Asymptotic enumeration of permutations avoiding generalized patterns

Sergi Elizalde

elizalde@msri.org

MSRI

Classical pattern avoidance

$$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n, \qquad \sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$$

Definition. π *avoids* σ if there do not exist $i_1 < \ldots < i_k$ s.t.

$$\rho(\pi_{i_1}\cdots\pi_{i_k}) = \sigma,$$

where ρ is the reduction that relabels the elements with $\{1,\ldots,k\}$ preserving the order relationships in π.

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Example.

$\underline{2}4\underline{53}1$ contains 132

42351 avoids 132

Generalized pattern avoidance (Babson, Steingrímsson '00)

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where $\varepsilon_j = \begin{cases} \text{-} & \text{or} \\ & \text{(empty string)} \end{cases}$

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Example. $\underline{35}42\underline{7}1\underline{6}$ avoids $12\text{-}43$, but it contains $12\text{-}4\text{-}3$

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Example.

Descents	$\longleftrightarrow$	Occurrences of 21
Alternating permutations	$\longleftrightarrow$	Permutations simultaneously avoiding 123 and 321

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$a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$

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$\sum_{n\geq 0} f_n z^n < \sum_{n\geq 0} g_n z^n$	means	$f_n < g_n \; \forall n$

Asymptotic enumeration: classical patterns

Theorem (*Stanley-Wilf conjecture*, proved by Marcus and Tardos '03). *For every classical pattern $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$, there is a constant λ s.t.*

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Note. A lower bound for $k \geq 3$ is given by

$$\alpha_n(\sigma) \geq \alpha_n(\rho(\sigma_1\text{-}\sigma_2\text{-}\sigma_3)) = \mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}} 4^n \qquad \text{(Catalan number)}$$

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The theorem implies that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)}$ exists.
Finding it for given patterns is an interesting problem.

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Theorem. *Let $k \geq 3$ and let $\sigma \in \mathcal{S}_k$ be a consecutive pattern.*

- *There exist constants $0 < c, d < 1$ such that*

$$c^n n! \leq \alpha_n(\sigma) \leq d^n n!$$

for all $n \geq k$.

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- *There exists a constant $0 < w < 1$ such that*

$$\lim_{n \to \infty} \left(\frac{\alpha_n(\sigma)}{n!} \right)^{1/n} = w.$$

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Proof sketch:

Upper bound: We have

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For consecutive patterns of length 3, $\alpha_n(\tau)$ is known.

Asymptotic enumeration: consecutive patterns (3)

Theorem (E., Noy '00).

$$A_{123}(z) = A_{321}(z) = \frac{\sqrt{3}}{2}\frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2}z + \frac{\pi}{6})}$$

$$A_{132}(z) = A_{231}(z) = A_{312}(z) = A_{213}(z) = \frac{1}{1 - \int_0^z e^{-t^2/2}dt}$$

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Their coefficients satisfy

$$\alpha_n(123) \sim \gamma_1 \cdot (\rho_1)^n \cdot n! \qquad \alpha_n(132) \sim \gamma_2 \cdot (\rho_2)^n \cdot n!$$

where

$$\rho_1 = 0.8269933, \quad \gamma_1 = 1.8305194, \quad \rho_2 = 0.7839769, \quad \gamma_2 = 2.2558142.$$

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Furthermore, $\forall n \geq 4$,

$$\alpha_n(123) > \alpha_n(132).$$

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It also follows from

$$\frac{\alpha_{m+n}(\sigma)}{(m+n)!} \leq \frac{\alpha_m(\sigma)}{m!} \frac{\alpha_n(\sigma)}{n!}$$

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We know that

$$1 > w \geq \lim_{n\to\infty} \left(\frac{\alpha_n(132)}{n!}\right)^{1/n} = 0.7839769$$

Asymptotic enumeration: $\sigma = \cdots \sigma_i \sigma_{i+1} \sigma_{i+2} \cdots$

The previous proof generalizes to any pattern having three consecutive elements without a dash:

$$\sigma = \cdots \sigma_i \sigma_{i+1} \sigma_{i+2} \cdots$$

(for example, $\sigma = 54\text{-}273\text{-}1\text{-}6$)

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Proposition. *Let σ be a generalized pattern as above. Then there exist constants $0 < c, d < 1$ such that*

$$c^n n! \leq \alpha_n(\sigma) \leq d^n n!$$

for all $n \geq k$.

Three cases of generalized patterns

Solved cases:

- σ has dashes everywhere: $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$ $\longrightarrow$ λ^n
- σ has two consecutive slots without a dash:
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Remaining case:

- σ has some slot without a dash, but not two consecutive slots without dashes (we call these *small-block patterns*).

  Example: $\sigma = 47\text{-}52\text{-}3\text{-}18\text{-}6$.

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We now focus on small-block patterns.

Proposition (Claesson '01). *Let σ be a generalized pattern of length 3 with one dash.*

- *If* $\sigma \in \{1\text{-}23, 3\text{-}21, 32\text{-}1, 12\text{-}3, 1\text{-}32, 23\text{-}1, 3\text{-}12, 21\text{-}3\}$, *then*

$$\alpha_n(\sigma) = \mathbf{B}_n \qquad \text{(Bell number)}$$

- *If* $\sigma \in \{2\text{-}13, 2\text{-}31, 31\text{-}2, 13\text{-}2\}$, *then*

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Note that

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Note that

$$\frac{\ln \mathbf{B}_n}{n} = \ln n - \ln\ln n + O\left(\frac{\ln\ln n}{\ln n}\right).$$

In particular, $\delta^n \ll \mathbf{B}_n \ll c^n n!$ for any constants $\delta, c > 0$.

