Cylindric growth diagrams, walks in simplices, and exclusion processes

Sergi Elizalde

Dartmouth College

Richard P. Stanley Seminar in Combinatorics
September 2025

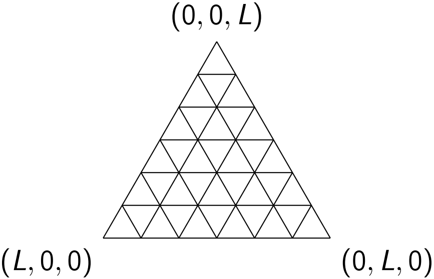
Overview

- Background
    - Walks in simplicial regions
    - Standard cylindric tableaux (SCT)
    - Exclusion processes on a cycle
- Connecting all three
- The cylindric Robinson–Schensted correspondence
- Cylindric growth diagrams
- Open problems

Walks in simplicial regions

For $d, L \geq 0$, consider the simplicial region

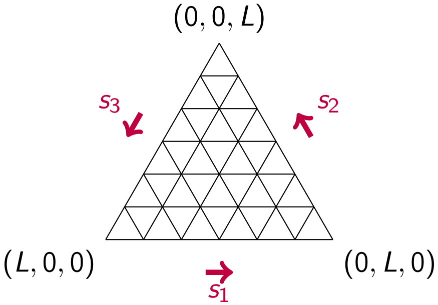
$$\Delta_{d,L} = \{(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d : x_1 + x_2 + \cdots + x_d = L\}.$$



Walks in simplicial regions

For $d, L \geq 0$, consider the simplicial region

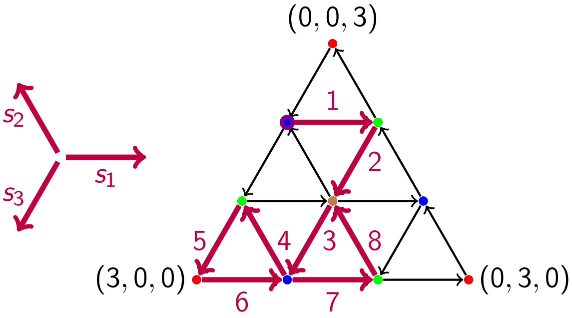
$$\Delta_{d,L} = \{(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d : x_1 + x_2 + \cdots + x_d = L\}.$$



For $1 \leq i \leq d$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 in position i, and let $s_i = e_{i+1} - e_i$, with the convention $e_{d+1} := e_1$.

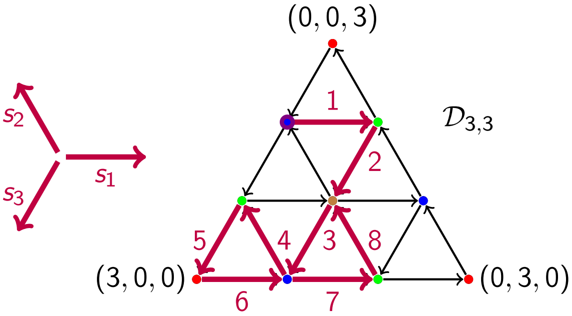
Walks in simplicial regions

Consider walks inside $\Delta_{d,L}$ with steps s_i for $1 \le i \le d$.



Walks in simplicial regions

Consider walks inside $\Delta_{d,L}$ with steps s_i for $1 \leq i \leq d$.



Denote by $\mathcal{D}_{d,L}$ the corresponding directed graph.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

For $d = 2$, these are one-dimensional walks, and their enumeration is relatively easy.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

For $d = 2$, these are one-dimensional walks, and their enumeration is relatively easy.

For $d = 3$, the problem was solved by Mortimer and Prellberg '15.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

For $d = 2$, these are one-dimensional walks, and their enumeration is relatively easy.

For $d = 3$, the problem was solved by Mortimer and Prellberg '15.

$M_{n,h} = \#$ Motzkin paths of length n and height at most h.

$M'_{n,h} = \#$ Motzkin paths of length n and height at most h that do not have any horizontal steps at height h.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

For $d = 2$, these are one-dimensional walks, and their enumeration is relatively easy.

For $d = 3$, the problem was solved by Mortimer and Prellberg '15.

$M_{n,h} = \#$ Motzkin paths of length n and height at most h.

$M'_{n,h} = \#$ Motzkin paths of length n and height at most h that do not have any horizontal steps at height h.

Theorem (Mortimer–Prellberg '15)

The number of n-step walks in $\mathcal{D}_{3,L}$ starting at $(L, 0, 0)$ equals

$$\begin{cases} M_{n,h} & \text{if } L = 2h+1, \\ M'_{n,h} & \text{if } L = 2h. \end{cases}$$

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

To prove this, Mortimer and Prellberg found a functional equation and used the kernel method to solve it.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

To prove this, Mortimer and Prellberg found a functional equation and used the kernel method to solve it.

A complicated bijective proof was later found by Courtiel, Elvey Price, and Marcovici in 2021.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

To prove this, Mortimer and Prellberg found a functional equation and used the kernel method to solve it.

A complicated bijective proof was later found by Courtiel, Elvey Price, and Marcovici in 2021.

In the same paper, they prove the following surprising result.

Theorem (Courtiel–Elvey Price–Marcovici '21)

For any $\mathrm{x} \in \Delta_{d,L}$*, there is a bijection between the set of n-step walks in* $\mathcal{D}_{d,L}$ **starting** *at* x *and the set of n-step walks in* $\mathcal{D}_{d,L}$ **ending** *at* x*.*

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

To prove this, Mortimer and Prellberg found a functional equation and used the kernel method to solve it.

A complicated bijective proof was later found by Courtiel, Elvey Price, and Marcovici in 2021.

In the same paper, they prove the following surprising result.

Theorem (Courtiel–Elvey Price–Marcovici '21)

For any $\mathrm{x} \in \Delta_{d,L}$, *there is a bijection between the set of n-step walks in* $\mathcal{D}_{d,L}$ **starting** *at* x *and the set of n-step walks in* $\mathcal{D}_{d,L}$ **ending** *at* x.

We will call this the Forward-Backward Theorem.

Enumeration of walks in $\mathcal{D}_{d,L}$ starting at a given point

To prove this, Mortimer and Prellberg found a functional equation and used the kernel method to solve it.

A complicated bijective proof was later found by Courtiel, Elvey Price, and Marcovici in 2021.

In the same paper, they prove the following surprising result.

Theorem (Courtiel–Elvey Price–Marcovici '21)

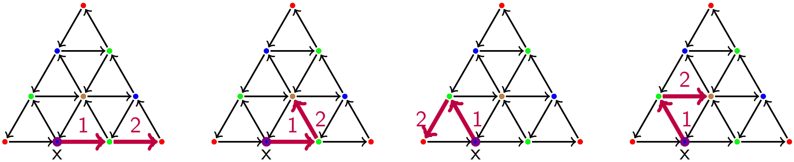
For any $\mathrm{x} \in \Delta_{d,L}$*, there is a bijection between the set of* n*-step walks in* $\mathcal{D}_{d,L}$ **starting** *at* x *and the set of* n*-step walks in* $\mathcal{D}_{d,L}$ **ending** *at* x*.*

We will call this the Forward-Backward Theorem.

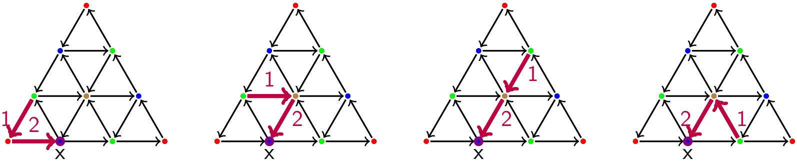
Their bijection repeatedly applies certain flips to adjacent steps and to the last step of the walk.

