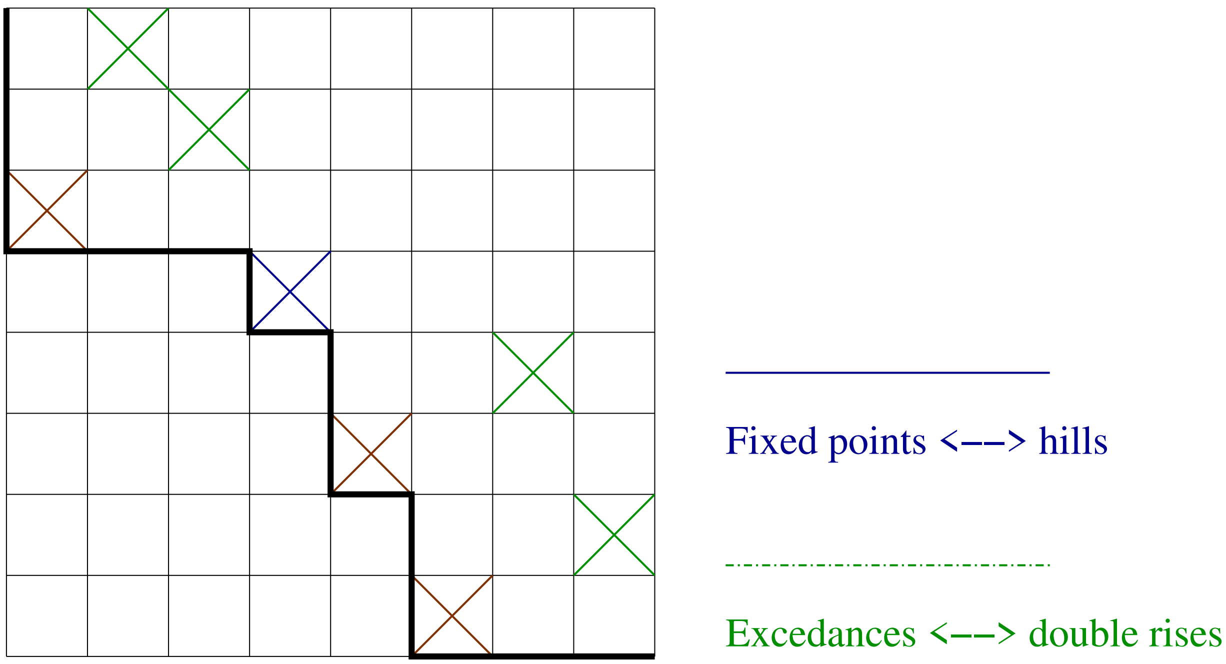
$$x^4 + 3x^2q + 2xq^2 + 2xq + q^3 + 4q^2 + q$$

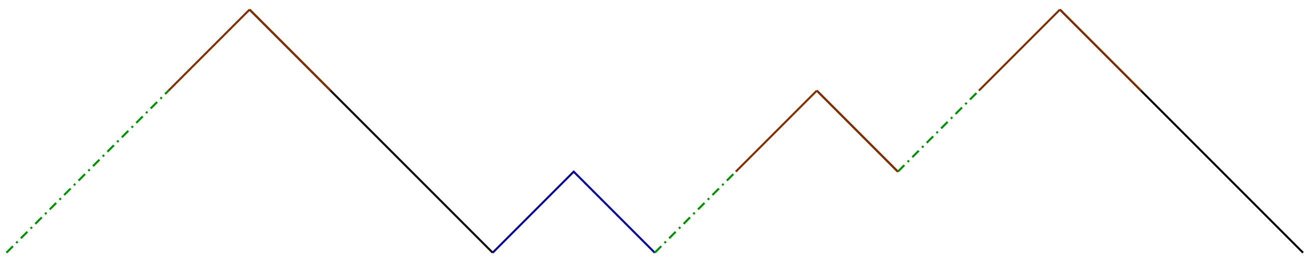
Sketch of the proof

Step 1

Bijection $\mathcal{S}_n(321) \longleftrightarrow \mathcal{D}_n$

Example: $\pi = \overline{23}14\overline{7}5\overline{8}6 \in \mathcal{S}_8(321)$