Proposition. *Let* $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ *be a consecutive pattern, and let* 1-σ *denote the generalized pattern* 1-$(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_k+1)$. *Then,*

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Proposition. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ be a consecutive pattern, and let $1\text{-}\sigma$ denote the generalized pattern $1\text{-}(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_k+1)$. Then,*

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$$A_{1\text{-}324}(z) = \exp\left(\int_0^z \frac{dt}{1-\int_0^t e^{-u^2/2}du}\right) \qquad A_{1\text{-}432}(z) = \exp\left(\frac{\sqrt{3}}{2}\int_0^z \frac{e^{t/2}\,dt}{\cos(\frac{\sqrt{3}}{2}t+\frac{\pi}{6})}\right)$$

Proposition. *For* $k \geq 1$*, let*

$$h_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \qquad b_k(z) = \sum_{i=0}^{k} \binom{k}{i}^2 \left[z + 2(h_{k-i} - h_i)\right] e^{iz},$$

$$c_k(z) = \frac{e^{(k+1)z}}{k+1} - \sum_{i=0}^{k} \binom{k}{i}\binom{k+1}{i} \left[z + 2(h_{k-i} - h_i) + \frac{1}{k+1-i}\right] e^{iz},$$

$$S(z) = \sum_{k \geq 1} b_k(z) + \sum_{k \geq 1} c_k(z).$$

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$$S(z) = \sum_{k\geq 1} b_k(z) + \sum_{k \geq 1} c_k(z).$$

Then

$$e^{S(z)} < A_{12\text{-}34}(z) < e^{S(z)+e^z+z-1}$$

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Then

$$e^{S(z)} < A_{12\text{-}34}(z) < e^{S(z) + e^z + z - 1}$$

Observation. The lower bound implies that $\alpha_n(12\text{-}34) \gg \mathbf{B}_n$.

Proof idea

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  They can be written uniquely as $\pi = B_0 a_1 B_1 a_2 B_2 a_3 B_3 \cdots$, where each $B_i = w_{i,0} w_{i,1} w_{i,2} \cdots w_{i,r_i}$, s.t.

  - each $w_{i,j}$ is a decreasing word,
  - for $j \geq 1$, $w_{i,j}$ is nonempty and its first element is bigger than a_i,
  - the last element of each $w_{i,j}$ is less than a_i,
  - the last element of B_i is less than a_{i+1}.

Proof idea

- Analyze the structure of $12\text{-}34$-avoiding permutations:

  They can be written uniquely as $\pi = B_0 a_1 B_1 a_2 B_2 a_3 B_3 \cdots$, where each $B_i = w_{i,0} w_{i,1} w_{i,2} \cdots w_{i,r_i}$, s.t.

  - each $w_{i,j}$ is a decreasing word,
  - for $j \geq 1$, $w_{i,j}$ is nonempty and its first element is bigger than a_i,
  - the last element of each $w_{i,j}$ is less than a_i,
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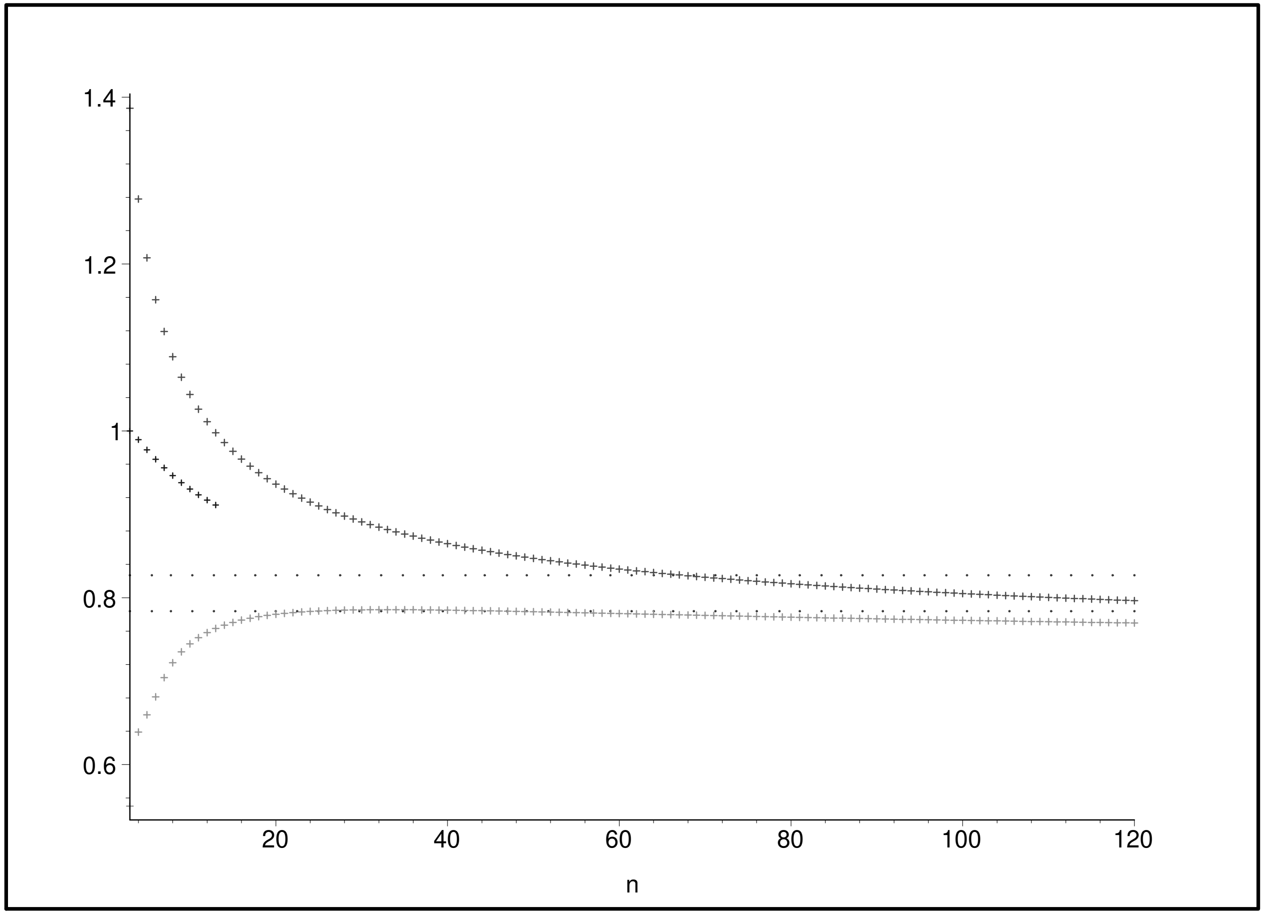
Upper and lower bounds for $\alpha_n(12\text{-}34)$