Small example

Starting at x:



Ending at x:



Standard cylindric tableaux

Standard cylindric tableaux

A **cylindric shape** of period (d, L) is a doubly infinite weakly decreasing sequence of integers, $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$, such that $\alpha_i = \alpha_{i+d} + L$ for all $i \in \mathbb{Z}$.

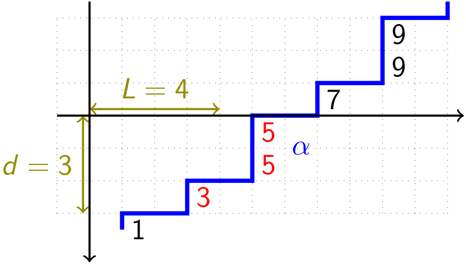
$\Lambda_{d,L}$= set of cylindric shapes of period (d, L).

Standard cylindric tableaux

A **cylindric shape** of period (d, L) is a doubly infinite weakly decreasing sequence of integers, $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$, such that $\alpha_i = \alpha_{i+d} + L$ for all $i \in \mathbb{Z}$.

$\Lambda_{d,L}$= set of cylindric shapes of period (d, L).

Ex: $\alpha = (\ldots, 9, 9, 7, 5, 5, 3, 1, 1, -1, \ldots) \in \Lambda_{3,4}$

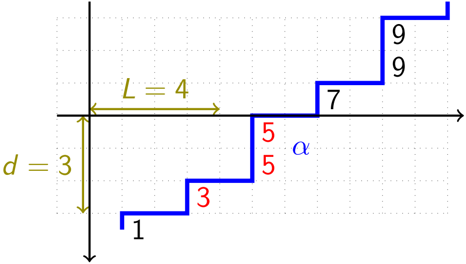


Standard cylindric tableaux

A **cylindric shape** of period (d, L) is a doubly infinite weakly decreasing sequence of integers, $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$, such that $\alpha_i = \alpha_{i+d} + L$ for all $i \in \mathbb{Z}$.

$\Lambda_{d,L}$= set of cylindric shapes of period (d, L).

Ex: $\alpha = (\dots, 9, 9, 7, 5, 5, 3, 1, 1, -1, \dots) \in \Lambda_{3,4}$



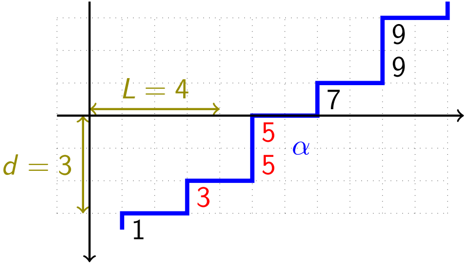
Uniquely determined by $\alpha_1, \alpha_2, \dots, \alpha_d$.

Standard cylindric tableaux

A **cylindric shape** of period (d, L) is a doubly infinite weakly decreasing sequence of integers, $\alpha = (\alpha_i)_{i \in \mathbb{Z}}$, such that $\alpha_i = \alpha_{i+d} + L$ for all $i \in \mathbb{Z}$.

$\Lambda_{d,L}$= set of cylindric shapes of period (d, L).

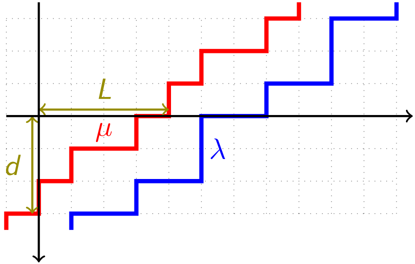
Ex: $\alpha = (\dots, 9, 9, 7, 5, 5, 3, 1, 1, -1, \dots) = [5, 5, 3] \in \Lambda_{3,4}$



Uniquely determined by $\alpha_1, \alpha_2, \dots, \alpha_d$. Write $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$.

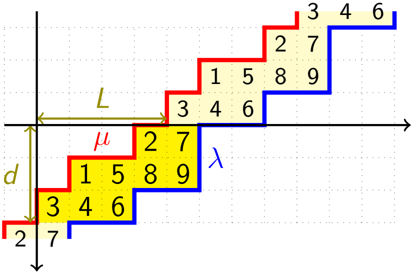
Standard cylindric tableaux

Let $\lambda, \mu \in \Lambda_{d,L}$ such that $\mu_i \leq \lambda_i$ for all i.



Standard cylindric tableaux

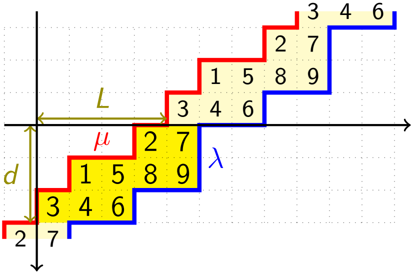
Let $\lambda, \mu \in \Lambda_{d,L}$ such that $\mu_i \leq \lambda_i$ for all i.



A **standard cylindric tableau** of shape λ/μ is a filling of the cells in between with $1, 2, \ldots, n$ preserving the periodicity, and so that entries increase along rows (from left to right) and columns (from top to bottom).

Standard cylindric tableaux

Let $\lambda, \mu \in \Lambda_{d,L}$ such that $\mu_i \leq \lambda_i$ for all i.



A **standard cylindric tableau** of shape λ/μ is a filling of the cells in between with $1, 2, \ldots, n$ preserving the periodicity, and so that entries increase along rows (from left to right) and columns (from top to bottom).

One can think of them as skew SYT with additional restrictions.

History and notation

Cylindric tableaux are a special case of *cylindric partitions*, introduced by Gessel and Krattenthaler '97. They were studied by Postnikov '05 in connection to Gromov–Witten invariants.

History and notation

Cylindric tableaux are a special case of *cylindric partitions*, introduced by Gessel and Krattenthaler '97. They were studied by Postnikov '05 in connection to Gromov–Witten invariants.

Let $\lambda, \mu \in \Lambda_{d,L}$ such that $\mu_i \leq \lambda_i$ for all i.

$\mathrm{SCT}_{d,L}(\lambda/\mu) =$ set of standard cylindric tableaux of shape λ/μ

History and notation

Cylindric tableaux are a special case of *cylindric partitions*, introduced by Gessel and Krattenthaler '97. They were studied by Postnikov '05 in connection to Gromov–Witten invariants.

Let $\lambda, \mu \in \Lambda_{d,L}$ such that $\mu_i \leq \lambda_i$ for all i.

$\mathrm{SCT}_{d,L}(\lambda/\mu) =$ set of standard cylindric tableaux of shape λ/μ

$\mathrm{SCT}^{n}_{d,L}(\cdot/\mu) =$ set of standard cylindric tableaux with n cells and inner shape μ

History and notation

Cylindric tableaux are a special case of *cylindric partitions*, introduced by Gessel and Krattenthaler '97. They were studied by Postnikov '05 in connection to Gromov–Witten invariants.

Let $\lambda, \mu \in \Lambda_{d,L}$ such that $\mu_i \leq \lambda_i$ for all i.

