Allowed patterns of $\beta$-shifts

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Allowed patterns of a map

Let $X$ be a linearly ordered set, $f : X \rightarrow X$. For each $w \in X$ and $n \geq 1$, consider the sequence

$$w, f(w), f^2(w), \ldots, f^{n-1}(w)$$
Allowed patterns of a map

Let $X$ be a linearly ordered set, $f : X \to X$. For each $w \in X$ and $n \geq 1$, consider the sequence

$$w, f(w), f^2(w), \ldots, f^{n-1}(w)$$

If there are no repetitions, the relative order of the entries determines a permutation, called an allowed (or realized) pattern of $f$. 
Example

Let $f : [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = \{2x\}$ (fractional part)
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For $w = 0.39$ and $n = 4$, we get

\[ 0.39 \xrightarrow{f} 0.78 \xrightarrow{f} 0.56 \xrightarrow{f} 0.12 \]
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0.39 \quad \overset{f}{\rightarrow} \quad 0.78 \quad \overset{f}{\rightarrow} \quad 0.56 \quad \overset{f}{\rightarrow} \quad 0.12
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\( \Rightarrow \) 2431 is an allowed pattern of \( f \)
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For $w = 0.39$ and $n = 4$, we get

\begin{align*}
0.39 & \xrightarrow{f} 0.78 & & \xrightarrow{f} 0.56 & & \xrightarrow{f} 0.12 \\
\Rightarrow & & & \Rightarrow & & 2431 \text{ is an allowed pattern of } f
\end{align*}

We say that $w$ induces 2431.
“Another” example

\[ X = \{0, 1\}^\mathbb{N} \] infinite binary words, ordered lexicographically
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Let \( f : X \to X \) be the shift

\[ f(w_1w_2w_3\ldots) = w_2w_3w_4\ldots \]
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\[ w = 0110010111\ldots \]
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\[ f(w) = 110010111 \ldots \]
\[ f^2(w) = 10010111 \ldots \]
\[ f^3(w) = 0010111 \ldots \]
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\[ f(w_1 w_2 w_3 \ldots) = w_2 w_3 w_4 \ldots \]

\[
\begin{align*}
  w &= 0110010111\ldots & 2 \\
  f(w) &= 110010111\ldots & 4 \\
  f^2(w) &= 10010111\ldots & 3 \\
  f^3(w) &= 0010111\ldots & 1 \\
\end{align*}
\]

\[ \Rightarrow \quad 2431 \text{ is an allowed pattern of } f \]
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f^3(w) = 0010111\ldots
f^4(w) = 010111\ldots
f^5(w) = 10111\ldots
f^6(w) = 0111\ldots
\]

\[
\begin{align*}
3 & \quad f(w) \\
7 & \quad f^2(w) \\
5 & \quad f^3(w) \\
1 & \quad f^4(w) \\
2 & \quad f^5(w) \\
6 & \quad f^6(w)
\end{align*}
\]

lexicographic order of the shifted words
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On the other hand, \( 1423 \) is a forbidden pattern of \( f \):
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\[ \pi = \begin{array}{cccc}
1 & 4 & 2 & 3 \\
\end{array} \]

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On the other hand, 1423 is a forbidden pattern of $f$:

$$\pi = \begin{pmatrix} 1 & 4 & 2 & 3 \\ w = \begin{pmatrix} 0 & w_2 & w_3 & w_4 & w_5 & \ldots \end{pmatrix} \end{pmatrix}$$
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\[ \pi = 1 \quad 4 \quad 2 \quad 3 \]
\[ w = 0 \quad w_2 \quad 0? \quad w_4 \quad w_5 \ldots \]

\[ w_2 w_3 w_4 \ldots >_{\text{lex}} w_4 w_5 w_6 \ldots \Rightarrow 0w_2 w_3 w_4 \ldots >_{\text{lex}} 0w_4 w_5 w_6 \ldots \]
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\end{array} \]

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\[ w_2w_3w_4\ldots >_{\text{lex}} w_4w_5w_6\ldots \Rightarrow 0w_2w_3w_4\ldots >_{\text{lex}} 0w_4w_5w_6\ldots \]

But then \( 1w_4w_5\ldots \geq_{\text{lex}} w_4w_5\ldots \quad \rightarrow \quad \text{contradiction!} \]
Allowed and forbidden patterns

\[ \text{Allow}_n(f) = \text{set of allowed patterns of } f \text{ of length } n. \]
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**Fact:** \[ \text{Allow}(f) = \bigcup_{n \geq 0} \text{Allow}_n(f) \] is closed under consecutive pattern containment.

