A generalized Lalanne–Kreweras involution for rectangular tableaux

Sergi Elizalde

Dartmouth

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Peaks and valleys of Dyck paths

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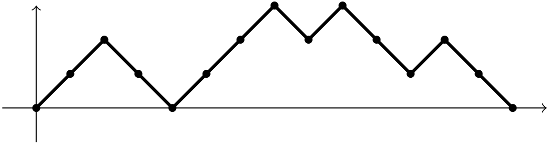

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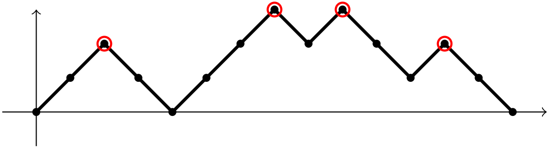

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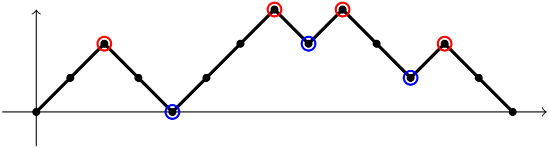

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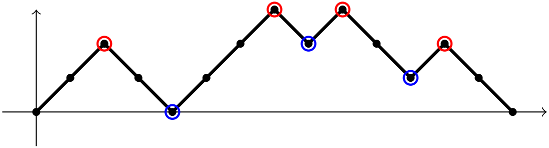

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Classical result

The number of Dyck paths of length $2n$ with h valleys (or equivalently, $h+1$ peaks), is the *Narayana number*

$$N(n,h)=\frac{1}{n}\binom{n}{h}\binom{n}{h+1}.$$

Symmetry of the Narayana numbers

From the formula

$$N(n,h) = \frac{1}{n}\binom{n}{h}\binom{n}{h+1},$$

we see the symmetry

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Biijective proof?

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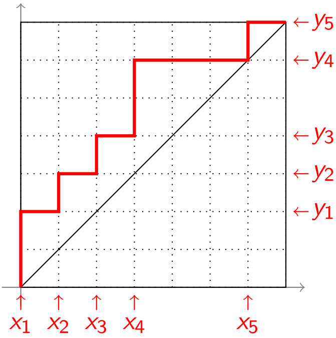

Draw $D \in \mathcal{D}_n$ with north and east steps, and suppose that its peaks have coordinates $(x_1, y_1), \ldots, (x_{h+1}, y_{h+1})$.

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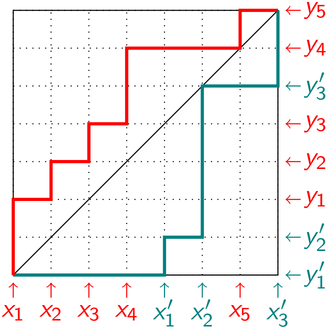

Draw $D \in \mathcal{D}_n$ with north and east steps, and suppose that its peaks have coordinates $(x_1, y_1), \ldots, (x_{h+1}, y_{h+1})$. Let

$$\{0, 1, \ldots, n\} \setminus \{x_1, x_2, \ldots, x_{h+1}\} = \{x'_1, x'_2, \ldots, x'_{n-h}\},$$
$$\{0, 1, \ldots, n\} \setminus \{y_1, y_2, \ldots, y_{h+1}\} = \{y'_1, y'_2, \ldots, y'_{n-h}\},$$

and let $\mathrm{LK}(D)$ be the Dyck path with peaks at $(x'_1, y'_1), \ldots, (x'_{n-h}, y'_{n-h})$.

Standard Young tableaux

Let λ be a partition of N. A standard Young tableau of shape λ is a filling of the boxes of the Young diagram of λ with the numbers $1, 2, \ldots, N$ so that rows and columns are increasing.

$\mathrm{SYT}(\lambda)$ = set of standard Young tableaux of shape λ

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 4 & 8 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array} \in \mathrm{SYT}(4,3,1)$$

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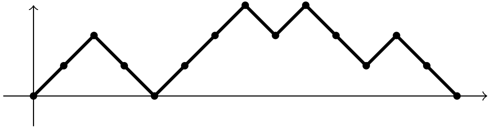

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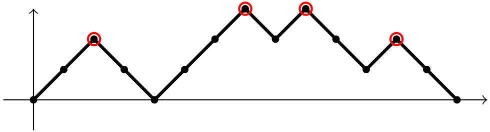

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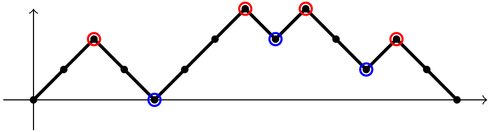

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Peaks become jumps to a lower row. Valleys become jumps to a higher row.

