Allowed patterns of β-shifts

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Allowed patterns of a map

Let X be a linearly ordered set, $f : X \to X$. For each $w \in X$ and $n \geq 1$, consider the sequence

$$w, f(w), f^2(w), \dots, f^{n-1}(w)$$

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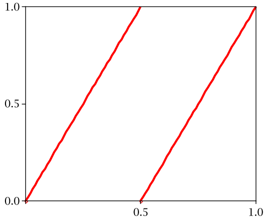
If there are no repetitions, the relative order of the entries determines a permutation, called an allowed (or realized) pattern of f.

Example

Let $f : [0,1] \to [0,1]$ be defined by $f(x) = \{2x\}$ (fractional part)



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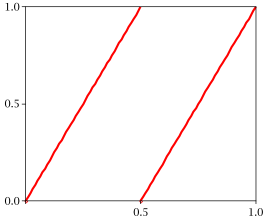

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$$0.39 \overset{f}{\mapsto} 0.78 \overset{f}{\mapsto} 0.56 \overset{f}{\mapsto} 0.12$$

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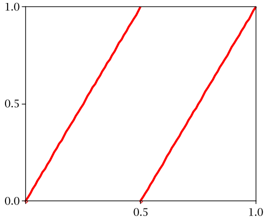

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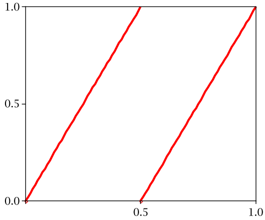

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We say that w induces 2431.

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$X = \{0,1\}^{\mathbb{N}}$ infinite binary words, ordered lexicographically

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But then $1 w_4 w_5 \dots \geq_{lex} w_4 w_5 \dots \longrightarrow$ contradiction!

Allowed and forbidden patterns

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Describing the set of allowed patterns of a given map is a difficult problem in general.

Maps on a one-dimensional interval

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Theorem (Bandt-Keller-Pompe '02)

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Provides a combinatorial way to compute the topological entropy, which is a measure of the complexity of the dynamical system.

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Shift maps

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Shift on N letters (for $N \geq 2$):

$$\begin{array}{rccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

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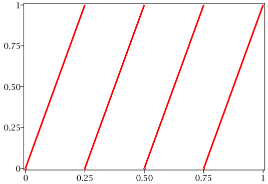
Thinking of words as expansions in base N of numbers in $[0, 1)$, Σ_N is "equivalent" to the sawtooth map

$$\begin{array}{rccc} M_N : & [0,1) & \rightarrow & [0,1) \\ & x & \mapsto & \{Nx\} \end{array}$$

(fractional part)



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Example

The shortest forbidden patterns of Σ_4 are

$$615243, 324156, 342516, 162534, 453621, 435261.$$

The minimum # of letters needed to realize a pattern

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An example of what $\hat{\pi}$ is:

$\pi = 892364157 \rightsquigarrow (8,9,2,3,6,4,1,5,7) \rightsquigarrow 536174892 \rightsquigarrow 536174\,92 = \hat{\pi}$

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Can deduce an ugly formula for $|\text{Allow}_n(\Sigma_N)|$, for given n and N.

Why β-shifts?

- Natural generalization of shifts.
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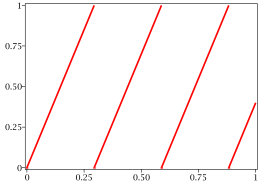
- Natural generalization of shifts.
- Good prototypes of chaotic dynamical systems.
- Originated in the study of expansions of real numbers in an arbitrary real base $\beta > 1$.
- Widely studied in the literature from different perspectives: measure theory, computability theory, number theory, etc.

β-shifts

For a real number $\beta > 1$, let M_β be the β-sawtooth map

$$M_\beta : \begin{array}{ccc} [0,1) & \longrightarrow & [0,1) \\ x & \longmapsto & \{\beta x\} \end{array}$$



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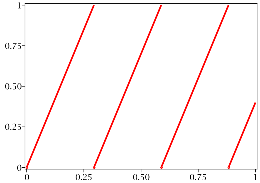

We'd like to define the β-shift as

$$\Sigma_\beta : \begin{array}{ccc} W(\beta) & \longrightarrow & W(\beta) \\ w_1\, w_2\, w_3 \ldots & \mapsto & w_2\, w_3\, w_4 \ldots \end{array}$$

for some set $W(\beta)$.