E.g., if 4156273 \( \in \) Allow(\( f \)), then 2314 \( \in \) Allow(\( f \)).
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E.g., if \(4156273 \in \text{Allow}(f)\), then \(2314 \in \text{Allow}(f)\).

Describing the set of allowed patterns of a given map is a difficult problem in general.
Maps on a one-dimensional interval

Let $I \subseteq \mathbb{R}$ be a closed interval.

**Theorem (Bandt-Keller-Pompe ’02)**

Let $f : I \rightarrow I$ be a piecewise monotone map. Then

- $f$ has forbidden patterns,
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- $\lim_{n \to \infty} |\text{Allow}_n(f)|^{1/n}$ exists, and its logarithm equals the **topological entropy** of $f$. 
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1. $f$ has forbidden patterns,
2. $\lim_{n \to \infty} |\text{Allow}_n(f)|^{1/n}$ exists, and its logarithm equals the topological entropy of $f$.

Provides a combinatorial way to compute the topological entropy, which is a measure of the complexity of the dynamical system.
Some (mostly open) questions

- How are properties of $\text{Allow}(f)$ related to properties of $f$?
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  - when is the set of minimal forbidden patterns of $f$ finite?
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- Enumerate or characterize $\text{Allow}(f)$ for some families of maps.
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- What sets of permutations can be $\text{Allow}(f)$ for some $f$?
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- Use this to design \textit{tests} to distinguish random sequences from deterministic ones.
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- Use this to design tests to distinguish random sequences from deterministic ones.
Shift maps

$$\mathcal{W}_N = \{0, 1, \ldots, N-1\}^\mathbb{N}$$

infinite words on $N$ letters, ordered lexicographically
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infinite words on \( N \) letters, ordered lexicographically

**Shift** on \( N \) letters (for \( N \geq 2 \)):

\[ \Sigma_N : \quad \mathcal{W}_N \quad \rightarrow \quad \mathcal{W}_N \]

\[ w_1 w_2 w_3 \ldots \quad \mapsto \quad w_2 w_3 w_4 \ldots \]
Shift maps

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Shift on \( N \) letters (for \( N \geq 2 \)):

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Thinking of words as expansions in base \( N \) of numbers in \([0, 1)\), \( \Sigma_N \)
is “equivalent” to the sawtooth map

\[ M_N : \quad [0, 1) \quad \rightarrow \quad [0, 1) \]

\[ x \quad \longmapsto \quad \{Nx\} \]

(fractional part)
Forbidden patterns of shifts

**Theorem**
\[ \Sigma_N \text{ has no forbidden patterns of length } N + 1 \text{ or shorter, but it has forbidden patterns of length } N + 2. \]
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*In fact, \( \Sigma_N \) has exactly 6 forbidden patterns of length \( N + 2 \).*
Forbidden patterns of shifts

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\( \Sigma_N \) has no forbidden patterns of length \( N + 1 \) or shorter, but it has forbidden patterns of length \( N + 2 \).

In fact, \( \Sigma_N \) has exactly 6 forbidden patterns of length \( N + 2 \).

**Example**
The shortest forbidden patterns of \( \Sigma_4 \) are

\[
615243, 324156, 342516, 162534, 453621, 435261.
\]
The minimum $\#$ of letters needed to realize a pattern

For $\pi \in S_n$, let $N(\pi) = \min\{N : \pi \in \text{Allow}(\Sigma_N)\}$. 
The minimum \# of letters needed to realize a pattern

For \( \pi \in S_n \), let \( N(\pi) = \min\{ N : \pi \in \text{Allow}(\Sigma_N) \} \).

**Theorem:** \( N(\pi) = 1 + \text{des}(\hat{\pi}) + \epsilon(\hat{\pi}) \).

\( 0 \) or \( 1 \)
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0 or 1

An example of what \( \hat{\pi} \) is:

\( \pi = 892364157 \leadsto (8,9,2,3,6,4,1,5,7) \leadsto 536174892 \leadsto 53617492 = \hat{\pi} \)
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$\text{des}(\hat{\pi}) = \text{des}(53617492) = 4$
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\( N(892364157) = 1 + 4 + 0 = 5 \)
The minimum \# of letters needed to realize a pattern

For \( \pi \in S_n \), let \( N(\pi) = \min\{N : \pi \in \text{Allow}(\Sigma_N)\} \).