Ascents and descents on standard Young tableaux

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- i is an ascent if $i+1$ is higher than i,

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$$\begin{aligned} N(k,n,h) &= |\{T \in \mathrm{SYT}(n^k) : \mathrm{asc}(T) = h\}| \\ &= |\{T \in \mathrm{SYT}(n^k) : \mathrm{des}(T) = h+k-1\}| \quad \text{(Sulanke '04)} \end{aligned}$$

Theorem (Sulanke '04)

$$N(k,n,h) = \sum_{\ell=0}^{h} (-1)^{h-\ell} \binom{kn+1}{h-\ell} \prod_{i=0}^{n-1} \prod_{j=0}^{k-1} \frac{i+j+1+\ell}{i+j+1}.$$

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Generalized Narayana numbers

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Our goal is to give a bijective proof of this symmetry.

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In 2005, Farley turned Stanley's proof into a recursive bijection based on Garsia and Milne's involution principle, and again asked for a direct bijection for the case $P = \mathrm{k} \times \mathrm{n}$.

The bijection φ

We will describe a bijection $\varphi : \mathsf{SYT}(n^k) \to \mathsf{SYT}(n^k)$ such that

$$\mathrm{des}(T) + \mathrm{des}(\varphi(T)) = (k-1)(n+1)$$

for all $T \in \mathsf{SYT}(n^k)$.

Arrow encodings

We encode standard Young tableaux using arrows describing the placement of the entries:

1	2	6	9	10	15
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1	2	6	9	10	15
3	5	11	17	18	22
4	8	16	21	24	27
7	12	19	23	26	28
13	14	20	25	29	30

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1	2 ↓	6 ↓	9	10 ↓	15 ↓
↑ 3 ↓	5 ↑↓	↑ 11 ↓	↑↓ 17	18 ↓	22 ↓
↑ 4 ↑↓	8 ↑↓	↑ 16 ↑↓	21 ↑↓	24 ↓	27 ↓
↑ 7 ↑	12 ↓	↑ 19 ↓	↑ 23 ↑↓	26 ↑	28 ↓
↑ 13	14 ↑	20 ↑	25 ↑	29	30

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↑	3	↓	5	↑↓↑	11	↓↑↓	17		18	↓	22	↓
↑	4	↑↓	8	↑↓↑	16	↑↓	21	↑↓	24	↓	27	↓
↑	7	↑	12	↓↑	19	↓↑	23	↑↓	26	↑	28	↓
↑	13		14	↑	20	↑	25	↑	29		30	

This encoding is a bijection between $\mathrm{SYT}(n^k)$ and *valid arrow arrays* (i.e., satisfying certain conditions).

Leading and trailing arrows

- leading $\downarrow$ = leftmost $\downarrow$ in its block
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Leading ↓ are colored in red.

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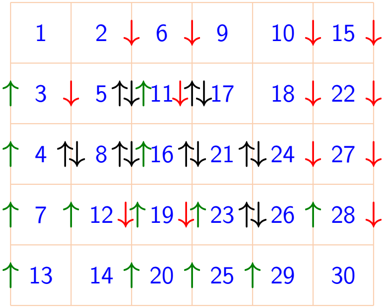

Leading ↓ are colored in red.

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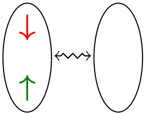

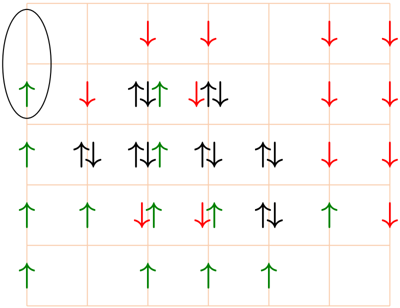

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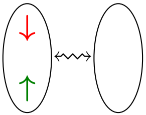

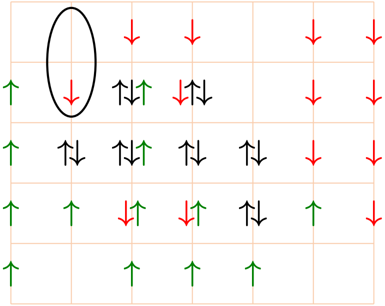

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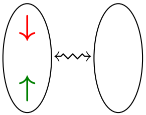

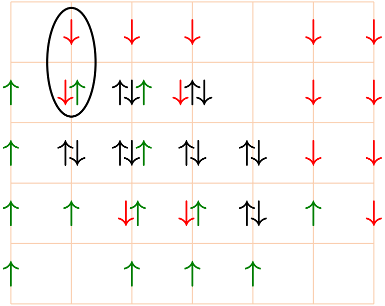

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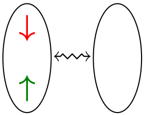