The domain of Σ_β

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with $w_0 = \lfloor x \rfloor, \quad w_1 = \lfloor \beta\{x\} \rfloor, \quad w_2 = \lfloor \beta\{\beta\{x\}\} \rfloor, \quad \dots$

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Example: $\quad 0.7 = \frac{2}{\pi} + \frac{0}{\pi^2} + \frac{1}{\pi^3} + \frac{3}{\pi^4} + \cdots$

The domain of Σ_β

Theorem (Parry '60)

Let

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots$$

be the β-expansion of β. Then

$$W(\beta) = \{w \ : \ w_k w_{k+1} w_{k+2} \ldots <_{lex} a_0 a_1 a_2 \ldots \ \text{for all } k \geq 1\}.$$

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Example

For $\beta = 1 + \sqrt{2}$, $\quad \beta = 2 + \frac{1}{\beta}$, so $\quad a_0 a_1 a_2 \ldots = 210^\infty$,

$W(\beta)$ = words over $\{0, 1, 2\}$ where every 2 is followed by a 0.

The shift-complexity of a permutation

Proposition. If $1 < \beta \leq \beta'$, then

- $W(\beta) \subseteq W(\beta')$,
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Definition (*shift-complexity*)

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Easy fact: $N(\pi) = \lfloor B(\pi) \rfloor + 1.$

Our goal is to be able to determine $B(\pi)$ for an arbitrary π.

From permutations to words

For a word w, let $b(w) = \inf\{\beta : w \in W(\beta)\}$. Then

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Proposition. There is an "easy" way to find $b(w)$ for any given w.

To compute $B(\pi)$, we'll find a word w inducing π such that $b(w)$ is as small as possible.

Example

Goal: find a word w inducing π such that $b(w)$ is small (in particular, want w not to have big letters).

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$\pi =$	8	9	3	1	4	6	2	7	5
$w =$				0			0		

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\end{array}
$$

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$\pi =$	8	9	3	1	4	6	2	7	5
$w =$	2		1	0	1	2	0	2	?

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Proposition: The entries $w_1 w_2 \dots w_{n-1}$ are forced (if minimizing # letters), and can be completed into a word w that induces π.

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$$\begin{array}{rcccccccccc} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 & \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 3^{\infty} \end{array}$$ (one possibility)

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Here, letting $w^{(m)} = 2310(1202)^m20^\infty$, we have

$$B(\pi) = \lim_{m\to\infty} b(w^{(m)}).$$

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Computation of $B(\pi)$ in general

Given a finite word $u_1 u_2 \dots u_r$, let

$$p_{u_1 u_2 \dots u_r}(\beta) = \beta^r - u_1 \beta^{r-1} - u_2 \beta^{r-2} - \dots - u_r.$$

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Theorem

For $\pi \in \mathcal{S}_n$, let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, $k = \pi^{-1}(c-1)$, and let $w_1 w_2 \dots w_{n-1}$ be the forced entries for w. Let

$$P_\pi(\beta) = \begin{cases} p_{w_\ell w_{\ell+1} \dots w_{n-1}}(\beta) & \text{if } c = 1, \\ p_{w_\ell w_{\ell+1} \dots w_{n-1} w_k w_{k+1} \dots w_{\ell-1}}(\beta) - 1 & \text{if } c \neq 1,\ \ell > k, \\ p_{w_\ell w_{\ell+1} \dots w_{n-1}}(\beta) - p_{w_\ell w_{\ell+1} \dots w_{k-1}}(\beta) & \text{if } c \neq 1,\ \ell < k. \end{cases}$$

Then $B(\pi)$ is the unique real root $\beta \geq 1$ of $P_\pi(\beta)$.

Example

$$\begin{array}{rccccccccc} \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & \ldots \end{array}$$

Example

$$\begin{array}{rccccccccc} & & \ell & & & k & & & & \\ \pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ w = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & \dots \end{array}$$

By the theorem, $B(\pi)$ is the root $\beta \geq 1$ of

$$P_\pi(\beta) = p_{3101202}(\beta) - p_{310}(\beta) = \beta^7 - 3\beta^6 - \beta^5 - 2\beta^3 + \beta^2 + \beta - 2$$

Example

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$$\Rightarrow \quad B(893146275) \approx 3.343618091$$

The shift-complexity of permutations of length 4

$\pi \in \mathcal{S}_4$	$B(\pi)$	$B(\pi)$ is a root of
$1234, 2341, 3412, 4123$	1	$\beta - 1$
$1342, 2413, 3124, 4231$	1.46557	$\beta^3 - \beta^2 - 1$
$1243, 1324, 2431, 3142, 4312$	$\frac{1+\sqrt{5}}{2} \approx 1.61803$	$\beta^2 - \beta - 1$
4213	1.80194	$\beta^3 - \beta^2 - 2\beta + 1$
$1432, 2143, 3214, 4321$	1.83929	$\beta^3 - \beta^2 - \beta - 1$
$2134, 3241$	2	$\beta - 2$
4132	2.24698	$\beta^3 - 2\beta^2 - \beta + 1$
$2314, 3421$	$1 + \sqrt{2} \approx 2.41421$	$\beta^2 - 2\beta - 1$
1423	$\frac{3+\sqrt{5}}{2} \approx 2.61803$	$\beta^2 - 3\beta + 1$

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