FIXED POINTS AND EXCEDANCES IN RESTRICTED PERMUTATIONS

Sergi Elizalde

MIT

FPSAC'03, Sweden

Updated versions available at

http://www-math.mit.edu/~sergi

Pattern-avoidance

$\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_k$

π *contains* σ if there exist $i_1 < \ldots < i_k$ s.t.
$\rho(\pi_{i_1} \cdots \pi_{i_k}) = \sigma$
where ρ is the relabeling with $\{1, \ldots, k\}$ keeping the same order relationships

Otherwise, we say that π *avoids* σ

Example:
24531 contains 132
42351 avoids 132

$\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n | \ \pi \text{ avoids } \sigma\}$

Patterns of length 3

$$\#\mathcal{S}_n(123) = \#\mathcal{S}_n(321)$$

$\pi_1\pi_2\cdots\pi_n$ avoids 123 $\Leftrightarrow$ $\pi_n\cdots\pi_2\pi_1$ avoids 321

$$\#\mathcal{S}_n(132) = \#\mathcal{S}_n(231) = \#\mathcal{S}_n(312) = \#\mathcal{S}_n(213)$$

$$\begin{array}{ccc}
\pi_1\pi_2\cdots\pi_n \text{ avoids } 132 & \Leftrightarrow & \pi_n\cdots\pi_2\pi_1 \text{ avoids } 231 \\
\Updownarrow & & \Updownarrow \\
\begin{array}{c}(n+1-\pi_1)\cdots(n+1-\pi_n) \\ \text{avoids } 312\end{array} & \Leftrightarrow & \begin{array}{c}(n+1-\pi_n)\cdots(n+1-\pi_1) \\ \text{avoids } 213\end{array}
\end{array}$$

Classical result:

$$\#\mathcal{S}_n(123) = \#\mathcal{S}_n(132) = C_n$$

Several bijections between $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ are known

Refinements

i is a *fixed point* of π if $\pi_i = i$
$\mathrm{fp}(\pi) :=$ number of fixed points of π

i is an *excedance* of π if $\pi_i > i$
$\mathrm{exc}(\pi) :=$ number of excedances of π

Ex: if $\pi = 42351$, then $\mathrm{fp}(\pi) = 2$, $\mathrm{exc}(\pi) = 2$

Theorem (Robertson-Saracino-Zeilberger '02).
For any $k \leq n$

$$\#\{\pi \in \mathcal{S}_n(321) | \ \mathrm{fp}(\pi) = k\} = \#\{\pi \in \mathcal{S}_n(132) | \ \mathrm{fp}(\pi) = k\}$$

The proof is not bijective

Questions:

- Is there a simple bijective proof?
- Can this theorem be generalized, considering other statistics in permutations?

Main Theorem. For any $k, l \le n$

$$\#\{\pi \in \mathcal{S}_n(321) \mid \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}$$
$$= \#\{\pi \in \mathcal{S}_n(132) \mid \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}$$

$\lrcorner$

Equivalently,

$$\sum_{\pi \in \mathcal{S}_n(321)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} = \sum_{\pi \in \mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)}$$

Example (n=4):

$\mathcal{S}_4(321)$

π	fp	exc
1234	4	0
1243	2	1
1324	2	1
1342	1	2
1423	1	1
2134	2	1
2143	0	2
2314	1	2
2341	0	3
2413	0	2
3124	1	1
3142	0	2
3412	0	2
4123	0	1

$\mathcal{S}_4(132)$

π	fp	exc
1234	4	0
2134	2	1
2314	1	2
2341	0	3
3124	1	1
3214	2	1
3241	1	2
3412	0	2
3421	0	2
4123	0	1
4213	1	1
4231	2	1
4312	0	2
4321	0	2

$$x^4 + 3x^2q + 2xq^2 + 2xq + q^3 + 4q^2 + q$$

First proof (in the proceedings):