$\mathrm{SCT}_{d,L}(\lambda/\mu)$ = set of standard cylindric tableaux of shape λ/μ

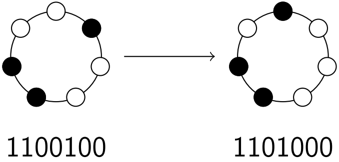
$\mathrm{SCT}^n_{d,L}(\cdot/\mu)$ = set of standard cylindric tableaux with n cells and inner shape μ

$\mathrm{SCT}^n_{d,L}(\lambda/\cdot)$ = set of standard cylindric tableaux with n cells and outer shape λ

The totally asymmetric simple exclusion process (TASEP)

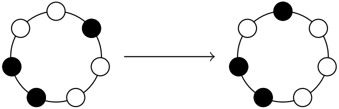
The totally asymmetric simple exclusion process (TASEP)

States of the **TASEP** on the cycle $\mathbb{Z}_N$ are encoded by binary words with d ones (representing the positions of particles) and $N - d$ zeros. At each time step, a particle can jump to the next site in counterclockwise direction if this site is empty.



The totally asymmetric simple exclusion process (TASEP)

States of the **TASEP** on the cycle $\mathbb{Z}_N$ are encoded by binary words with d ones (representing the positions of particles) and $N - d$ zeros. At each time step, a particle can jump to the next site in counterclockwise direction if this site is empty.

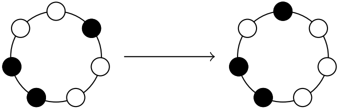


1100100 1101000

Typically, one associates transition probabilities to these particle jumps to define a Markov chain (Liggett '99, Ferrari–Martin '07).

The totally asymmetric simple exclusion process (TASEP)

States of the **TASEP** on the cycle $\mathbb{Z}_N$ are encoded by binary words with d ones (representing the positions of particles) and $N-d$ zeros. At each time step, a particle can jump to the next site in counterclockwise direction if this site is empty.

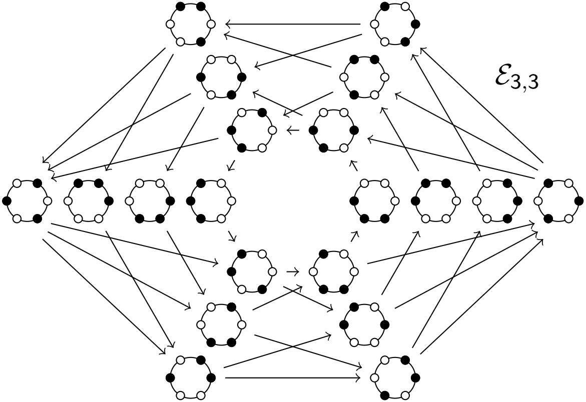


1100100 1101000

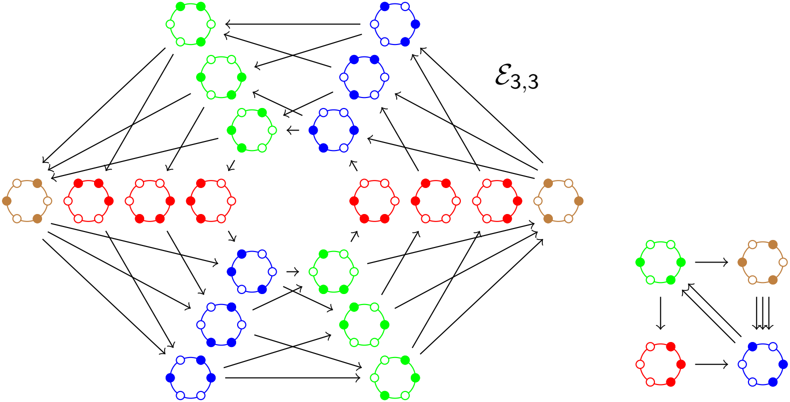
Typically, one associates transition probabilities to these particle jumps to define a Markov chain (Liggett '99, Ferrari–Martin '07).

Here we consider the underlying directed graph $\mathcal{E}_{d,N-d}$ whose vertices are the states, and whose edges correspond to valid jumps of a particle.

The totally asymmetric simple exclusion process (TASEP)



The totally asymmetric simple exclusion process (TASEP)



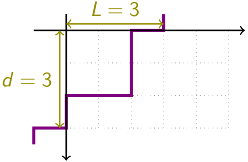
One can also consider the directed multigraph obtained as the quotient of $\mathcal{E}_{d,N-d}$ under cyclic rotations.

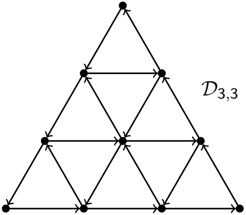
Connecting all three

$$\begin{matrix}\text{cylindric shape} \\ \alpha \in \Lambda_{d,L}\end{matrix} \rightsquigarrow \begin{matrix}\text{vertex in simplicial region} \\ \mathsf{x} = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L} \\ x_i = \alpha_{i-1} - \alpha_i \quad \forall i\end{matrix} \rightsquigarrow \begin{matrix}\text{state in TASEP} \\ u = 0^{x_1} 1 0^{x_2} 1 \ldots 0^{x_d} 1\end{matrix}$$

Example

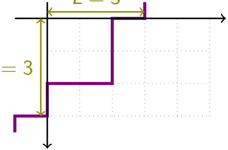
$\alpha = [2, 2, 0] = (\dots, 5, 5, 3, 2, 2, 0, \dots)$



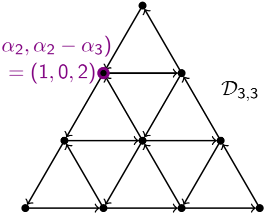


Example

$\alpha = [2,2,0] = (\dots, 5, 5, 3, 2, 2, 0, \dots)$

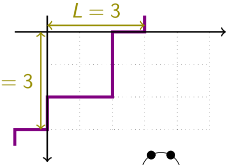


$\mathsf{x} = (\alpha_0 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3)$
$= (1, 0, 2)$

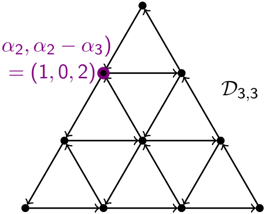


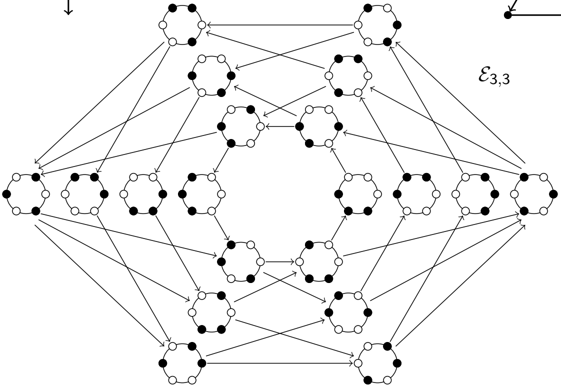
Example

$\alpha = [2,2,0] = (\dots,5,5,3,2,2,0,\dots)$

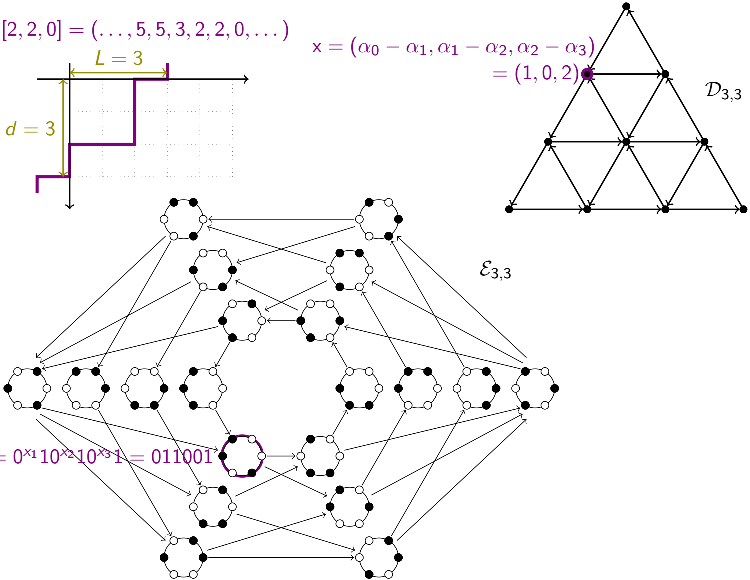


$\mathrm{x} = (\alpha_0 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3)$
$= (1,0,2)$