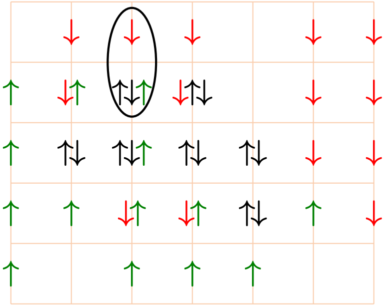

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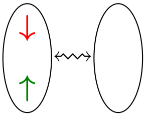

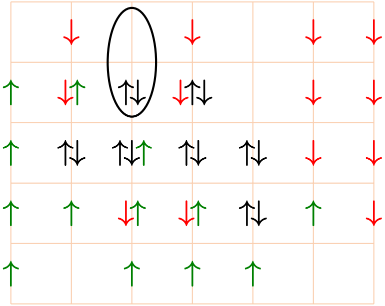

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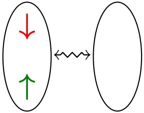

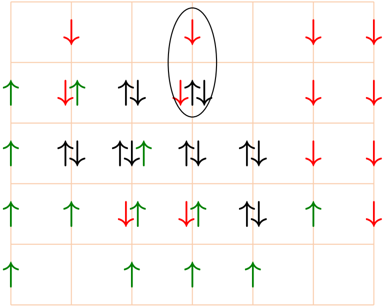

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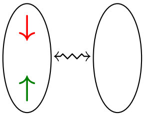

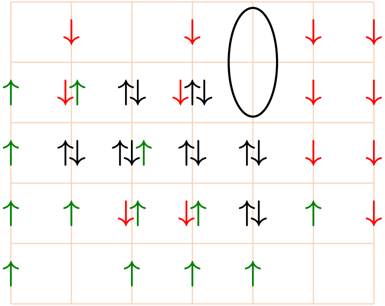

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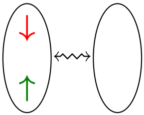

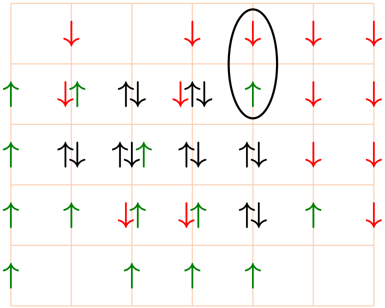

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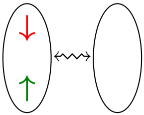

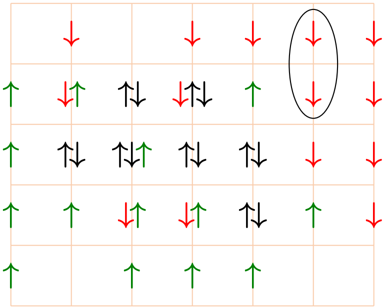

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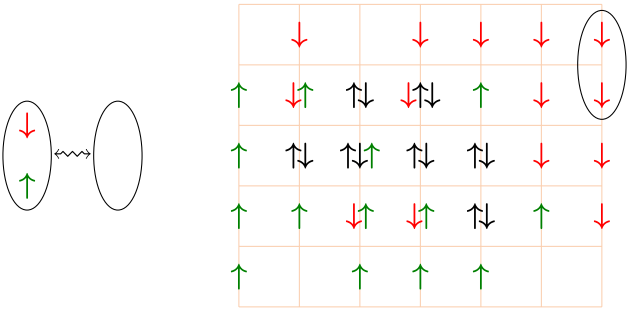

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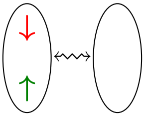

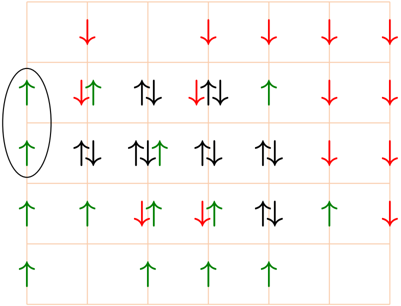

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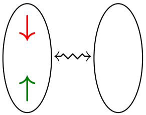

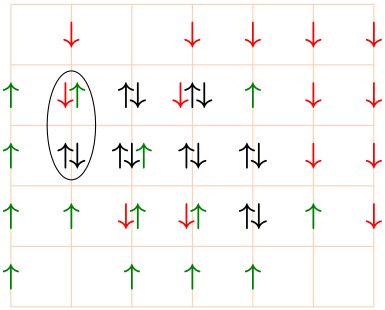

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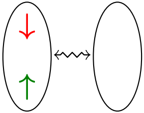