**Theorem:** \( N(\pi) = 1 + \text{des}(\hat{\pi}) + \begin{cases} 0 & \text{or} \ 1 \end{cases} \sum_{\pi} \in S_n \).

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\( N(892364157) = 1 + 4 + 0 = 5 \)

Can deduce an ugly formula for \( |\text{Allow}_n(\Sigma_N)| \), for given \( n \) and \( N \).
Why $\beta$-shifts?

- Natural generalization of shifts.
- Good prototypes of chaotic dynamical systems.
Why $\beta$-shifts?

- Natural generalization of shifts.
- Good prototypes of chaotic dynamical systems.
- Originated in the study of expansions of real numbers in an arbitrary real base $\beta > 1$.
- Widely studied in the literature from different perspectives: measure theory, computability theory, number theory, etc.
β-shifts

For a real number $\beta > 1$, let $M_\beta$ be the $\beta$-sawtooth map

$$M_\beta : [0, 1) \rightarrow [0, 1)$$

$$x \mapsto \{ \beta x \}$$
\(\beta\)-shifts

For a real number \(\beta > 1\), let \(M_\beta\) be the \(\beta\)-sawtooth map

\[
M_\beta : \ [0, 1) \rightarrow [0, 1) \\
x \mapsto \ \{\beta x\}
\]

We’d like to define the \(\beta\)-shift as

\[
\Sigma_\beta : \ W(\beta) \quad \rightarrow \quad W(\beta) \\
w_1 w_2 w_3 \ldots \quad \mapsto \quad w_2 w_3 w_4 \ldots
\]

for some set \(W(\beta)\).
The domain of $\Sigma_\beta$

$$\Sigma_\beta : \quad W(\beta) \quad \mapsto \quad W(\beta)$$

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The domain of $\Sigma_\beta$

$\Sigma_\beta : \ W(\beta) \rightarrow W(\beta)$

$w_1 w_2 w_3 \ldots \mapsto w_2 w_3 w_4 \ldots$

$W(\beta)$ is the set of words given by expansions in base $\beta$ of numbers $x \in [0, 1)$,
The domain of $\Sigma_{\beta}$

$$\Sigma_{\beta} : \quad W(\beta) \quad \mapsto \quad W(\beta)$$

$w_1 w_2 w_3 \ldots \quad \mapsto \quad w_2 w_3 w_4 \ldots$

$W(\beta)$ is the set of words given by expansions in base $\beta$ of numbers $x \in [0, 1)$,

$$x = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots,$$

with $w_0 = \lfloor x \rfloor$, $w_1 = \lfloor \beta \{x\} \rfloor$, $w_2 = \lfloor \beta \{\beta \{x\}\} \rfloor$, \ldots
The domain of $\Sigma_\beta$

$$\Sigma_\beta : \quad W(\beta) \quad \longrightarrow \quad W(\beta)$$

$$w_1 w_2 w_3 \ldots \quad \mapsto \quad w_2 w_3 w_4 \ldots$$

$W(\beta)$ is the set of words given by expansions in base $\beta$ of numbers $x \in [0, 1)$,

$$x = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots,$$

with $w_0 = \lfloor x \rfloor, \quad w_1 = \lfloor \beta \{x\} \rfloor, \quad w_2 = \lfloor \beta \{\beta \{x\}\} \rfloor, \quad \ldots$

Example: $\quad 0.7 = \frac{2}{\pi} + \frac{0}{\pi^2} + \frac{1}{\pi^3} + \frac{3}{\pi^4} + \cdots$
The domain of $\Sigma_\beta$

**Theorem (Parry ’60)**

Let

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots$$

be the $\beta$-expansion of $\beta$. Then

$$W(\beta) = \{w : w_k w_{k+1} w_{k+2} \cdots <_{\text{lex}} a_0 a_1 a_2 \cdots \text{ for all } k \geq 1\}.$$
The domain of $\Sigma_\beta$

**Theorem (Parry ’60)**

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**Example**

For $\beta = 1 + \sqrt{2}$, $\beta = 2 + \frac{1}{\beta}$, so $a_0 a_1 a_2 \cdots = 210\infty$, $W(\beta) =$ words over $\{0, 1, 2\}$ where every 2 is followed by a 0.
The shift-complexity of a permutation

Proposition. If $1 < \beta \leq \beta'$, then

- $W(\beta) \subseteq W(\beta')$, 
- $\text{Allow}(\Sigma_\beta) \subseteq \text{Allow}(\Sigma_{\beta'})$. 
The shift-complexity of a permutation

**Proposition.** If $1 < \beta \leq \beta'$, then

- $W(\beta) \subseteq W(\beta')$,
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**Definition (shift-complexity)**

For any permutation $\pi$, let

$$B(\pi) = \inf \{ \beta : \pi \in \text{Allow}(\Sigma_{\beta}) \}.$$
The shift-complexity of a permutation

Proposition. If $1 < \beta \leq \beta'$, then

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Definition (shift-complexity)

For any permutation $\pi$, let

$$B(\pi) = \inf\{\beta : \pi \in \text{Allow}(\Sigma_\beta)\}.$$ 

Easy fact: $N(\pi) = \lceil B(\pi) \rceil + 1$.