Example



$\alpha = [2, 2, 0] = (\dots, 5, 5, 3, 2, 2, 0, \dots)$

$\mathsf{x} = (\alpha_0 - \alpha_1, \alpha_1 - \alpha_2, \alpha_2 - \alpha_3)$
$= (1, 0, 2)$

$u = 0^{x_1}10^{x_2}10^{x_3}1 = 011001$

Connecting all three

Theorem

Let $\alpha \in \Lambda_{d,L}$*, let* $\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1}10^{x_2}1\ldots0^{x_d}1$*.*

Connecting all three

Theorem

Let $\alpha \in \Lambda_{d,L}$*, let* $\mathrm{x} = (x_1, x_2, \dots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1}10^{x_2}1\dots0^{x_d}1$*.*

There are bijections between the following:

- ⓐ *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*

Connecting all three

Theorem

Let $\alpha \in \Lambda_{d,L}$*, let* $\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1} 1 0^{x_2} 1 \ldots 0^{x_d} 1$*.*

There are bijections between the following:

- (a) *The set* $\mathrm{SCT}^{n}_{d,L}(\cdot/\alpha)$*.*
- (b) *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*

Connecting all three

Theorem

Let $\alpha \in \Lambda_{d,L}$*, let* $\mathrm{x} = (x_1, x_2, \dots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \le i \le d$*, and let* $u = 0^{x_1} 1 0^{x_2} 1 \dots 0^{x_d} 1$*.*

There are bijections between the following:

- a *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*
- b *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*
- c *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state u.*

The basic bijections: idea

One can view a SCT as a sequence of cylindric shapes, each one obtained from the previous one by adding a cell.

The basic bijections: idea

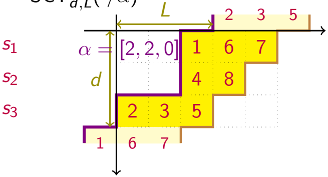
One can view a SCT as a sequence of cylindric shapes, each one obtained from the previous one by adding a cell.

Idea for the bijections:

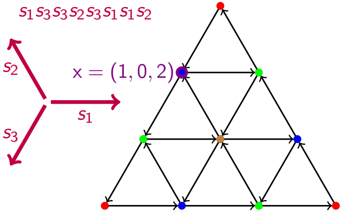
$$\begin{array}{c}\text{adding a cell in row } i \text{ of the cylindric shape}\\ \updownarrow \\ \text{taking step } s_i \text{ in the simplicial walk}\\ \updownarrow \\ i\text{th particle jumping to the next site}\end{array}$$

The basic bijections: example

$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$

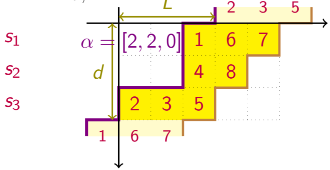


$s_1 s_3 s_3 s_2 s_3 s_1 s_1 s_2$

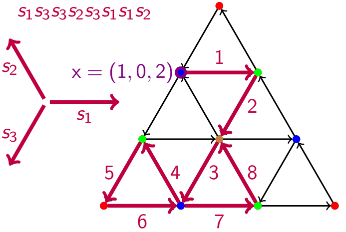


The basic bijections: example

$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$

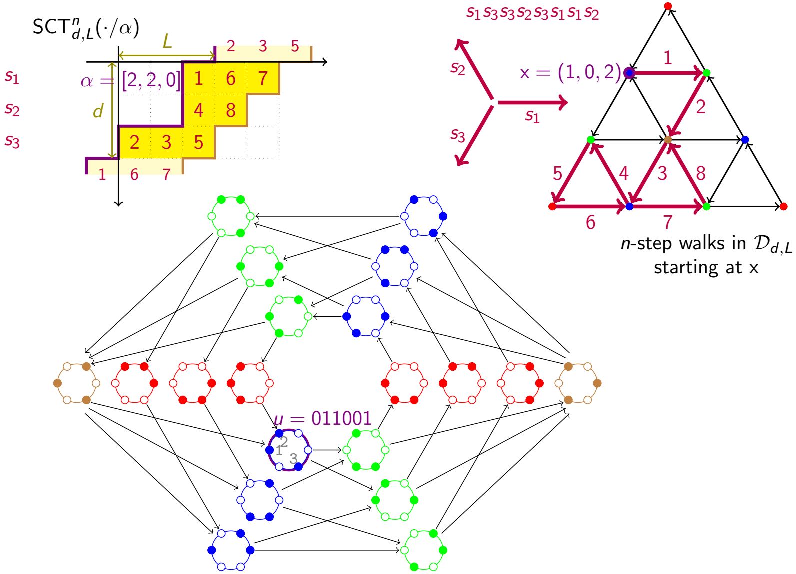


$s_1 s_3 s_3 s_2 s_3 s_1 s_1 s_2$

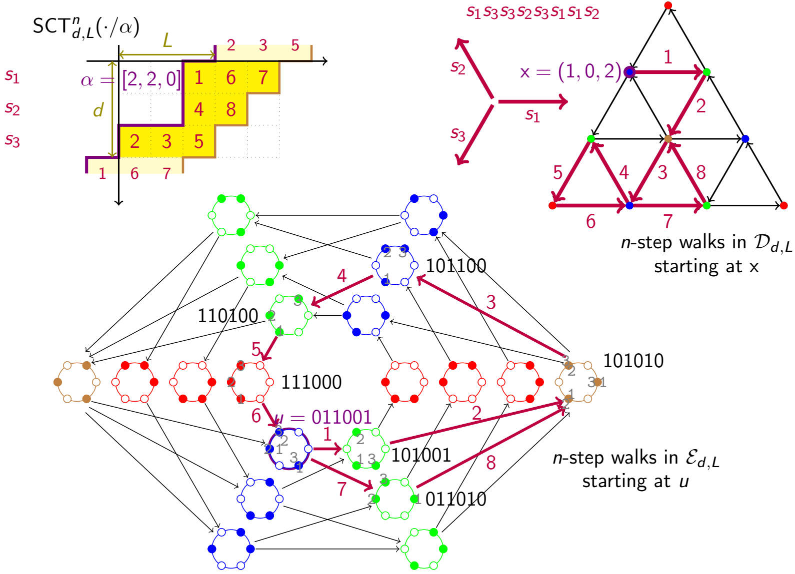


n-step walks in $\mathcal{D}_{d,L}$
starting at x

The basic bijections: example



The basic bijections: example

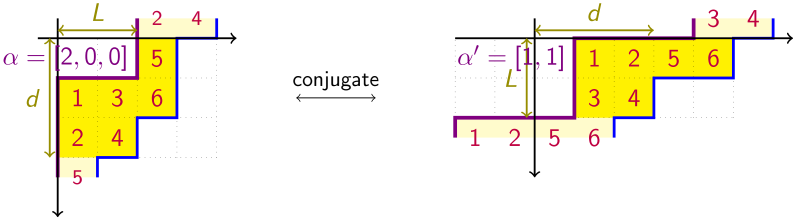


n-step walks in $\mathcal{D}_{d,L}$
starting at x

n-step walks in $\mathcal{E}_{d,L}$
starting at u

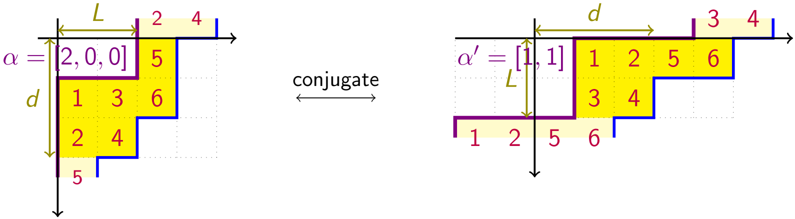
Conjugation of SCT

The **conjugate** of a cylindric shape (or a SCT) is obtained by reflecting it along the diagonal:



Conjugation of SCT

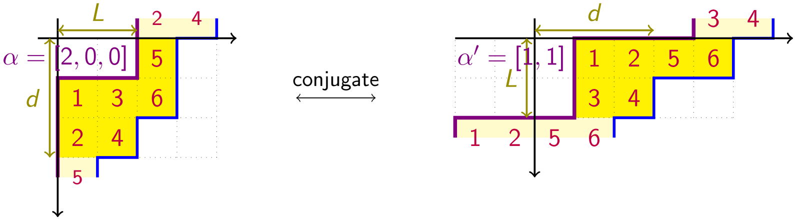
The **conjugate** of a cylindric shape (or a SCT) is obtained by reflecting it along the diagonal:



Note that conjugation swaps the parameters *d* and *L*.

Conjugation of SCT

The **conjugate** of a cylindric shape (or a SCT) is obtained by reflecting it along the diagonal:



Note that conjugation swaps the parameters *d* and *L*.

Let us use our bijections to translate this symmetry to the other settings.

Symmetries

Theorem

Let $\alpha \in \Lambda_{d,L}$ *, let* $\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \le i \le d$*, and let* $u = 0^{x_1}10^{x_2}1\ldots0^{x_d}1$*.*

There are natural bijections between the following:

- (a) *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*
- (b) *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*
- (c) *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state* u*.*

Symmetries

Theorem

Let $\alpha \in \Lambda_{d,L}$, let $\mathrm{x} = (x_1, x_2, \dots, x_d) \in \Delta_{d,L}$ where $x_i = \alpha_{i-1} - \alpha_i$ for $1 \leq i \leq d$, and let $u = 0^{x_1} 1 0^{x_2} 1 \dots 0^{x_d} 1$.

Let $\alpha' \in \Lambda_{L,d}$ be the conjugate of α, let $\mathrm{y} = (y_1, y_2, \dots, y_L) \in \Delta_{L,d}$ where $y_j = \alpha'_{j-1} - \alpha'_j$ for $1 \leq j \leq L$.

There are natural bijections between the following:

a. *The set $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$.*
b. *The set of n-step walks in $\mathcal{D}_{d,L}$ starting at vertex x.*
c. *The set of n-step walks in $\mathcal{E}_{d,L}$ starting at state u.*

Symmetries

Theorem

Let $\alpha \in \Lambda_{d,L}$, let $\mathrm{x} = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ where $x_i = \alpha_{i-1} - \alpha_i$ for $1 \leq i \leq d$, and let $u = 0^{x_1}10^{x_2}1\ldots0^{x_d}1$.

Let $\alpha' \in \Lambda_{L,d}$ be the conjugate of α, let $\mathrm{y} = (y_1, y_2, \ldots, y_L) \in \Delta_{L,d}$ where $y_j = \alpha'_{j-1} - \alpha'_j$ for $1 \leq j \leq L$.

There are natural bijections between the following:

- a *The set $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$.*
- b *The set of n-step walks in $\mathcal{D}_{d,L}$ starting at vertex x.*
- c *The set of n-step walks in $\mathcal{E}_{d,L}$ starting at state u.*
- a' *The set $\mathrm{SCT}^n_{L,d}(\cdot/\alpha')$.*

Symmetries

Theorem

Let $\alpha \in \Lambda_{d,L}$, let $x = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ where $x_i = \alpha_{i-1} - \alpha_i$ for $1 \leq i \leq d$, and let $u = 0^{x_1}10^{x_2}1 \ldots 0^{x_d}1$.

Let $\alpha' \in \Lambda_{L,d}$ be the conjugate of α, let $y = (y_1, y_2, \ldots, y_L) \in \Delta_{L,d}$ where $y_j = \alpha'_{j-1} - \alpha'_j$ for $1 \leq j \leq L$.

There are natural bijections between the following:

- a *The set $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$.*
- b *The set of n-step walks in $\mathcal{D}_{d,L}$ starting at vertex x.*
- c *The set of n-step walks in $\mathcal{E}_{d,L}$ starting at state u.*
- a' *The set $\mathrm{SCT}^n_{L,d}(\cdot/\alpha')$.*
- b' *The set of n-step walks in $\mathcal{D}_{L,d}$ starting at vertex y.*

Symmetries

Theorem

Let $\alpha \in \Lambda_{d,L}$ *, let* $\mathrm{x} = (x_1, x_2, \dots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1}10^{x_2}1\dots0^{x_d}1$*.*

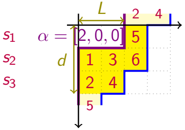
Let $\alpha' \in \Lambda_{L,d}$ *be the conjugate of* α*, let* $\mathrm{y} = (y_1, y_2, \dots, y_L) \in \Delta_{L,d}$ *where* $y_j = \alpha'_{j-1} - \alpha'_j$ *for* $1 \leq j \leq L$*.*

There are natural bijections between the following:

- a *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*
- b *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*
- c *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state* u*.*
- a' *The set* $\mathrm{SCT}^n_{L,d}(\cdot/\alpha')$*.*
- b' *The set of n-step walks in* $\mathcal{D}_{L,d}$ *starting at vertex* y*.*
- c' *The set of n-step walks in* $\mathcal{E}_{L,d}$ *starting at state* $u^{rc} = 01^{x_d}\dots01^{x_2}01^{x_1}$ *(reverse-complement of* u*).*

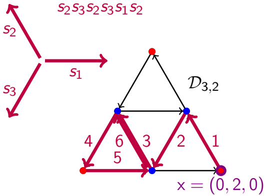
Symmetries: example

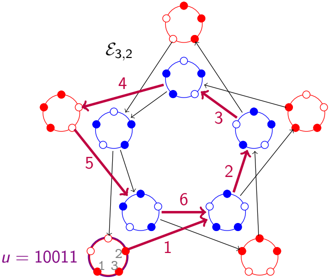
$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$