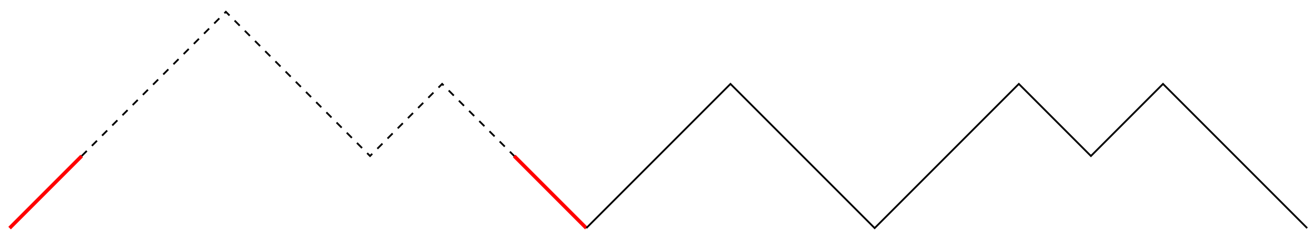




Let $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$ be the class of Dyck paths

Let $size(D) := n$ if $D \in \mathcal{D}_n$

$$\begin{aligned} A(x,q,t) &:= \sum_{n \geq 0} \Big(\sum_{\pi \in \mathcal{S}_n(321)} x^{fp(\pi)} q^{exc(\pi)} \Big) t^n \\ &= \sum_{D \in \mathcal{D}} x^{h(D)} q^{dr(D)} t^{size(D)} \end{aligned}$$



$$\mathcal{D} = \{\varepsilon\} \cup \{/\} \star (\{\varepsilon\} \cup (\mathcal{D} - \{\varepsilon\})) \star \{\backslash\} \star \mathcal{D}$$