Our goal is to be able to determine $B(\pi)$ for an arbitrary $\pi$. 
From permutations to words

For a word $w$, let $b(w) = \inf\{\beta : w \in \mathcal{W}(\beta)\}$. Then

$$B(\pi) = \inf\{b(w) : w \text{ induces } \pi\}.$$
From permutations to words

For a word $w$, let $b(w) = \inf\{\beta : w \in W(\beta)\}$. Then

$$B(\pi) = \inf\{b(w) : w \text{ induces } \pi\}.$$ 

Example: $b(30003020^\infty) \approx 3.1958$, $b(212310^\infty) \approx 3.3028$. 
From permutations to words

For a word $w$, let $b(w) = \inf\{\beta : w \in W(\beta)\}$. Then

$$B(\pi) = \inf\{b(w) : w \text{ induces } \pi\}.$$  

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Proposition. There is an “easy” way to find $b(w)$ for any given $w$. 

From permutations to words

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$$B(\pi) = \inf\{b(w) : w \text{ induces } \pi\}.$$

Example: $b(30003020^\infty) \approx 3.1958$, $b(212310^\infty) \approx 3.3028$.

Proposition. There is an “easy” way to find $b(w)$ for any given $w$.

To compute $B(\pi)$, we’ll find a word $w$ inducing $\pi$ such that $b(w)$ is as small as possible.
Example

**Goal:** find a word $w$ inducing $\pi$ such that $b(w)$ is small (in particular, want $w$ not to have big letters).
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\[
\pi = \begin{pmatrix}
8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
\end{pmatrix}
\]

\[
w = 0
\]
Example

Goal: find a word $w$ inducing $\pi$ such that $b(w)$ is small (in particular, want $w$ not to have big letters).

$$\pi = \begin{array}{ccccccc} 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\ \end{array}$$

$$w = \begin{array}{c} 0 \\ 0 \end{array}$$
Example

**Goal:** find a word $w$ inducing $\pi$ such that $b(w)$ is small (in particular, want $w$ not to have big letters).

\[
\begin{align*}
\pi &= 8 \quad 9 \quad 3 \quad 1 \quad 4 \quad 6 \quad 2 \quad 7 \quad 5 \\
w &= 1 \quad 0 \quad 0
\end{align*}
\]
**Example**

**Goal:** find a word $w$ inducing $\pi$ such that $b(w)$ is small (in particular, want $w$ not to have big letters).

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\pi = \begin{array}{ccccccccc}
8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
\end{array}
\]

\[
w = \begin{array}{cccc}
1 & 0 & 1 & 0 \\
\end{array}
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w &= 2 \ 1 \ 0 \ 1 \ 2 \ 0 \ 2 \ ?
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Proposition: The entries \( w_1 w_2 \ldots w_{n-1} \) are forced (if minimizing \( \# \) letters), and can be completed into a word \( w \) that induces \( \pi \).
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$$\pi = 8 \ 9 \ 3 \ 1 \ 4 \ 6 \ 2 \ 7 \ 5$$

$$w = 2 \ 3 \ 1 \ 0 \ 1 \ 2 \ 0 \ 2 \ 1 \ 3^\infty \text{ (one possibility)}$$

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\[ w = 2 \ 3 \ 1 \ 0 \ 1 \ 2 \ 0 \ 2 \ 1 \ 2 \ 0 \ 2 \ 2 \ 0^\infty \quad \text{(smaller } b) \]

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$$\pi = \begin{array}{cccccccc}
8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
\end{array}$$

$$w = \begin{array}{cccccccc}
2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 3^\infty & \text{(one possibility)} \\
2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & 1 & 2 & 0 & 2 & 2 & 0^\infty & \text{(smaller } b) \\
\end{array}$$

Here, letting $w^{(m)} = 2310(1202)^m20^\infty$, we have

$$B(\pi) = \lim_{m \to \infty} b(w^{(m)}).$$

Proposition: The entries $w_1w_2 \ldots w_{n-1}$ are forced (if minimizing $\#$ letters), and can be completed into a word $w$ that induces $\pi$.