Plot of the first values of $\sqrt[n]{\alpha_n(12\text{-}34)/n!}$ between those given by the upper and lower bounds:



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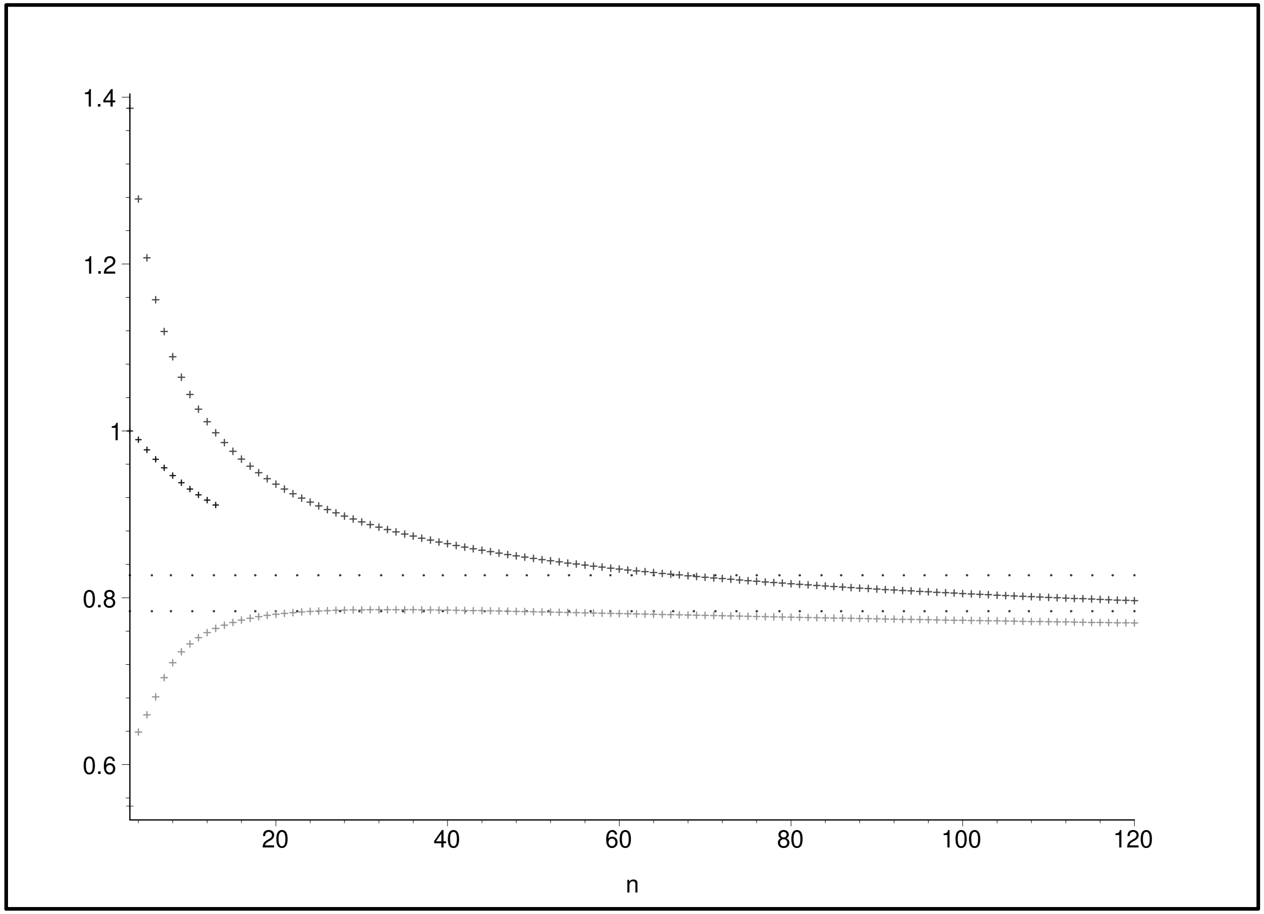

Question: Is it true that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(12\text{-}34)/n!} = 0$?

Proposition.

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Let $\mathbf{C}^{\exp}(z) := \sum_{n\geq 0} \mathbf{C}_n \frac{z^n}{n!}$ (exp. gen. fct. for Catalan numbers)

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The upper bound implies that

$$\lim_{n\to\infty}\left(\frac{\alpha_n(1\text{-}23\text{-}4)}{n!}\right)^{1/n} = 0$$