$$A(x,q,t) = 1 + t(x + q(A(1,q,t) - 1))A(x,q,t)$$

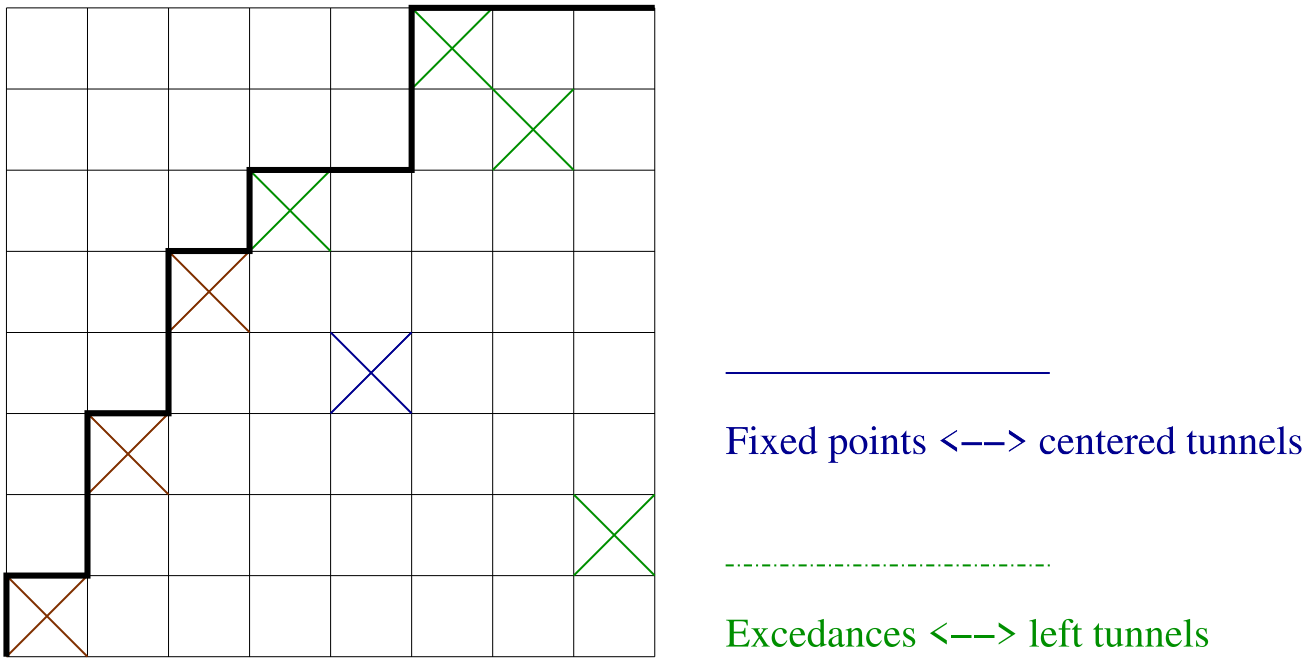
$$\Downarrow$$

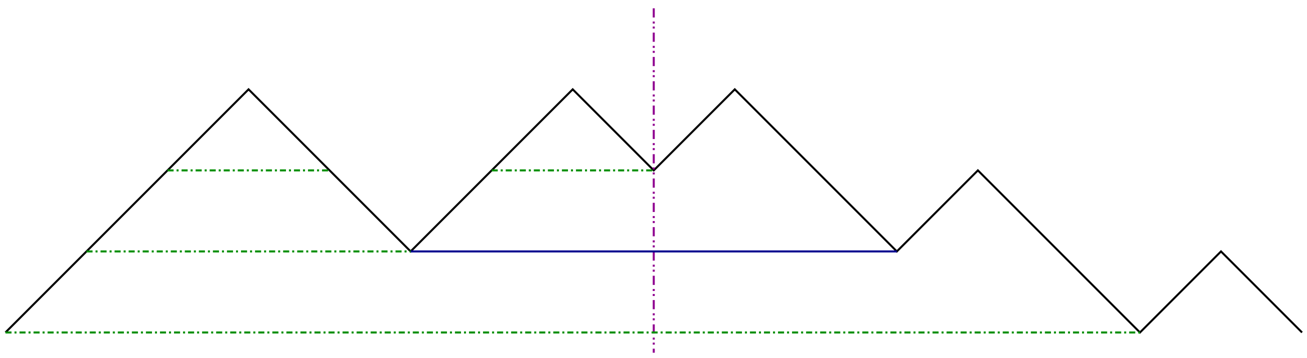
$$A(x,q,t) = \frac{2}{1 + t\ (1 + q - 2x) + \sqrt{1 - 2t(1+q) + t^2(1-q)^2}}$$

Step 2

Bijection $\mathcal{S}_n(132) \longleftrightarrow \mathcal{D}_n$

Example: $\pi = \overline{674}3\mathbf{5}2\overline{8}1 \in \mathcal{S}_8(132)$





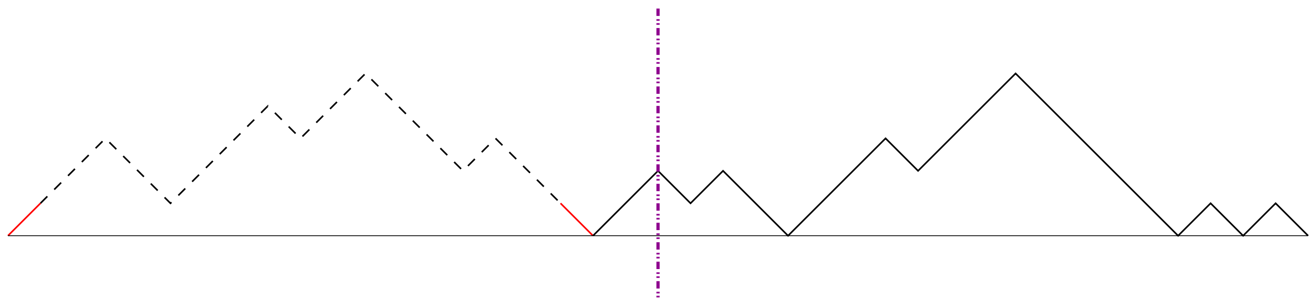
$CT(D) :=$ number of *centered tunnels* of D
$LT(D) :=$ number of *left tunnels* of D

$$
\begin{aligned}
B(x,q,t) &:= \sum_{n\geq 0} \Big(\sum_{\pi\in\mathcal{S}_n(132)} x^{fp(\pi)} q^{exc(\pi)} \Big) t^n \\
&= \sum_{D\in\mathcal{D}} x^{CT(D)} q^{LT(D)} t^{size(D)}
\end{aligned}
$$