Now we choose the entries $w_nw_{n+1} \ldots$ in order to minimize $b(w)$. 
Computation of $B(\pi)$ in general

Given a finite word $u_1u_2\ldots u_r$, let

$$p_{u_1u_2\ldots u_r}(\beta) = \beta^r - u_1\beta^{r-1} - u_2\beta^{r-2} - \ldots - u_r.$$
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**Theorem**

For $\pi \in S_n$, let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, $k = \pi^{-1}(c - 1)$, and let $w_1 w_2 \ldots w_{n-1}$ be the forced entries for $w$. Let

$$P_{\pi}(\beta) = \begin{cases} 
p_{w_\ell w_{\ell+1} \ldots w_{n-1}}(\beta) & \text{if } c = 1, \\
p_{w_\ell w_{\ell+1} \ldots w_{n-1} w_k w_{k+1} \ldots w_{\ell-1}}(\beta) - 1 & \text{if } c \neq 1, \ell > k, \\
p_{w_\ell w_{\ell+1} \ldots w_{n-1}}(\beta) - p_{w_\ell w_{\ell+1} \ldots w_{k-1}}(\beta) & \text{if } c \neq 1, \ell < k. 
\end{cases}$$

Then $B(\pi)$ is the unique real root $\beta \geq 1$ of $P_{\pi}(\beta)$. 
Example

\( \pi = 8 \ 9 \ 3 \ 1 \ 4 \ 6 \ 2 \ 7 \ 5 \)

\( w = 2 \ 3 \ 1 \ 0 \ 1 \ 2 \ 0 \ 2 \ \ldots \)
Example

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$ = 8 9 3 1 4 6 2 7 5</td>
<td></td>
</tr>
<tr>
<td>$w$ = 2 3 1 0 1 2 0 2 ...</td>
<td></td>
</tr>
</tbody>
</table>

By the theorem, $B(\pi)$ is the root $\beta \geq 1$ of

$$P_\pi(\beta) = p_{3101202}(\beta) - p_{310}(\beta) = \beta^7 - 3\beta^6 - \beta^5 - 2\beta^3 + \beta^2 + \beta - 2$$
Example

\[
\begin{array}{cccccccc}
\ell & k \\
\pi = & 8 & 9 & 3 & 1 & 4 & 6 & 2 & 7 & 5 \\
\omega = & 2 & 3 & 1 & 0 & 1 & 2 & 0 & 2 & \ldots
\end{array}
\]

By the theorem, \( B(\pi) \) is the root \( \beta \geq 1 \) of

\[
P_{\pi}(\beta) = p_{3101202}(\beta) - p_{310}(\beta) = \beta^7 - 3\beta^6 - \beta^5 - 2\beta^3 + \beta^2 + \beta - 2
\]

\[
\Rightarrow \quad B(893146275) \approx 3.343618091
\]
The shift-complexity of permutations of length 4

<table>
<thead>
<tr>
<th>$\pi \in S_4$</th>
<th>$B(\pi)$</th>
<th>$B(\pi)$ is a root of</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234, 2341, 3412, 4123</td>
<td>1</td>
<td>$\beta - 1$</td>
</tr>
<tr>
<td>1342, 2413, 3124, 4231</td>
<td>1.46557</td>
<td>$\beta^3 - \beta^2 - 1$</td>
</tr>
<tr>
<td>1243, 1324, 2431, 3142, 4312</td>
<td>$\frac{1+\sqrt{5}}{2} \approx 1.61803$</td>
<td>$\beta^2 - \beta - 1$</td>
</tr>
<tr>
<td>4213</td>
<td>1.80194</td>
<td>$\beta^3 - \beta^2 - 2\beta + 1$</td>
</tr>
<tr>
<td>1432, 2143, 3214, 4321</td>
<td>1.83929</td>
<td>$\beta^3 - \beta^2 - \beta - 1$</td>
</tr>
<tr>
<td>2134, 3241</td>
<td>2</td>
<td>$\beta - 2$</td>
</tr>
<tr>
<td>4132</td>
<td>2.24698</td>
<td>$\beta^3 - 2\beta^2 - \beta + 1$</td>
</tr>
<tr>
<td>2314, 3421</td>
<td>$1 + \sqrt{2} \approx 2.41421$</td>
<td>$\beta^2 - 2\beta - 1$</td>
</tr>
<tr>
<td>1423</td>
<td>$\frac{3+\sqrt{5}}{2} \approx 2.61803$</td>
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TAKK