Upper and lower bounds for $\alpha_n(1\text{-}23\text{-}4)$

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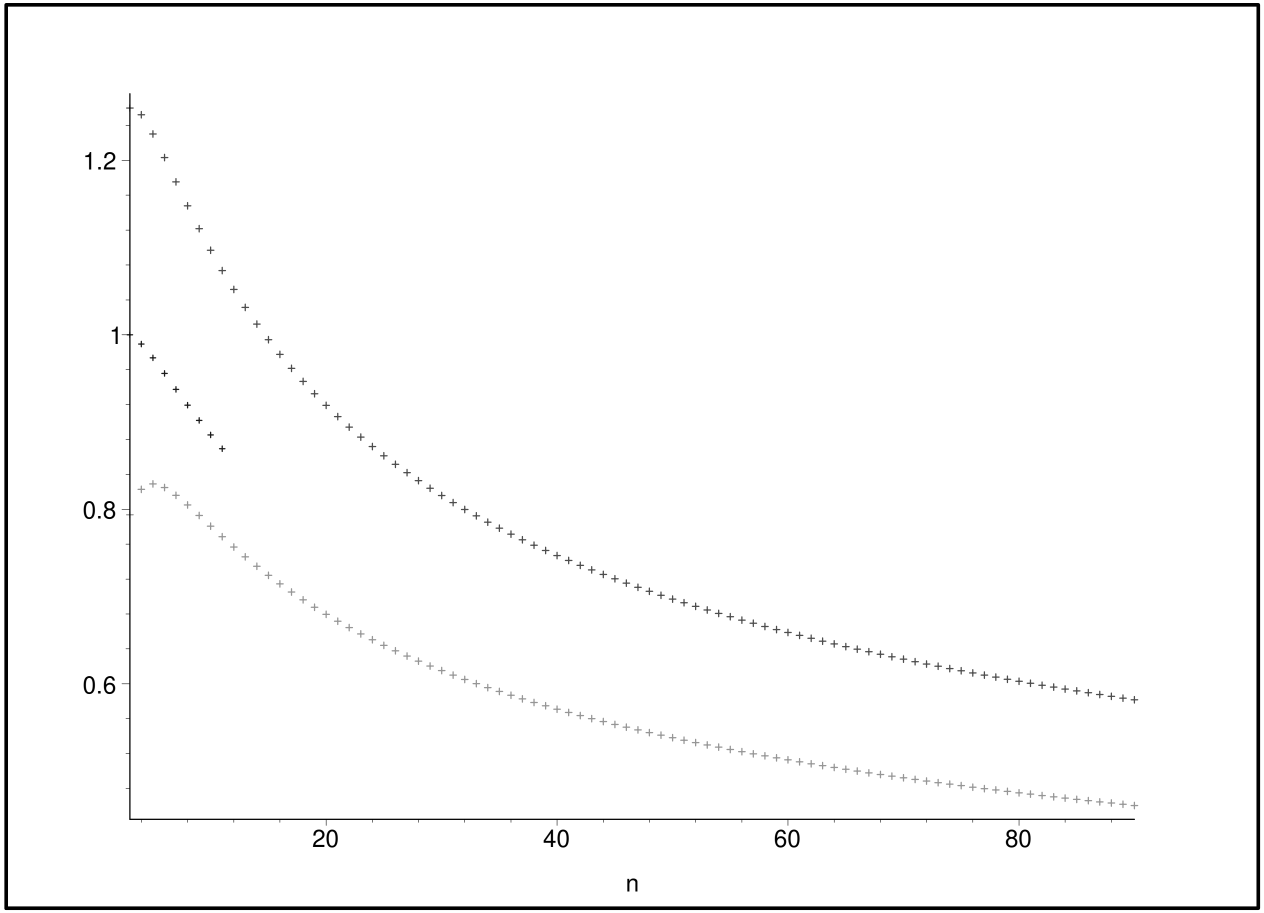

Proposition. *Let* $\sigma \in \mathcal{S}_{k-2}$ *be a consecutive pattern, and let* $1\text{-}\sigma\text{-}k$ *denote the generalized pattern* $1\text{-}(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_{k-2}+1)\text{-}k$. *Then,*

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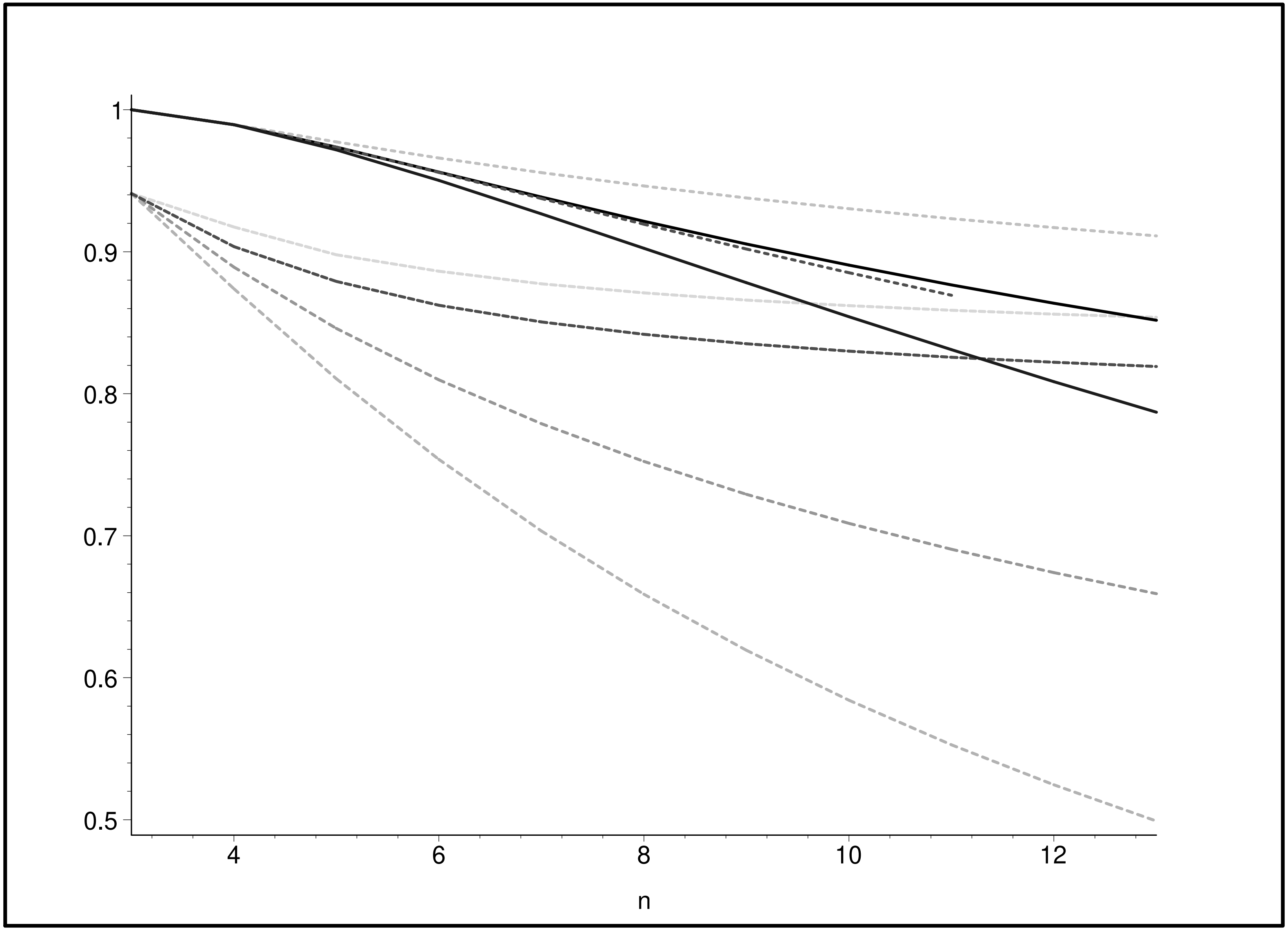
Plots of $\sqrt[n]{\alpha_n(\sigma)/n!}$ for some generalized patterns:



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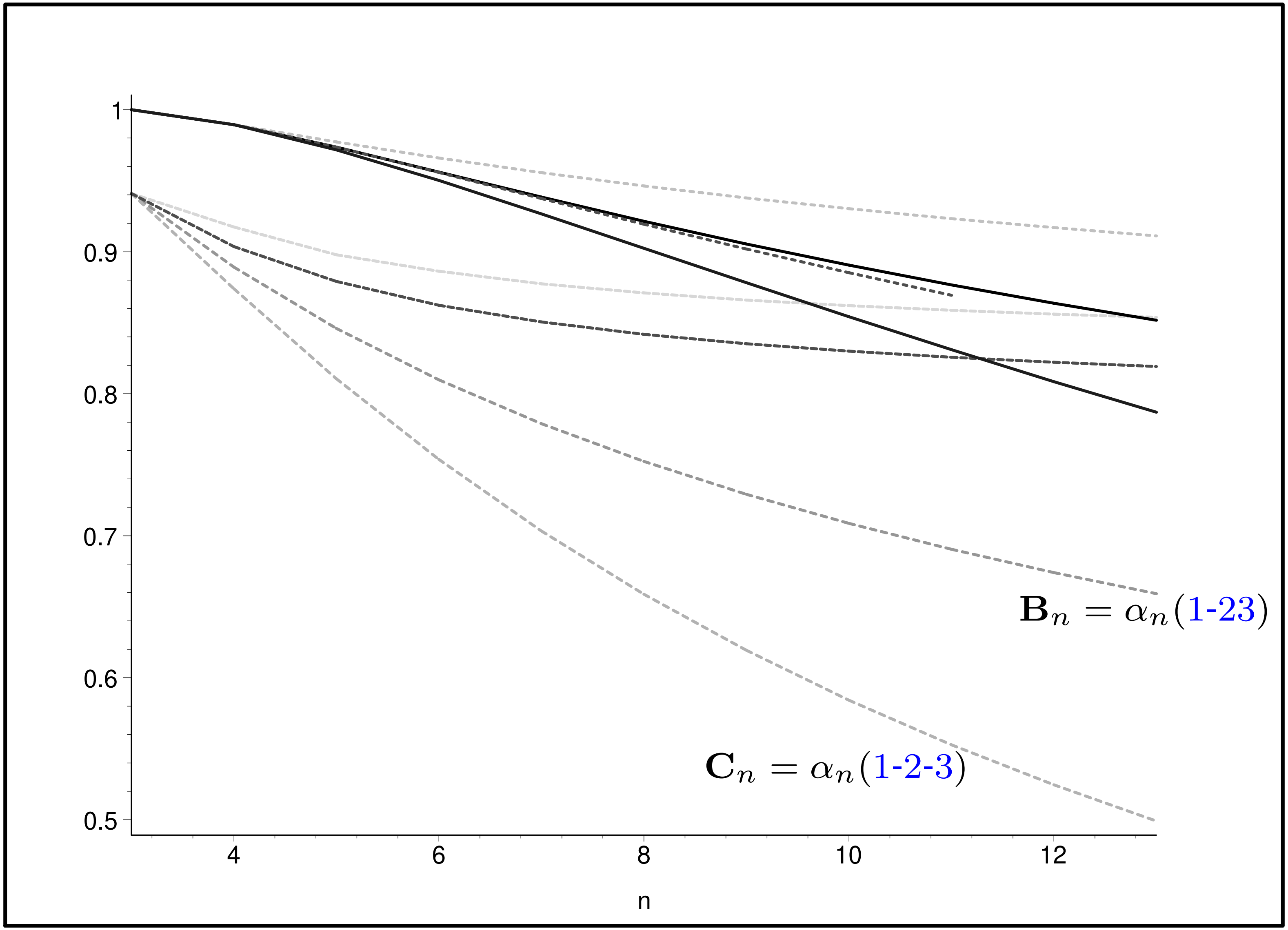