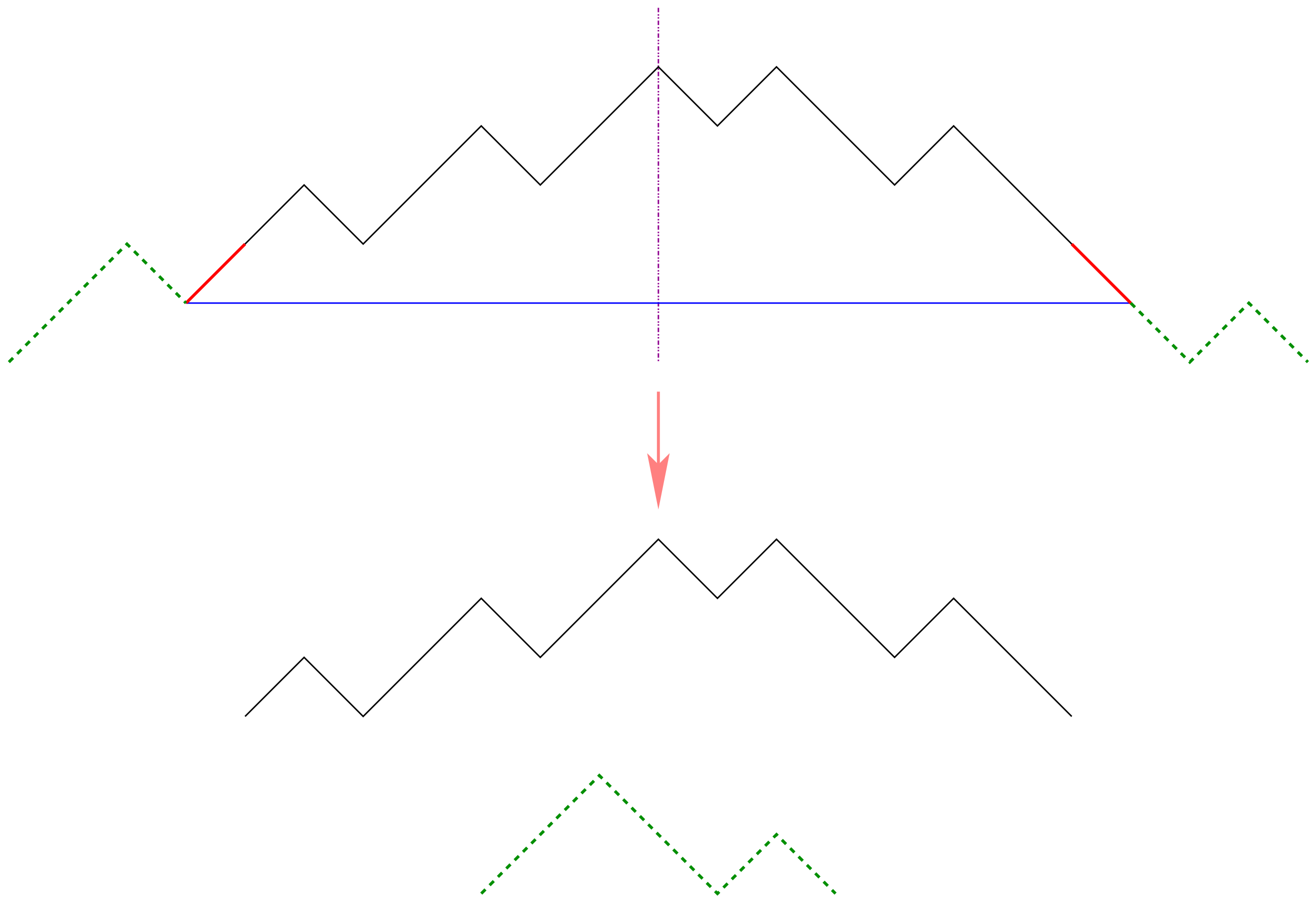
We want to show that

$$A(x,q,t) = B(x,q,t)$$

Problem: If we try to compute B decomposing Dyck paths as before, we lose the center of the path...



Solution for $q = 1$: We can use a different decomposition to count centered tunnels:

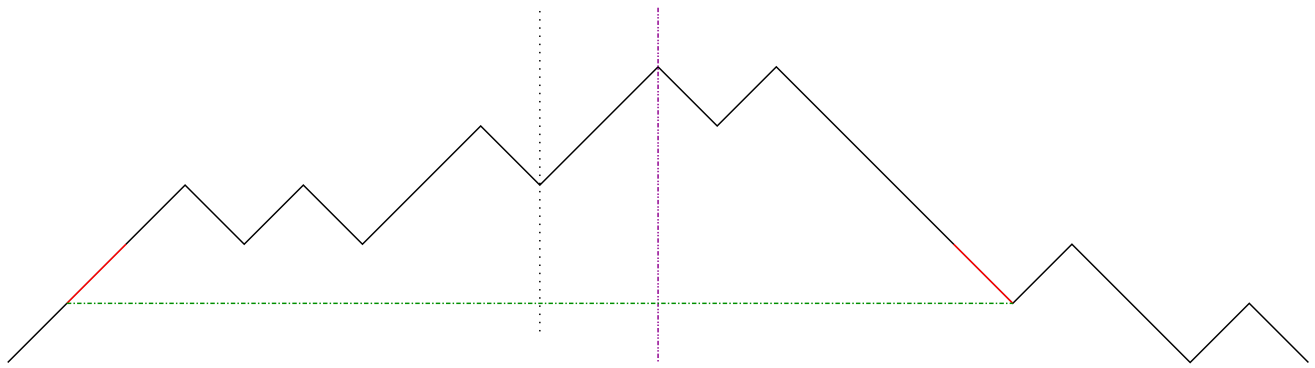


We get that

$$B(x, 1, t) = \frac{2}{1 + 2t(1 - x) + \sqrt{1 - 4t}} = A(x, 1, t)$$

This gives another proof of the theorem about fixed points.