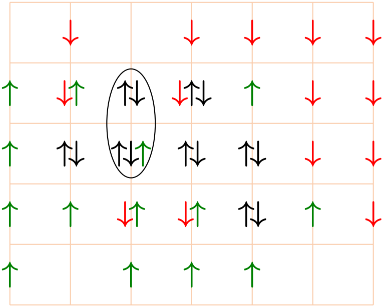

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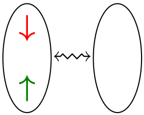

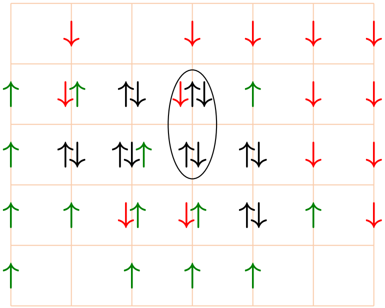

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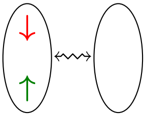

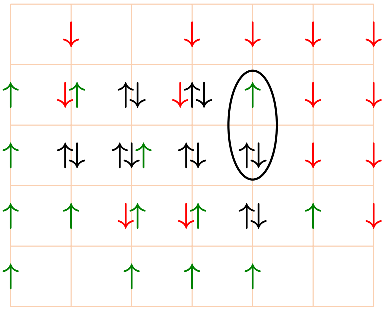

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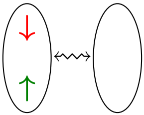

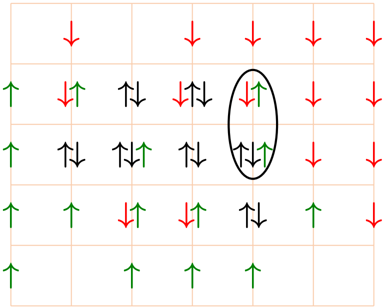

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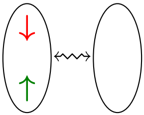

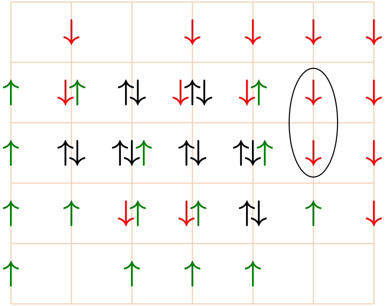

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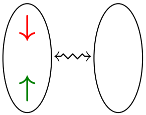

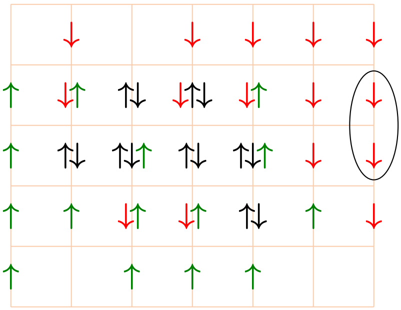

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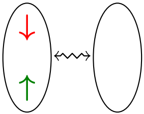

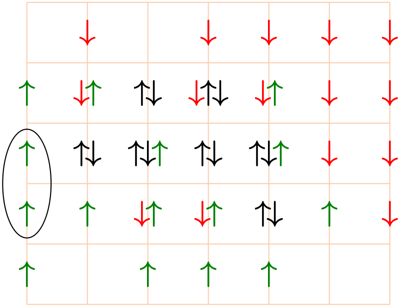

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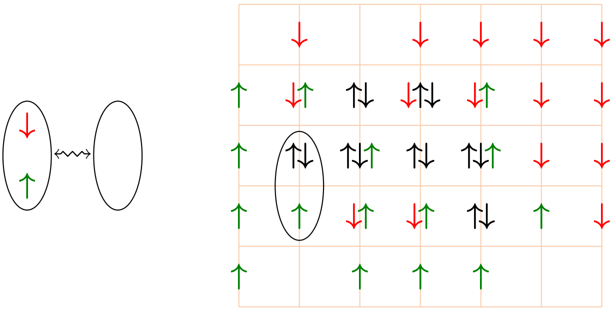

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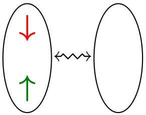

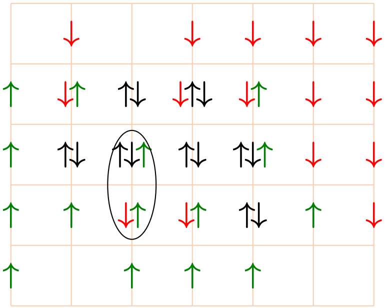

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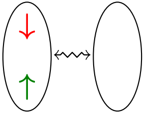