- Non-bijective

- Transforms the problem into a Dyck path enumeration problem

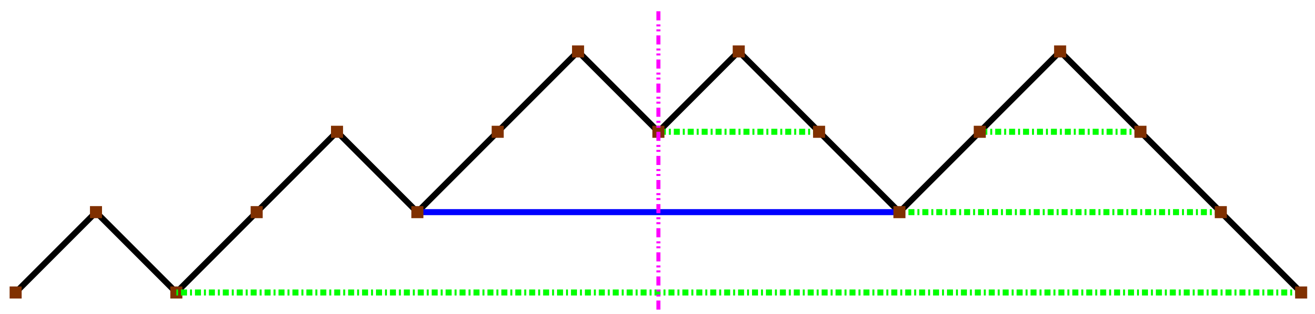
- Uses nonstandard techniques in generating functions: diagonal of a non-rational g.f.

Second proof (in this talk):

- Bijective

- Joint work with Igor Pak

Dyck path: path between $(0,0)$ and $(2n,0)$ with steps $(1,1)$ (*up-steps*) and $(1,-1)$ (*down-steps*) that never goes below the x-axis



$\mathcal{D}_n :=$ set of Dyck paths of length $2n$

$|\mathcal{D}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$ (Catalan number)

For $D \in \mathcal{D}_n$, define:

tunnel: horizontal segment between two lattice points of D that stays always below D
(D has n tunnels, one for each up-step)