However, this still doesn't work for left tunnels ($\sim$ excedances), because we also loose the center

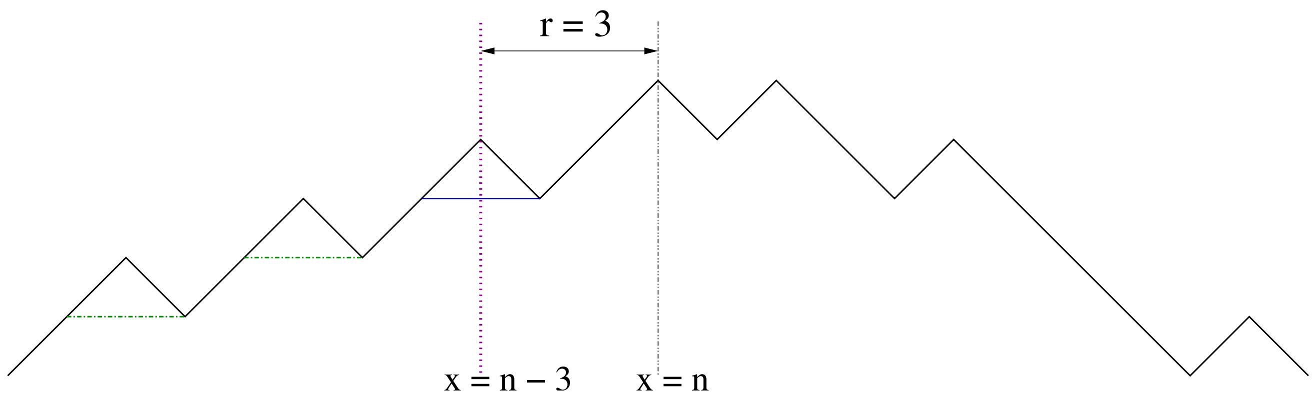


Solution: Generalize the concept of centered and left tunnels, allowing the center to be shifted

Step 3

For $r \in \mathbb{Z}$ and $D \in \mathcal{D}_n$, let
$CT_r(D) :=$ number of tunnels of D with center on $x = n - r$
$LT_r(D) :=$ number of tunnels of D with center on $x < n - r$

Example:



$CT_3(D) = 1 \qquad LT_3(D) = 2$

Consider

$$F(x, q, t, u) := \sum_{\substack{n \geq 0 \\ r \geq 0}} \Big(\sum_{D \in \mathcal{D}_n} x^{CT_r(D)} q^{LT_r(D)} \Big) u^r t^n$$

Intuitively, we sum over pairs *(Dyck path, vertical line)*

Conjecture:

$$F(x,q,t,u) = \frac{\frac{1-u+(q-1)tuA(1,1,tu)}{1-u+(q-1)tuA(1,q,t)} - (x-1)tuA(1,1,tu)}{(1-qt(A(1,q,t)-1)-xt)(1-u)}$$

(note that $A(1,1,t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} c_n t^n$)

Obs. By definition,

$$F(x,q,t,0) = \sum_{n\geq 0} \Big(\sum_{D\in\mathcal{D}_n} x^{CT_0(D)} q^{LT_0(D)} \Big) t^n = B(x,q,t)$$