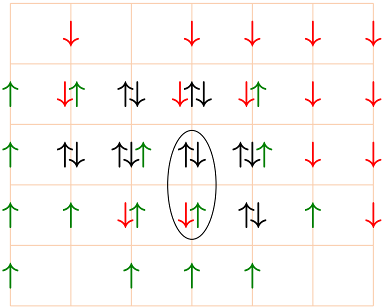

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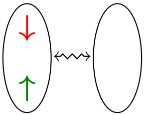

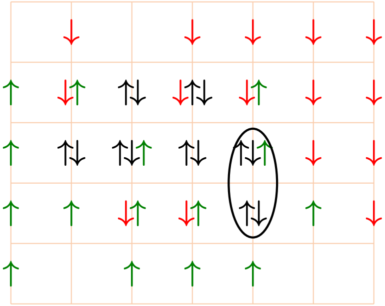

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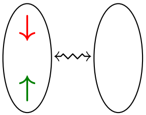

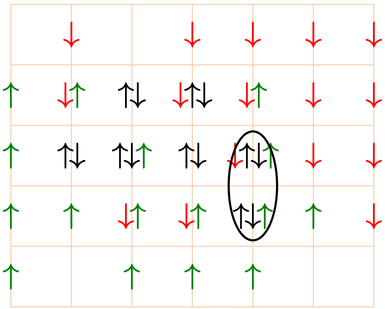

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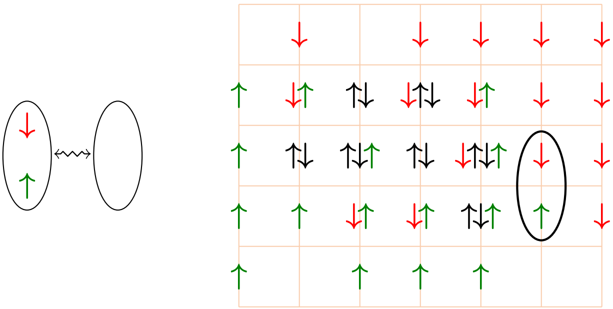

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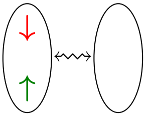

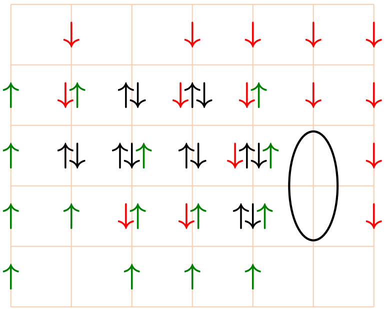

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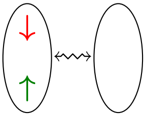

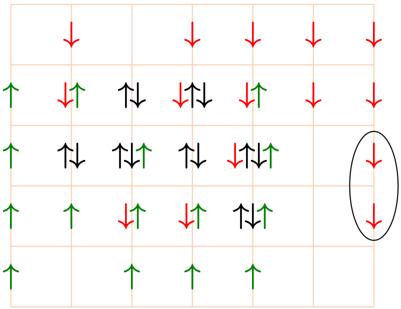

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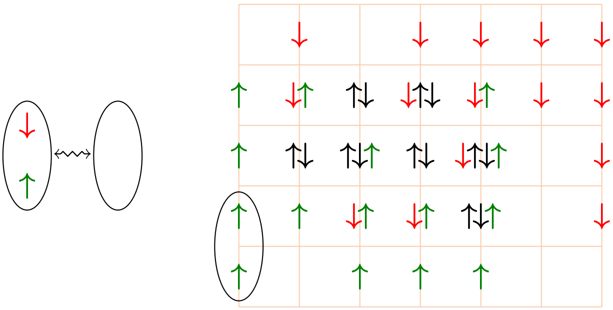

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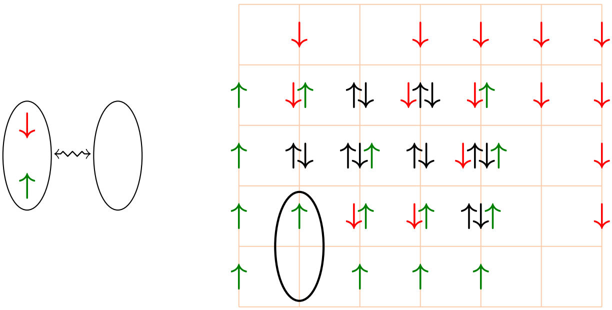