centered tunnel: x-coordinate of midpoint is n

right tunnel: x-coordinate of midpoint is $> n$

The Dyck path above has is 1 centered tunnel and 4 right tunnels

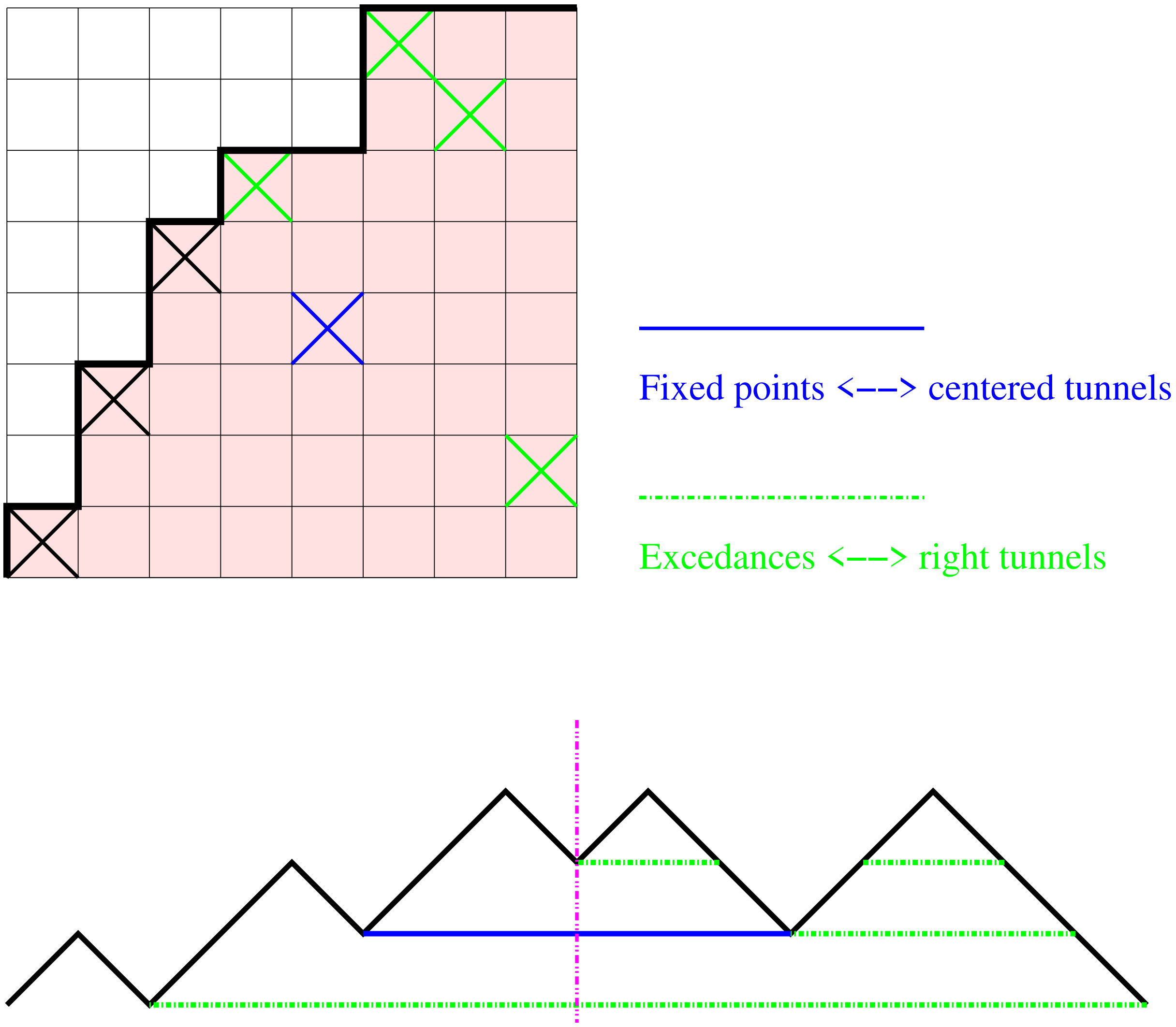
Bijective proof

Composition of two bijections into Dyck pahts:

$\mathcal{S}_n(132)$	$\xrightarrow{\Phi}$	$\mathcal{D}_n$	$\xleftarrow{\Psi}$	$S_n(321)$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	fixed points
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	excedances

$$\Phi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$$

Example: $\pi = 67435281 \in \mathcal{S}_8(132)$



$$\Psi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$$

Step 1: RSK correspondence

$$\pi \mapsto (P, Q)$$

Example: $\pi = 23514687$

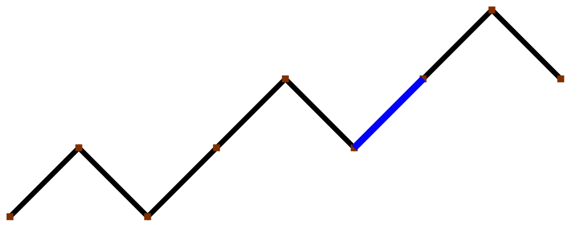
$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array}$$