n-step walks in $\mathcal{D}_{d,L}$ starting at x

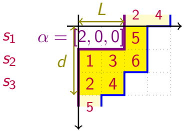
$s_2 s_3 s_2 s_3 s_1 s_2$





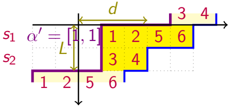
Symmetries: example

$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$

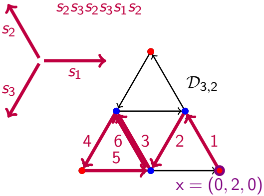


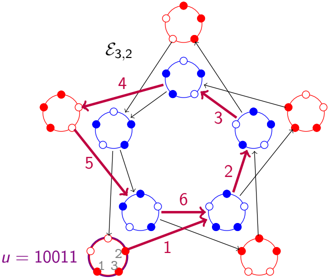
$\updownarrow$ conjugate

$\mathrm{SCT}^n_{d,L}(\cdot/\alpha')$

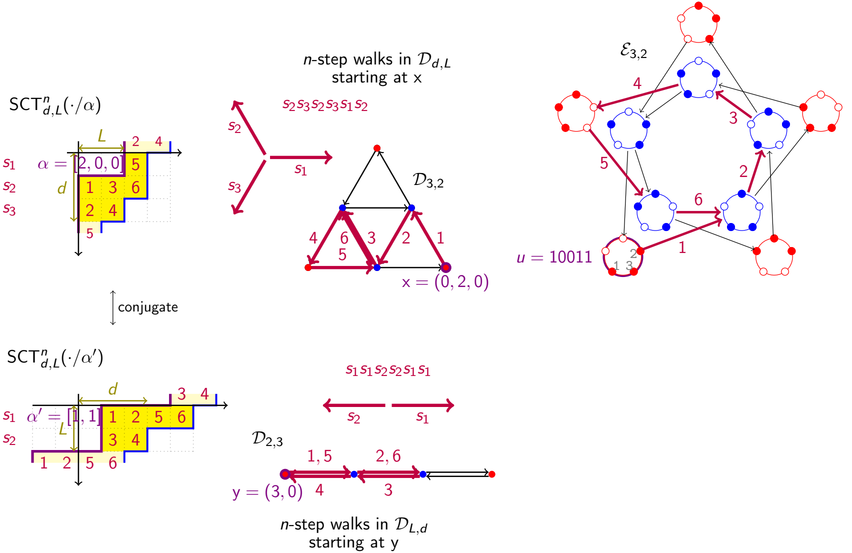


n-step walks in $\mathcal{D}_{d,L}$
starting at x

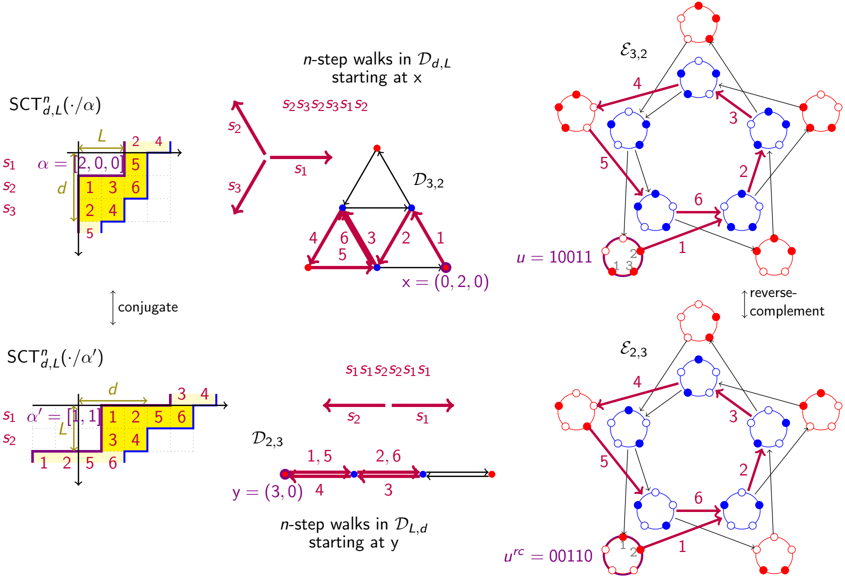




Symmetries: example



Symmetries: example



The cylindric Robinson–Schensted correspondence

Sagan and Stanley '90 introduced analogues of the Robinson–Schensted correspondence for skew tableaux.

The cylindric Robinson–Schensted correspondence

Sagan and Stanley '90 introduced analogues of the Robinson–Schensted correspondence for skew tableaux.

Neyman '15 extended these to cylindric tableaux.

The cylindric Robinson–Schensted correspondence

Sagan and Stanley '90 introduced analogues of the Robinson–Schensted correspondence for skew tableaux.

Neyman '15 extended these to cylindric tableaux.

Theorem (Neyman '15, adapting Sagan–Stanley '90)

Fix $\alpha, \beta \in \Lambda_{d,L}$ and $n, m \geq 0$. There is a bijection:

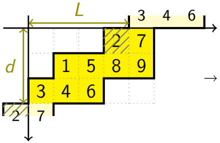
$$\mathrm{CRS} : \bigsqcup_{\substack{\mu \subseteq \alpha,\beta \\ |\alpha/\mu|=n, |\beta/\mu|=m}} \mathrm{SCT}_{d,L}(\alpha/\mu) \times \mathrm{SCT}_{d,L}(\beta/\mu)$$

$$\to \bigsqcup_{\substack{\lambda \supseteq \alpha,\beta \\ |\lambda/\beta|=n, |\lambda/\alpha|=m}} \mathrm{SCT}_{d,L}(\lambda/\beta) \times \mathrm{SCT}_{d,L}(\lambda/\alpha).$$

Internal row insertion

The description of CRS is based on row insertion operations.

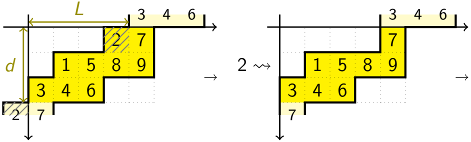
Here's an example of **internal row insertion** at row 1 of a SCT:



Internal row insertion

The description of CRS is based on row insertion operations.

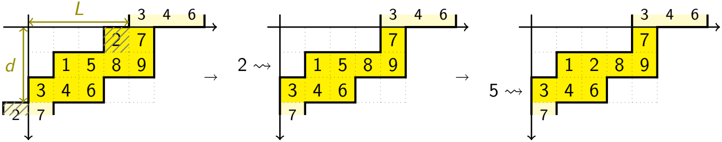
Here's an example of **internal row insertion** at row 1 of a SCT:



Internal row insertion

The description of CRS is based on row insertion operations.

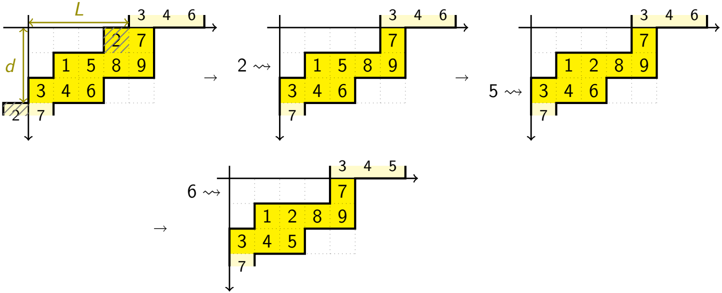
Here's an example of **internal row insertion** at row 1 of a SCT:



Internal row insertion

The description of CRS is based on row insertion operations.

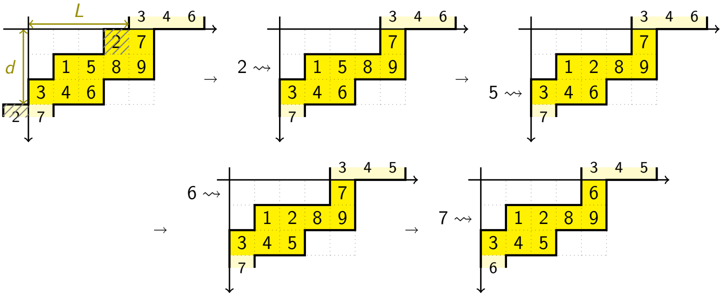
Here's an example of **internal row insertion** at row 1 of a SCT:



Internal row insertion

The description of CRS is based on row insertion operations.