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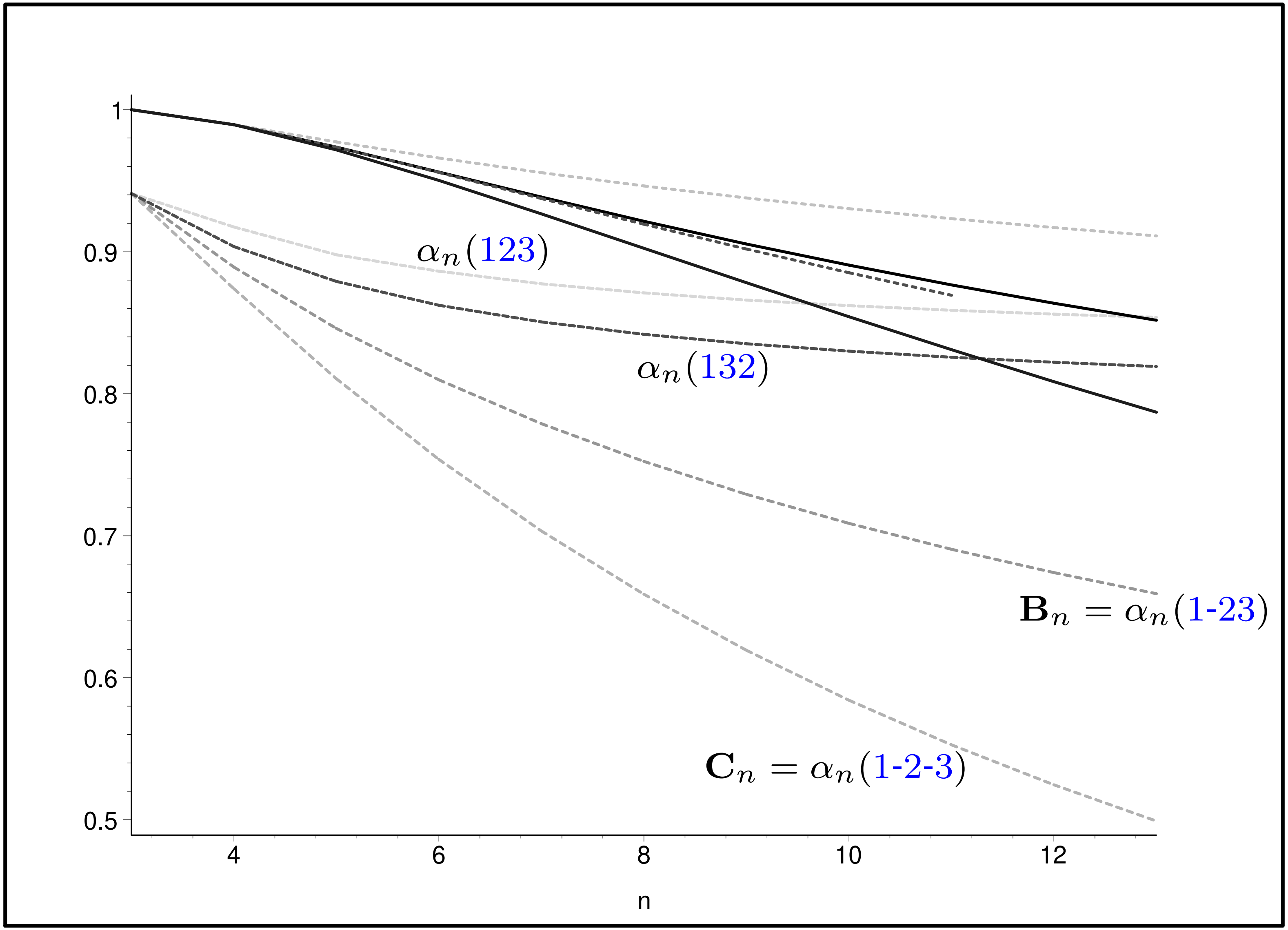

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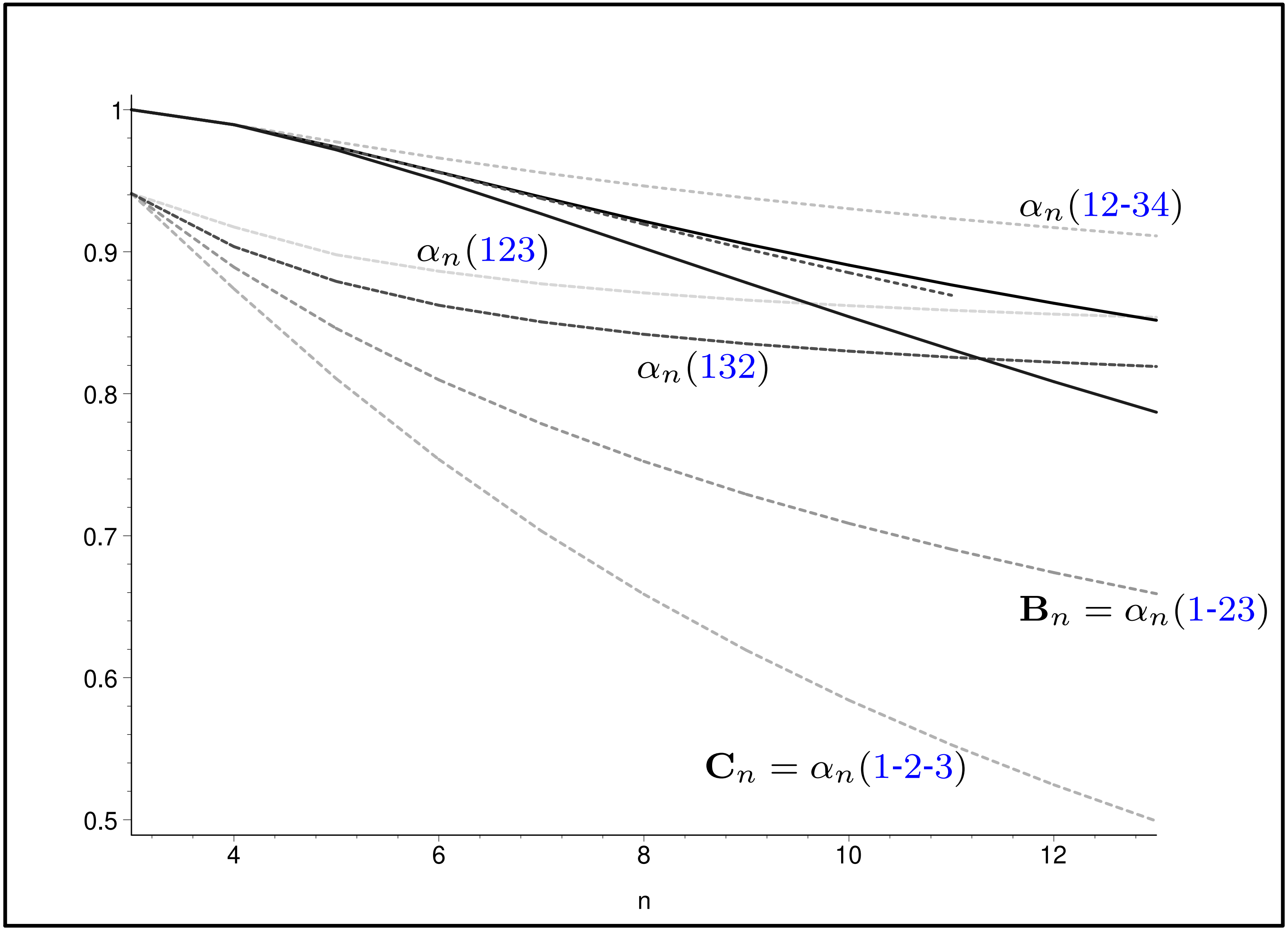