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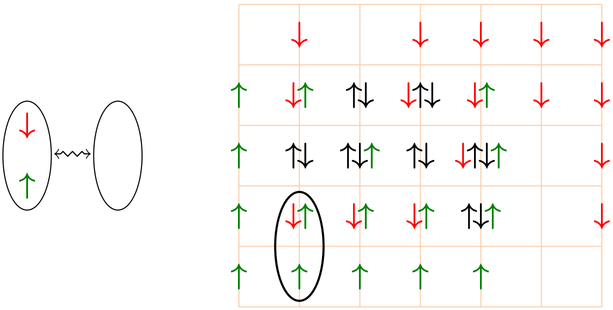

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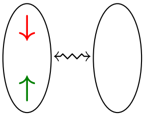

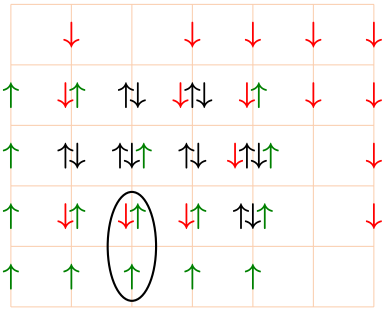

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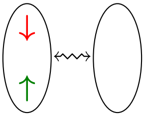

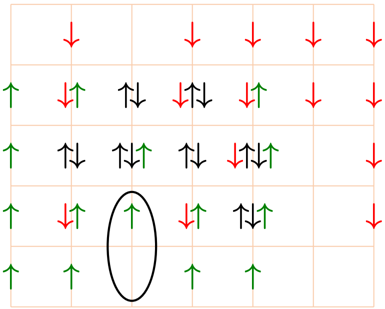

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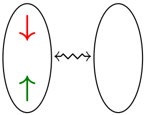

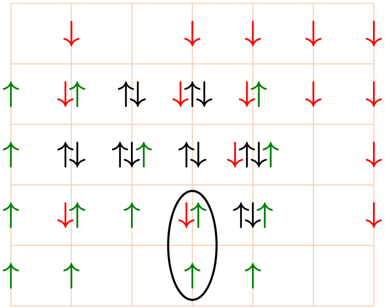

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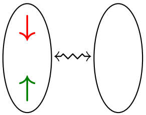

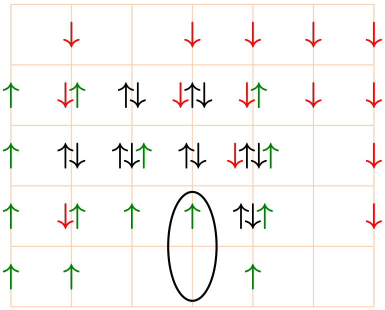

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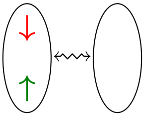

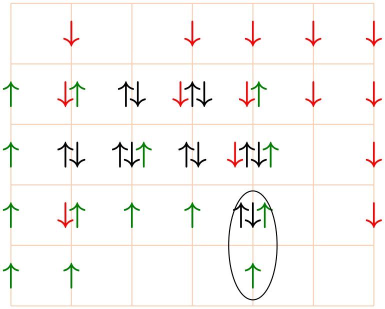

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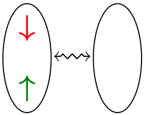

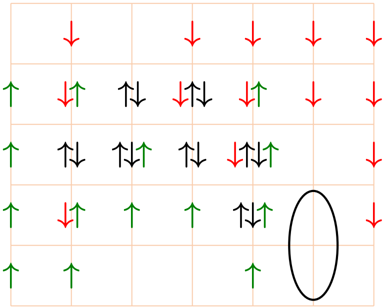

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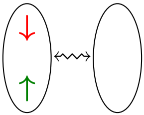

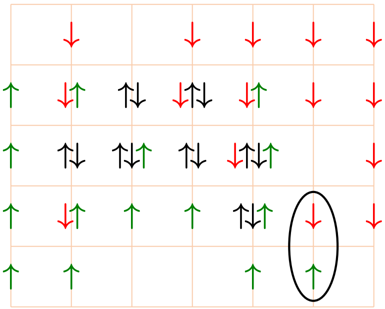

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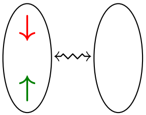