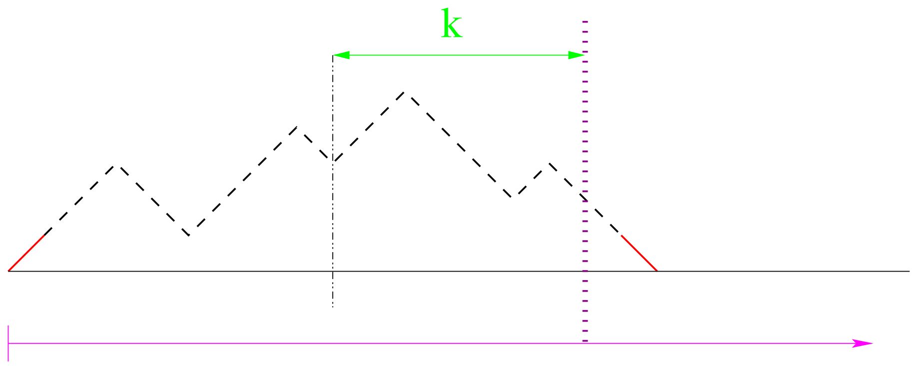
If the conjecture is true, then

$$F(x,q,t,0) = \frac{1}{1-qt(A(1,q,t)-1)-xt} = A(x,q,t)$$

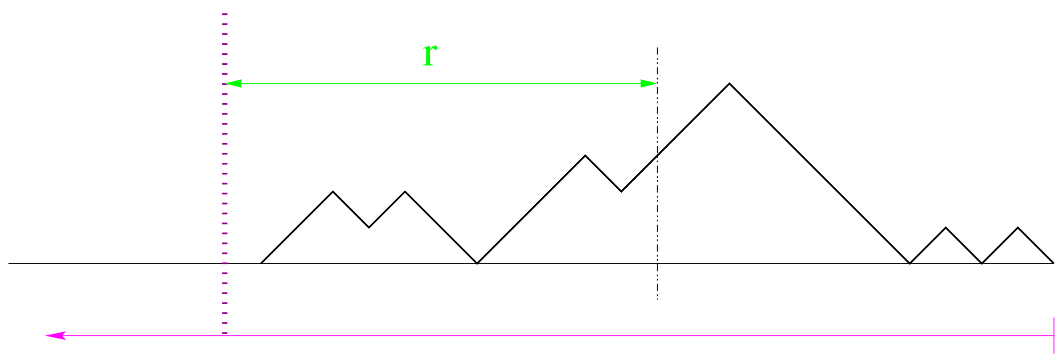
so it implies the theorem.

The conjecture also gives expressions for

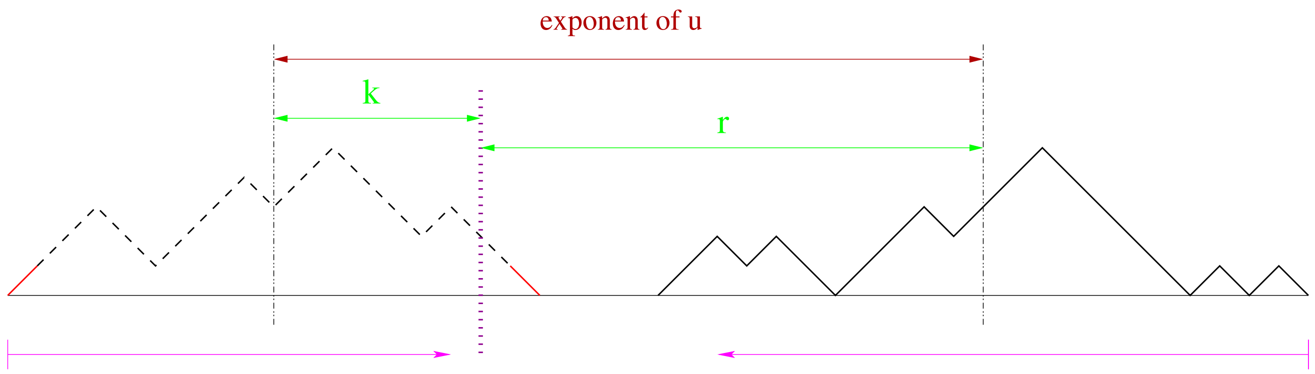
$$G_1(x,q,t,u) := \sum_{\substack{n\geq 1\\ k\geq -n}} \Big(\sum_{\substack{D\in\mathcal{D}_n \text{ with}\\ \text{no returns}}} x^{CT_{-k}(D)} q^{LT_{-k}(D)} \Big) u^k t^n$$



$$G_2(x,q,t,u) := \sum_{\substack{n\geq 0\\ r\geq -n}} \Big(\sum_{D\in\mathcal{D}_n} x^{CT_r(D)} q^{LT_r(D)} \Big) u^r t^n$$

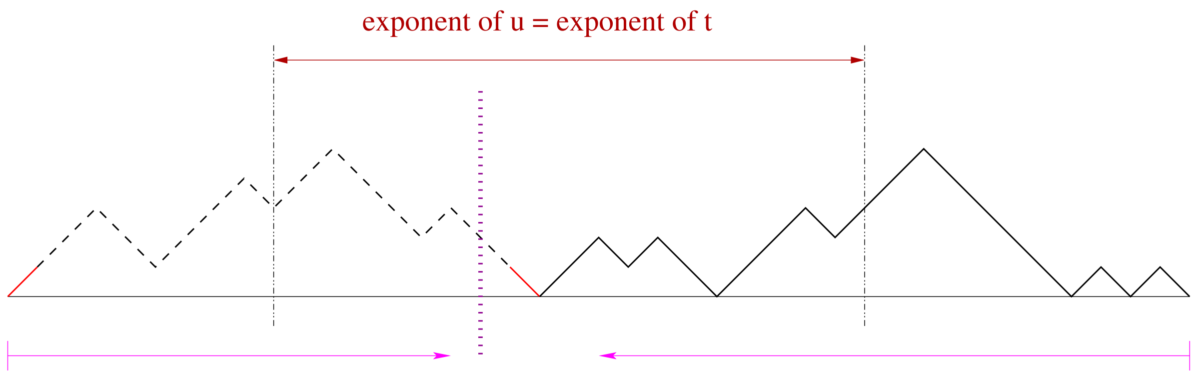


Multiplying these two functions, we get



But we want only the terms in which the two parts of the path are next to each other
That happens precisely when