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array}$$

Step 2: $(P, Q) \mapsto \Psi(\pi)$

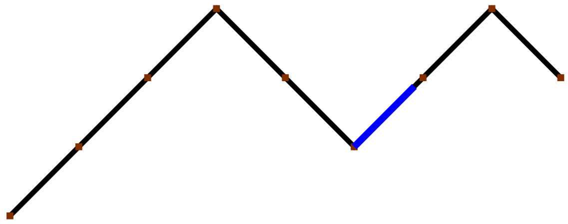
$P =$

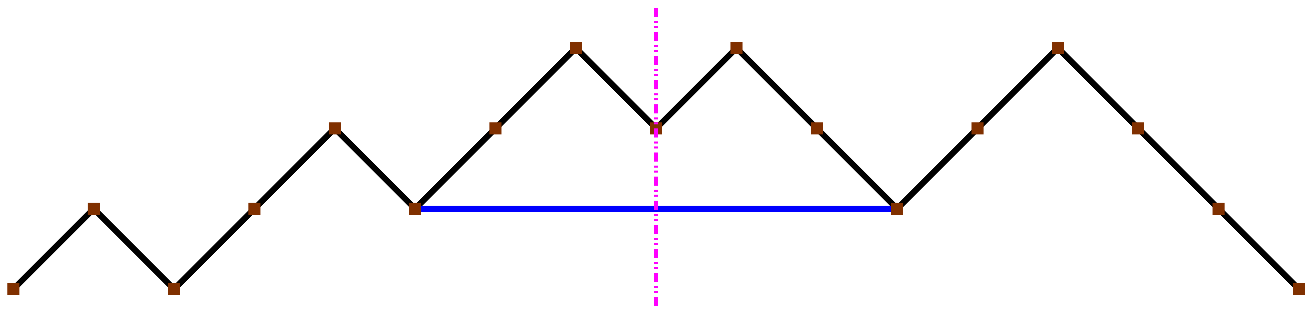
1	3	4	6	7
2	5	8		



$Q =$

1	2	3	6	7
4	5	8		

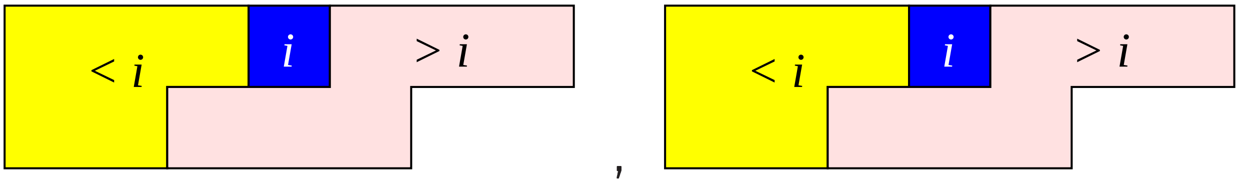




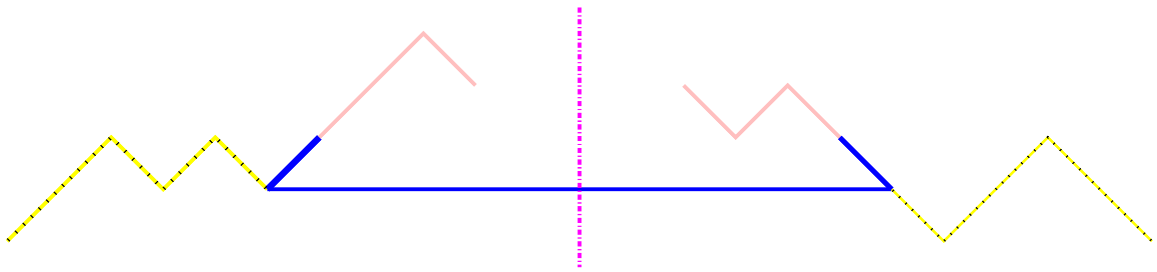
Ψ maps fixed points to centered tunnels:

$$\left.\begin{array}{c}\pi\ 321\text{-avoiding}\\ \pi_i = i\end{array}\right\} \Rightarrow \pi = \underbrace{\ldots\ldots}_{<i}\, i\, \underbrace{\ldots\ldots\ldots}_{>i}$$

Then, both P and Q have shape



and i produces a centered tunnel



It can also be checked that Ψ maps excedances to right tunnels

Generating function for fixed points and excedances:

$$\begin{aligned} A(x,q,t) &= \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} t^n \\ &= \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} t^n \end{aligned}$$