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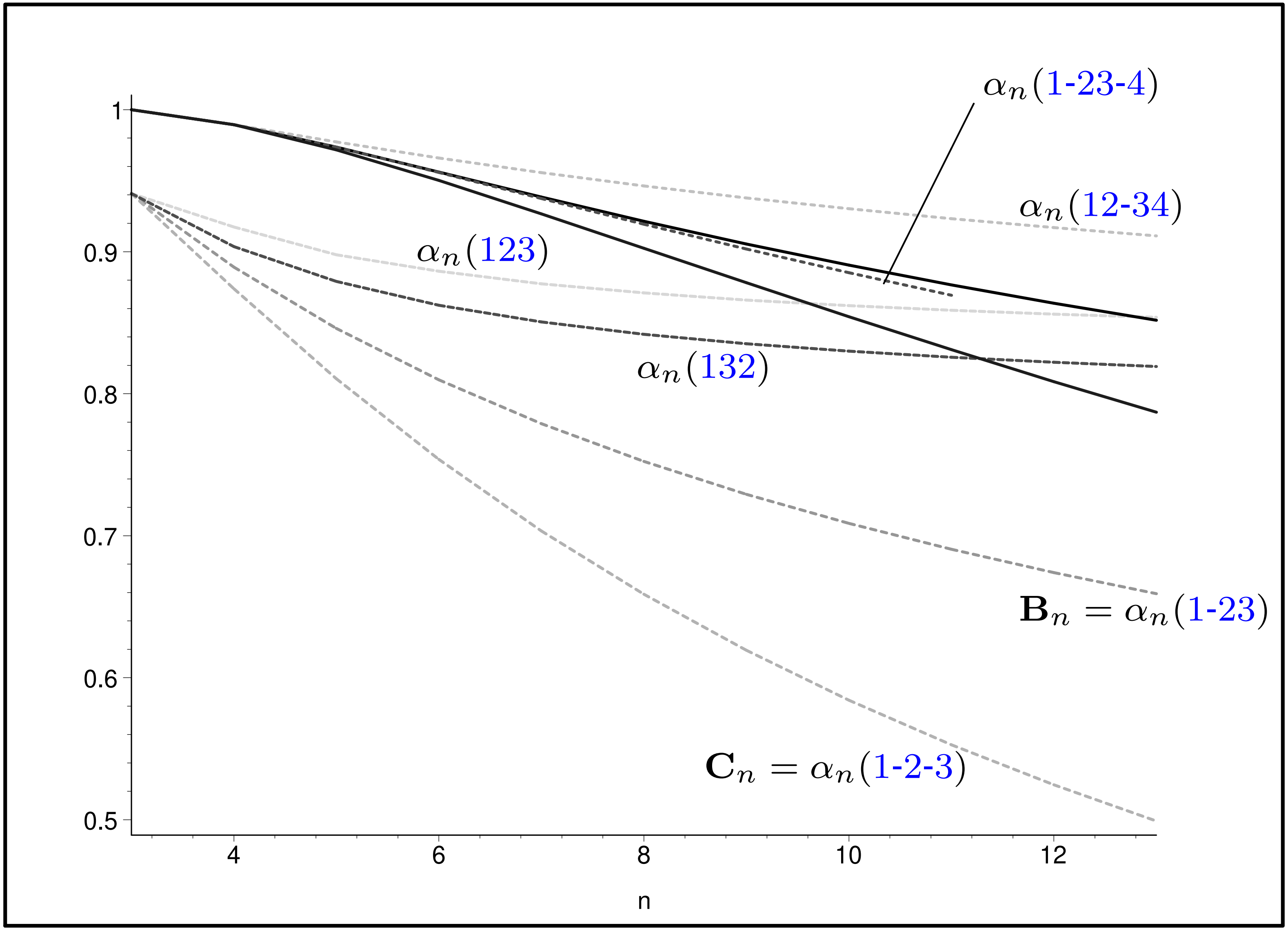

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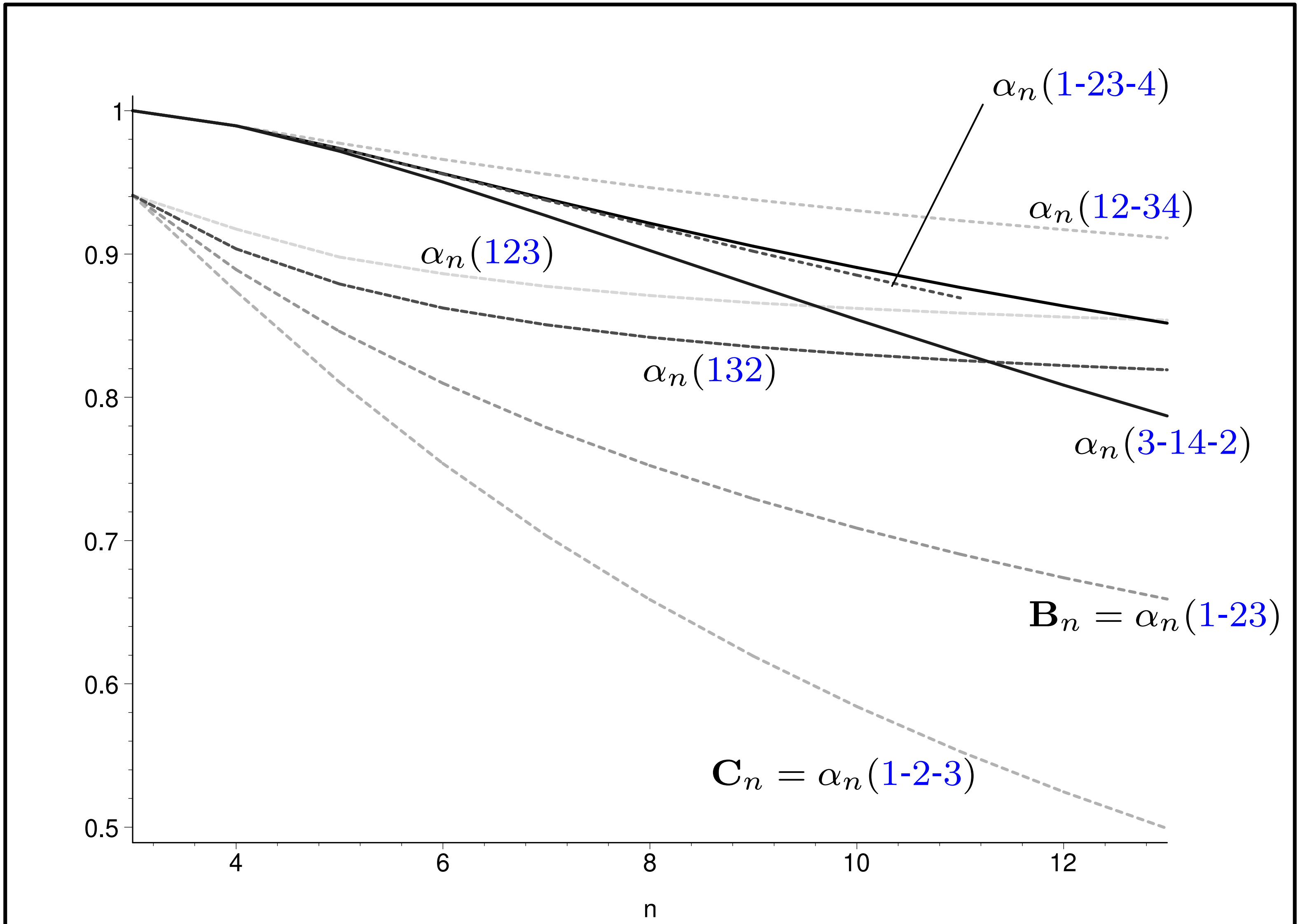