$$\text{exponent of } u = \text{exponent of } t$$



More formally, we have

$$\begin{aligned} H(x,q,t,u,w) &:= G_1(x,q,t,\frac{u}{w})\ G_2(x,q,tw,u) \\ &= \sum_{\substack{n \geq 0 \\ j \geq -n}} H_{j,n}(x,q,w) u^j t^n \end{aligned}$$

and we want

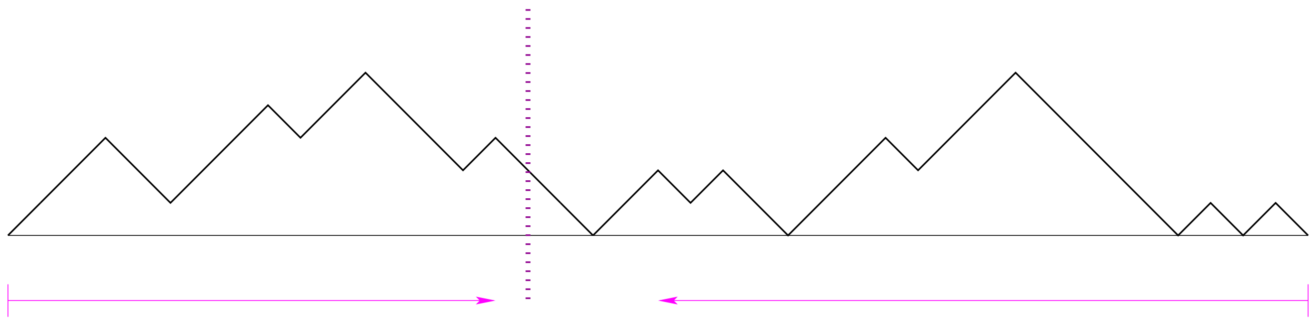
$$\text{diag } H := \sum_{n \geq 0} H_{n,n}(x,q,w) v^n$$

For rational generating functions, there is a general method for computing diagonals

Our case is a bit more complicated...
but it can be done.

For the conjecture to be true, we should get

$$\text{diag } H = \sum_{\substack{n \geq 1 \\ -n \leq r \leq n}} \Big(\sum_{D \in \mathcal{D}_n} x^{CT_r(D)} q^{LT_r(D)} \Big) w^r v^n$$



More precisely: if we use the conjectured expressions for G_1 and G_2, after multiplying them and taking the diagonal, we should get exactly the conjectured expression for this sum

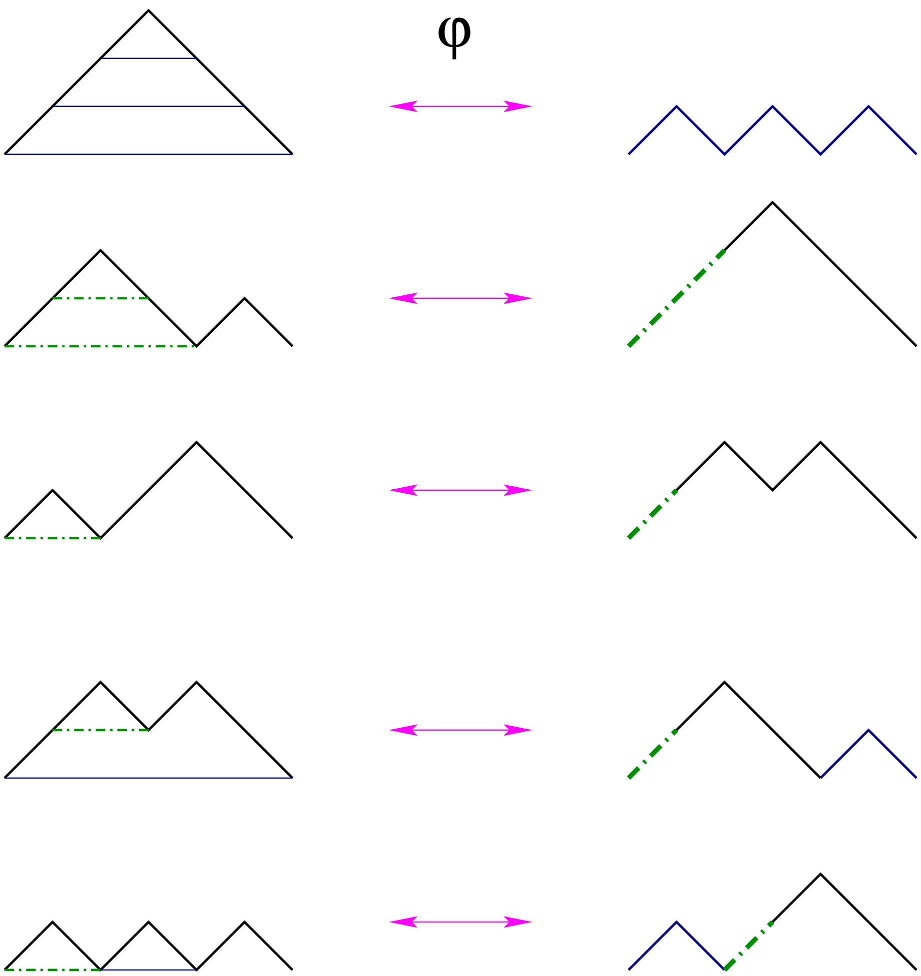
And this is exactly what we get!