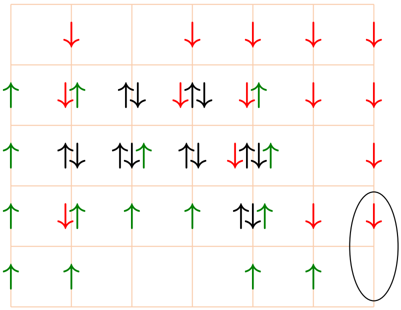

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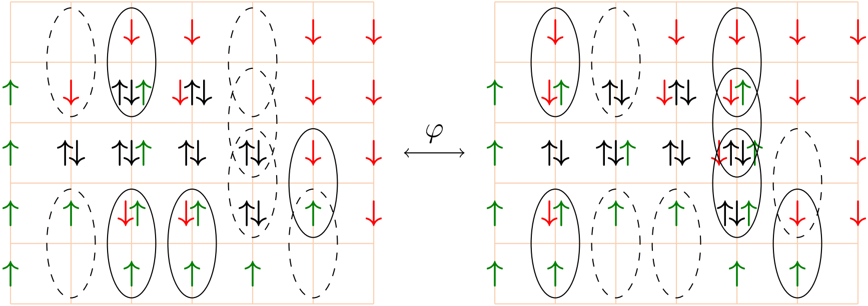

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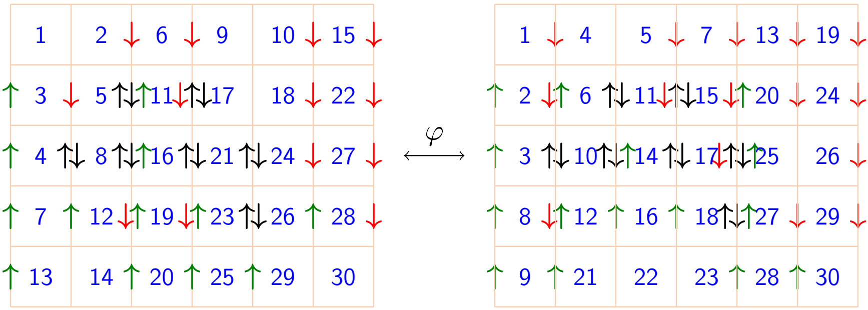

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Theorem

The map $\varphi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ *is an involution satisfying*

$$\mathrm{des}(T) + \mathrm{des}(\varphi(T)) = (k-1)(n+1).$$

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By switching the roles of leading and trailing arrows, we can similarly describe another bijection $\psi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$.

Theorem

The map $\psi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ *is an involution satisfying*

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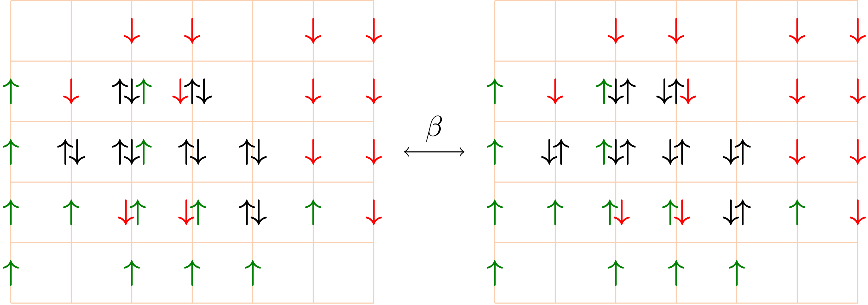
Consider a third involution $\beta : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ that simply reverses (i.e., reads from right to left) each block of arrows:



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1	2	6	9	10	15
3	5	11	17	18	22
4	8	16	21	24	27
7	12	19	23	26	28
13	14	20	25	29	30

$$\xleftrightarrow{\;\beta\;}$$

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The map $\beta : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ *is an involution satisfying*

$$\begin{aligned} \mathrm{des}(T) &= \mathrm{asc}(\beta(T)) + k - 1, \\ \mathrm{asc}(T) + k - 1 &= \mathrm{des}(\beta(T)). \end{aligned}$$

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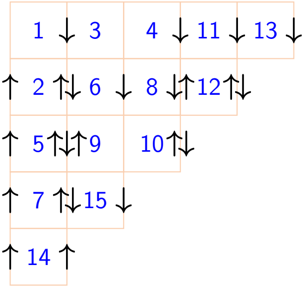
This gives a simpler proof of Sulanke's identity

$$|\{T \in \mathrm{SYT}(n^k) : \mathrm{asc}(T) = h\}| = |\{T \in \mathrm{SYT}(n^k) : \mathrm{des}(T) = h + k - 1\}|.$$

The three bijections are related by $\psi = \beta \circ \varphi \circ \beta$

Other shapes

Arrow encodings can be defined for arbitrary shapes:



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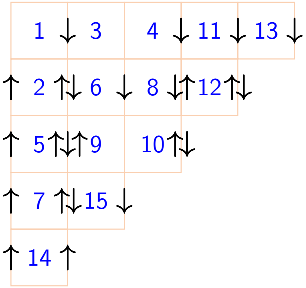

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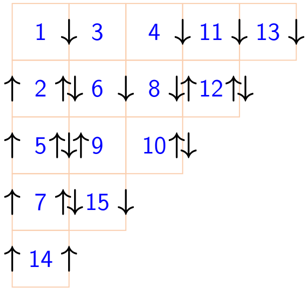

However, in general, the maps φ, ψ and β may produce invalid arrow arrays.