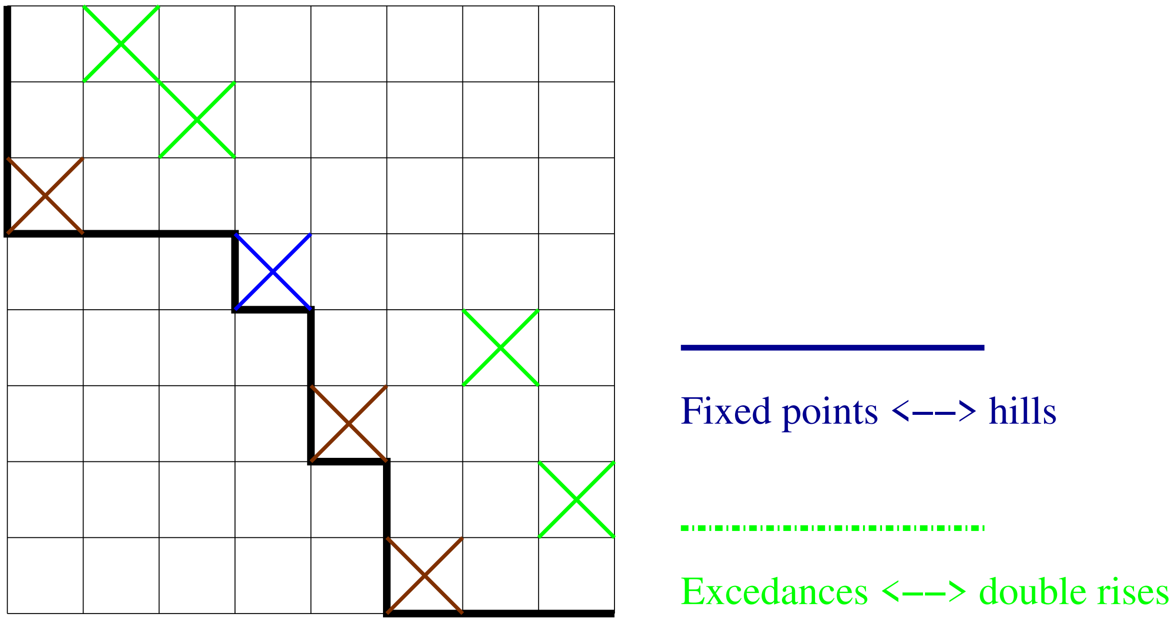
Theorem.

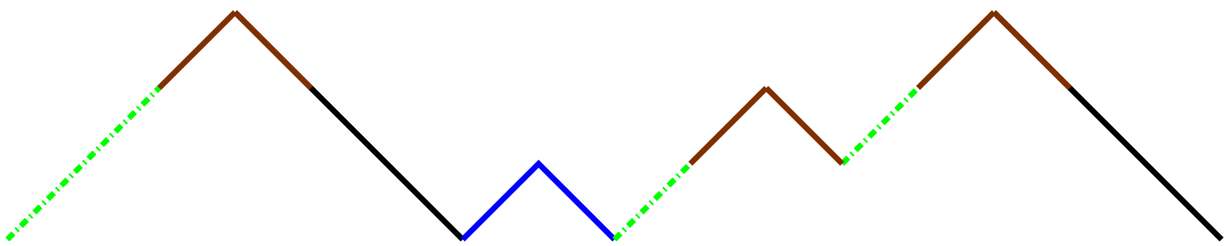
$$A(x,q,t) = \frac{2}{1+t\ (1+q-2x)+\sqrt{1-2t(1+q)+t^2(1-q)^2}}$$

Idea of proof. Another bijection

$$\varphi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$$

Example: $\pi = 23147586 \in \mathcal{S}_8(321)$





Hills and double rises in Dyck paths are easy to enumerate

Final remarks

We can restrict to involutions:

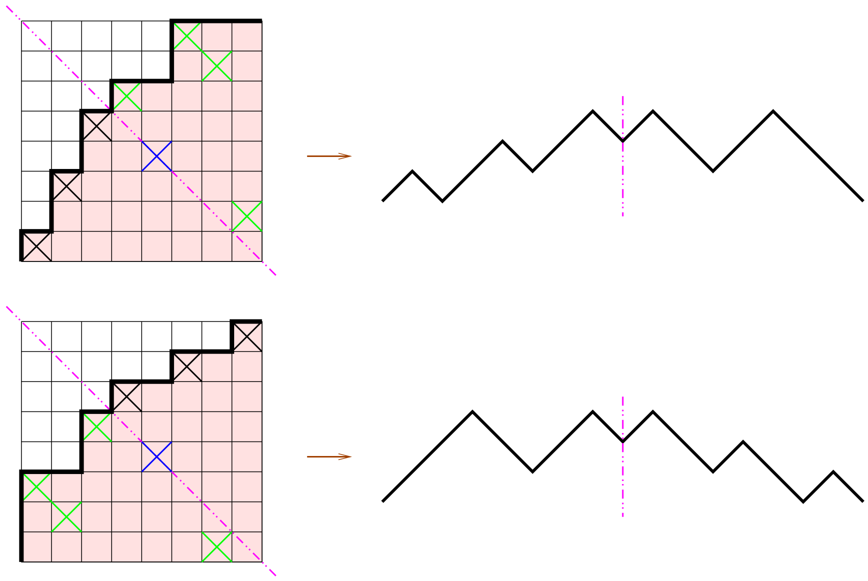
Corollary. For any $k, l \leq n$

$$\#\{\pi \in \mathcal{S}_n(321) \mid \pi = \pi^{-1},\ \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\} = \\ \#\{\pi \in \mathcal{S}_n(132) \mid \pi = \pi^{-1},\ \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}$$

Proof

Let D^* be the reflection of D over a vertical line

$$\Phi(\pi^{-1}) = \Phi(\pi)^*$$

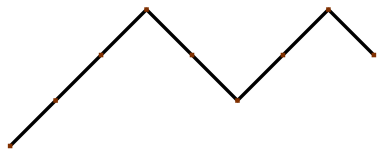


$$\color{red}{\Psi}(\pi^{-1}) = \color{red}{\Psi}(\pi)^*$$

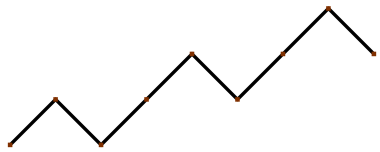
By the properties of RSK,

$$\pi \mapsto (P, Q) \;\Rightarrow\; \pi^{-1} \mapsto (Q, P)$$

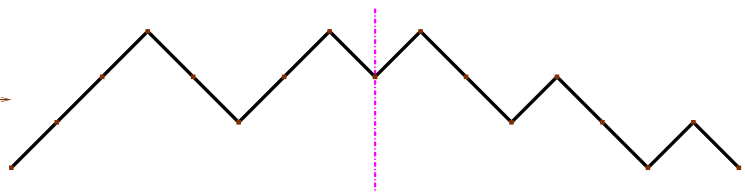
$$Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array} \longrightarrow$$



$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \longrightarrow$$



$(Q,P) \longrightarrow$



Involutions correspond to symmetric Dyck paths by both bijections

Double and triple restrictions

Theorem. Let

$$F_{\sigma_1,\sigma_2} := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\sigma_1,\sigma_2)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} t^n$$

(define $F_{\sigma_1,\sigma_2,\sigma_3}$ similarly)

We can give explicit expressions for F_{σ_1,σ_2} and $F_{\sigma_1,\sigma_2,\sigma_3}$, for any $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{S}_3$.

They are all rational

Examples:

$$F_{132,231} = \frac{1-t-qt^2+xqt^3}{(1-xt)(1-t-2qt^2)}$$

$$F_{132,213} = \frac{1-(1+q)t-2qt^2+4q(1+q)t^3-(xq^2+xq+5q^2)t^4+2xq^2t^5}{(1-t)(1-xt)(1-qt)(1-4qt^2)}$$

$$F_{123,132,213} = \frac{1+xt+(x^2-q)t^2+(-xq+q^2+q)t^3-x^2qt^4}{(1+qt^2)(1-3qt^2+q^2t^4)}$$

Open questions

- Further refinements?
  Other statistics on restricted permutations having the same distribution for different patterns?

- Similar theorems for permutations avoiding longer patterns (e.g., of length 4)?