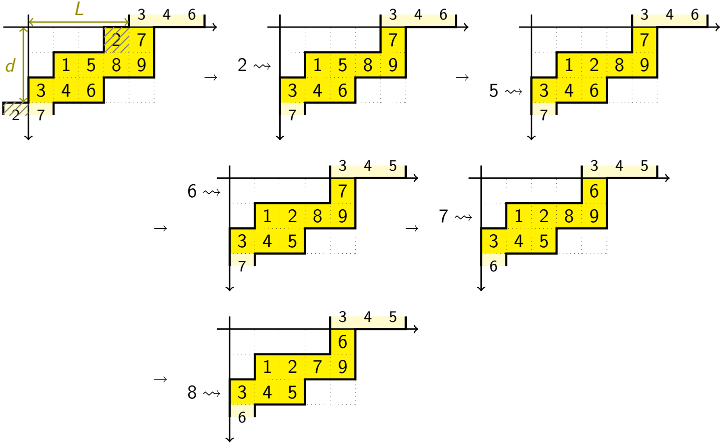
Here's an example of **internal row insertion** at row 1 of a SCT:



Internal row insertion

The description of CRS is based on row insertion operations.

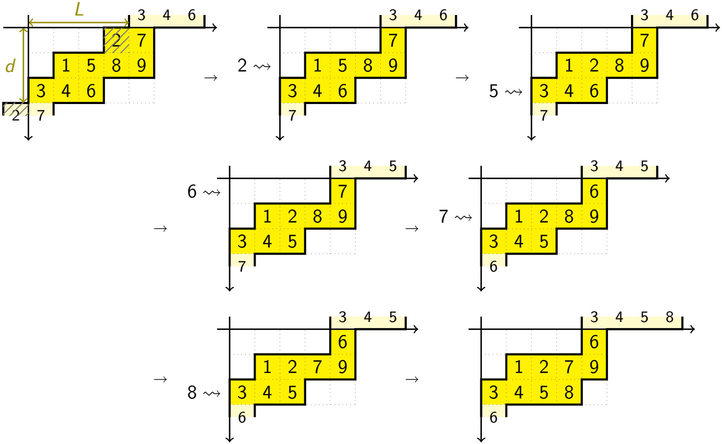
Here's an example of **internal row insertion** at row 1 of a SCT:



Internal row insertion

The description of CRS is based on row insertion operations.

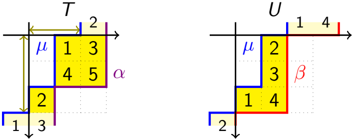
Here's an example of **internal row insertion** at row 1 of a SCT:



The CRS correspondence: example

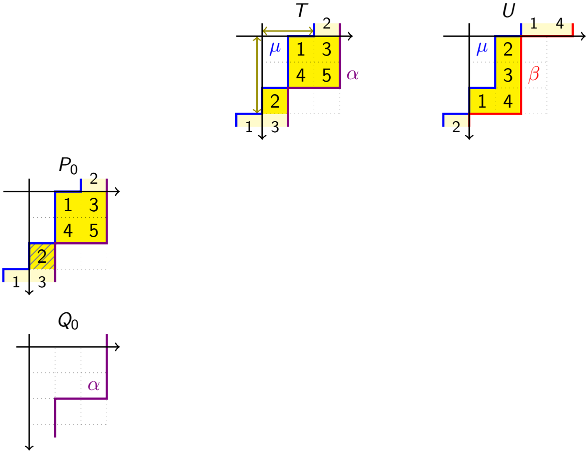
$$\mathrm{CRS}: \bigsqcup_{\mu\subseteq\alpha,\beta} \mathrm{SCT}_{d,L}(\alpha/\mu)\times\mathrm{SCT}_{d,L}(\beta/\mu) \rightarrow \bigsqcup_{\lambda\supseteq\alpha} \mathrm{SCT}_{d,L}(\lambda/\beta)\times\mathrm{SCT}_{d,L}(\lambda/\alpha)$$

$$(T,U) \mapsto (P,Q)$$



The CRS correspondence: example

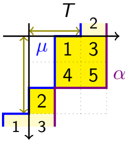
$$\begin{array}{rcl} \mathrm{CRS}: \displaystyle\bigsqcup_{\mu\subseteq\alpha,\beta} \mathrm{SCT}_{d,L}(\alpha/\mu)\times\mathrm{SCT}_{d,L}(\beta/\mu) & \to & \displaystyle\bigsqcup_{\lambda\supseteq\alpha} \mathrm{SCT}_{d,L}(\lambda/\beta)\times\mathrm{SCT}_{d,L}(\lambda/\alpha) \\ (T,U) & \mapsto & (P,Q) \end{array}$$

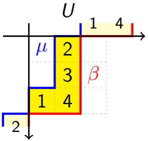


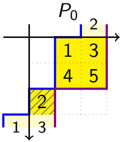
The CRS correspondence: example

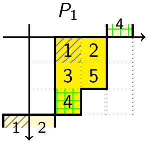
$$\mathrm{CRS}: \bigsqcup_{\mu\subseteq\alpha,\beta} \mathrm{SCT}_{d,L}(\alpha/\mu)\times \mathrm{SCT}_{d,L}(\beta/\mu) \quad\to\quad \bigsqcup_{\lambda\supseteq\alpha} \mathrm{SCT}_{d,L}(\lambda/\beta)\times\mathrm{SCT}_{d,L}(\lambda/\alpha)$$

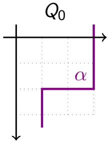
$$(T,U) \quad\mapsto\quad (P,Q)$$

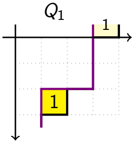






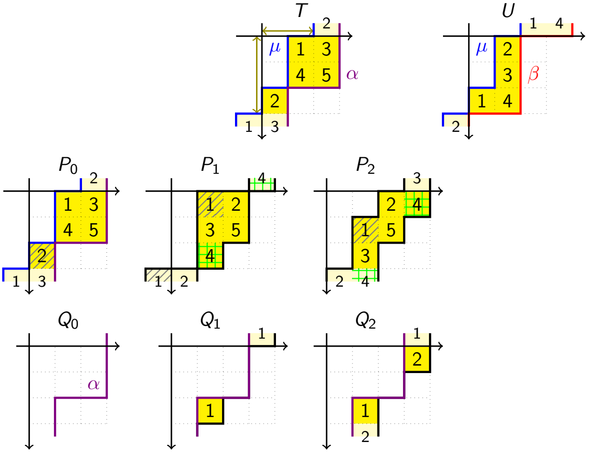






The CRS correspondence: example

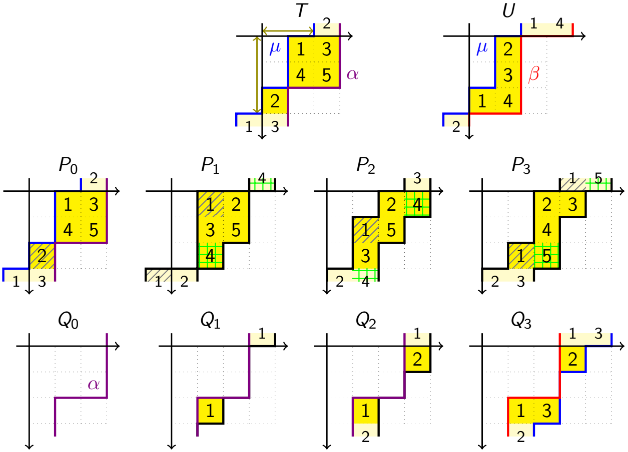
$$\begin{aligned}\mathrm{CRS}: \bigsqcup_{\mu\subseteq\alpha,\beta} \mathrm{SCT}_{d,L}(\alpha/\mu)\times \mathrm{SCT}_{d,L}(\beta/\mu) &\to \bigsqcup_{\lambda\supseteq\alpha}\mathrm{SCT}_{d,L}(\lambda/\beta)\times\mathrm{SCT}_{d,L}(\lambda/\alpha)\\ (T,U) &\mapsto (P,Q)\end{aligned}$$