One exception is for the shape $\lambda = (n, n-1, \ldots, n-k+1)$, called a *truncated staircase*.

The bijection ψ for truncated staircases

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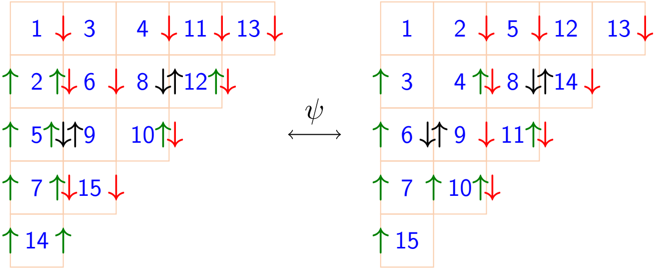

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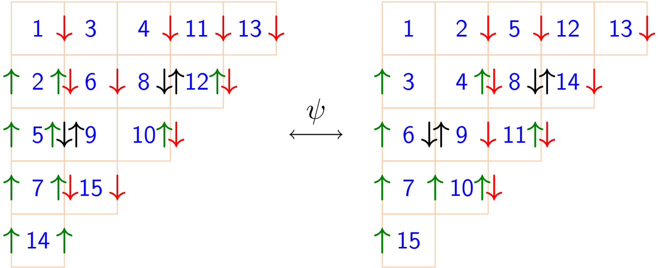

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The map $\psi : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$ *is an involution satisfying*

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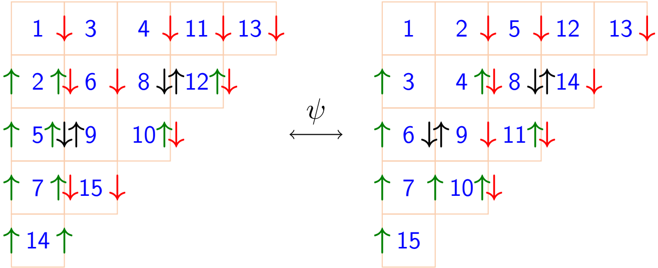

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In contrast, the distribution of des on $\mathrm{SYT}(\lambda)$ is **not** symmetric.

Another symmetry of the generalized Narayana numbers

Corollary (Sulanke '04)

$$N(k, n, h) = N(n, k, h).$$

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So we want a bijection that takes asc to hdes.

(For $k=2$, these statistics correspond to valleys and high peaks on Dyck paths, and several bijections are known.)

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Proposition

The map $\rho : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ defined by $\rho(T) = \beta(\varphi(\beta(T))')'$ is a bijection satisfying

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More generally...

Theorem

For **any** *shape λ, there is an bijection $\rho : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda)$ satisfying*
$$\mathrm{asc}(T) = \mathrm{hdes}(\rho(T)).$$

Open problems: major index

The major index of $T \in \mathsf{SYT}(\lambda)$ is defined as

$$\mathrm{maj}(T) = \sum_{i \text{ is a descent of } T} i.$$

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Problem

Describe a bijection $\Phi : \mathrm{SYT}(n^k) \to \mathrm{SYT}(n^k)$ *satisfying*

$$\mathrm{des}(T) + \mathrm{des}(\Phi(T)) = (k-1)(n+1),$$

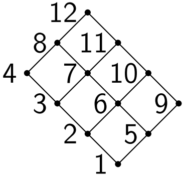
$$\mathrm{maj}(T) + \mathrm{maj}(\Phi(T)) = \frac{k(k-1)n(n+1)}{2}.$$

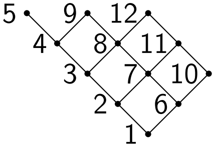
Open problems: generalization to other posets

Theorem (Stanley '72)

In any graded poset P, the distribution of the number of descents on linear extensions of P (with respect to any given natural labeling) is symmetric.

Our bijection ψ gives a bijective proof when $P = \mathrm{k} \times \mathrm{n}$ (rectangular shape) or comes from a truncated staircase shape:





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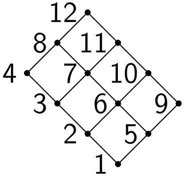

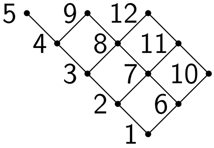

Problem

Give a bijective proof of Stanley's theorem for other graded posets P.
Interesting example: the product of three chains $\mathrm{k} \times \mathrm{n} \times \mathrm{m}$.