$\square$

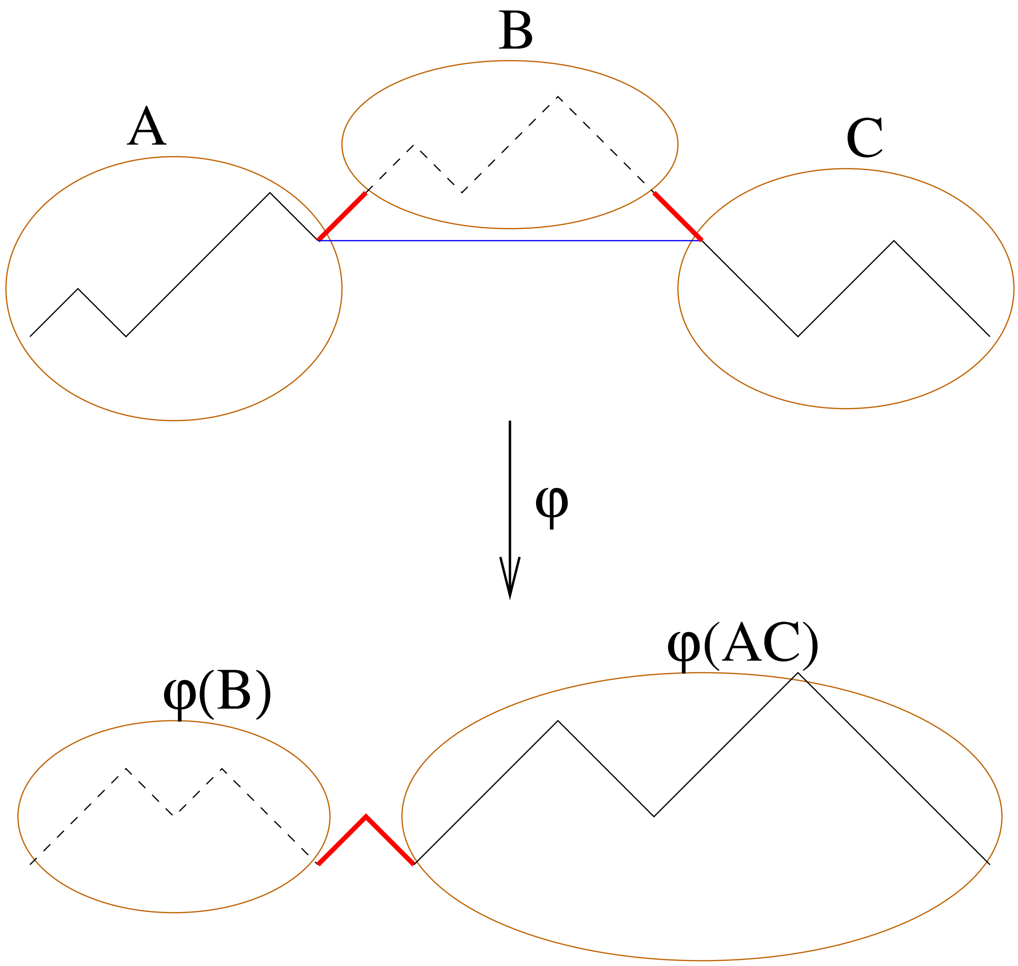
We would like to have a bijective proof. We want a bijection φ such that:

$\mathcal{S}_n(132)$	$\longleftrightarrow$	$\mathcal{D}_n$	$\overset{\varphi}{\longleftrightarrow}$	$\mathcal{D}_n$	$\longleftrightarrow$	$\mathcal{S}_n(321)$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills	$\leftrightarrow$	fixed points
exced- ances	$\leftrightarrow$	left tunnels	$\leftrightarrow$	double rises	$\leftrightarrow$	exced- ances

Example: $n = 3$



Observation: If D has some centered tunnel, $\varphi(D)$ can be defined recursively:



What if D has no centered tunnel?