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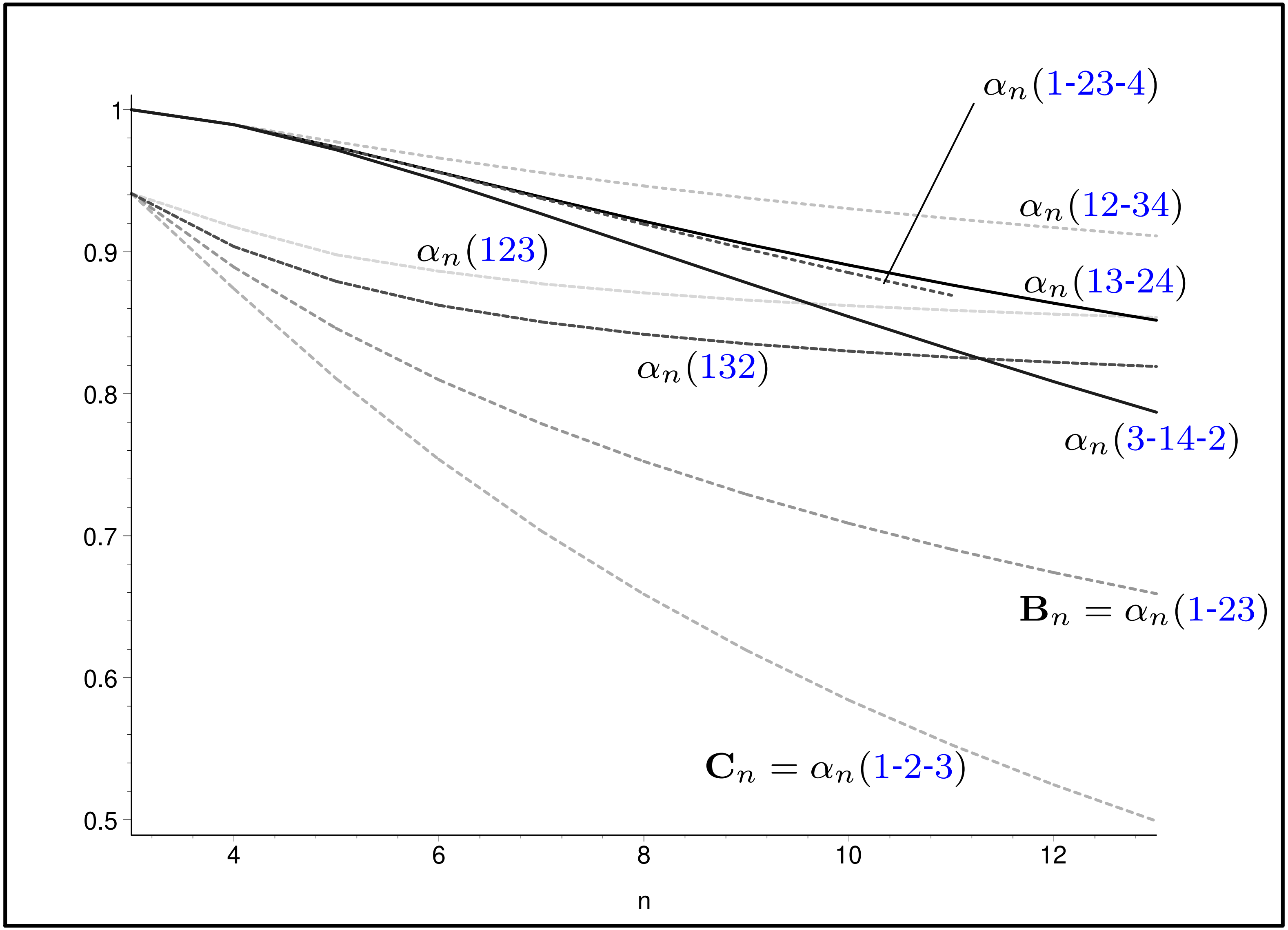

Open questions

- Ultimate goal: Classify generalized patterns according to the asymptotic behavior of $\alpha_n(\sigma)$ as $n \to \infty$.

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- Is there any small-block pattern σ such that $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!} > 0$?

- Study $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ for consecutive patterns σ (and other non-small-block generalized patterns).

  This is analogous to finding $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)}$ for classical patterns.