The CRS correspondence: example

$$\mathrm{CRS}: \bigsqcup_{\mu\subseteq\alpha,\beta} \mathrm{SCT}_{d,L}(\alpha/\mu)\times \mathrm{SCT}_{d,L}(\beta/\mu) \to \bigsqcup_{\lambda\supseteq\alpha} \mathrm{SCT}_{d,L}(\lambda/\beta)\times \mathrm{SCT}_{d,L}(\lambda/\alpha)$$

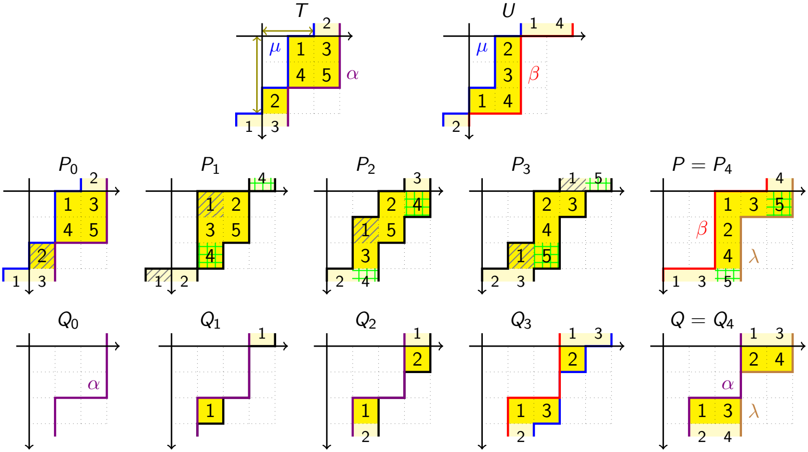
$$(T,U) \mapsto (P,Q)$$



The CRS correspondence: example

$$\mathrm{CRS}: \bigsqcup_{\mu\subseteq\alpha,\beta} \mathrm{SCT}_{d,L}(\alpha/\mu)\times\mathrm{SCT}_{d,L}(\beta/\mu) \to \bigsqcup_{\lambda\supseteq\alpha}\mathrm{SCT}_{d,L}(\lambda/\beta)\times\mathrm{SCT}_{d,L}(\lambda/\alpha)$$

$$(T,U)\mapsto(P,Q)$$



A special case of CRS

In the special case $\alpha = \beta$ and $T = U$, we have $\mathrm{CRS}(T, T) = (P, P)$.

A special case of CRS

In the special case $\alpha = \beta$ and $T = U$, we have $\mathrm{CRS}(T, T) = (P, P)$.

The resulting map $T \mapsto P$ is a bijection

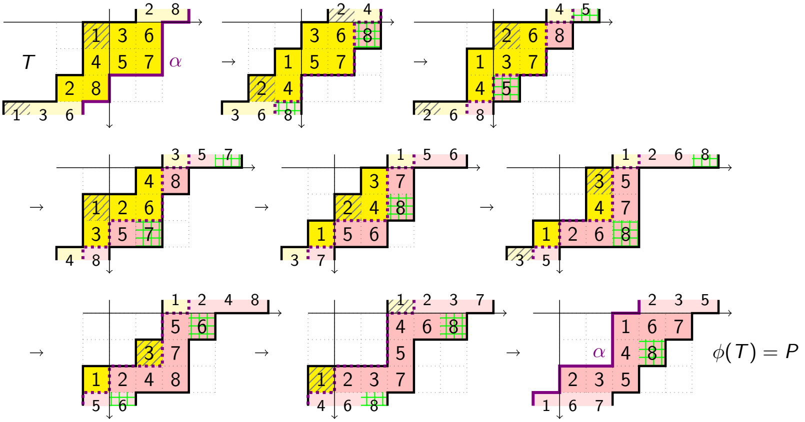
$$\phi : \mathrm{SCT}^n_{d,L}(\alpha/\cdot) \to \mathrm{SCT}^n_{d,L}(\cdot/\alpha).$$

A special case of CRS

In the special case $\alpha = \beta$ and $T = U$, we have $\mathrm{CRS}(T, T) = (P, P)$.

The resulting map $T \mapsto P$ is a bijection

$$\phi : \mathrm{SCT}^n_{d,L}(\alpha/\cdot) \to \mathrm{SCT}^n_{d,L}(\cdot/\alpha).$$



Translating ϕ to the other settings

Theorem

Let $\alpha \in \Lambda_{d,L}$, *let* $x = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$, *and let* $u = 0^{x_1} 1 0^{x_2} 1 \ldots 0^{x_d} 1$.

There are natural bijections between the following:

- a *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$.
- b *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x.
- c *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state* u.

Translating ϕ to the other settings

Theorem

Let $\alpha \in \Lambda_{d,L}$ *, let* $x = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1} 1 0^{x_2} 1 \ldots 0^{x_d} 1$*.*

There are natural bijections between the following:

- a *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*
- b *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*
- c *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state* u*.*
- a" *The set* $\mathrm{SCT}^n_{d,L}(\alpha/\cdot)$*.*

Translating ϕ to the other settings

Theorem

Let $\alpha \in \Lambda_{d,L}$ *, let* $x = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1}10^{x_2}1\ldots0^{x_d}1$*.*

There are natural bijections between the following:

- (a) *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*
- (b) *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*
- (c) *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state u.*
- (a'') *The set* $\mathrm{SCT}^n_{d,L}(\alpha/\cdot)$*.*
- (b'') *The set of n-step walks in* $\mathcal{D}_{d,L}$ **ending** *at vertex* x*.*

Translating ϕ to the other settings

Theorem

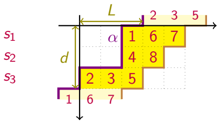
Let $\alpha \in \Lambda_{d,L}$ *, let* $x = (x_1, x_2, \ldots, x_d) \in \Delta_{d,L}$ *where* $x_i = \alpha_{i-1} - \alpha_i$ *for* $1 \leq i \leq d$*, and let* $u = 0^{x_1}10^{x_2}1\ldots0^{x_d}1$*.*

There are natural bijections between the following:

- a *The set* $\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$*.*
- b *The set of n-step walks in* $\mathcal{D}_{d,L}$ *starting at vertex* x*.*
- c *The set of n-step walks in* $\mathcal{E}_{d,L}$ *starting at state* u*.*
- a" *The set* $\mathrm{SCT}^n_{d,L}(\alpha/\cdot)$*.*
- b" *The set of n-step walks in* $\mathcal{D}_{d,L}$ **ending** *at vertex* x*.*
- c" *The set of n-step walks in* $\mathcal{E}_{d,L}$ **ending** *at state* u*.*

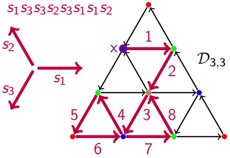
Translating ϕ to the other settings

$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$



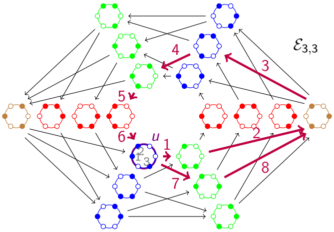
$\longleftrightarrow$

n-step walks in $\mathcal{D}_{d,L}$ starting at x

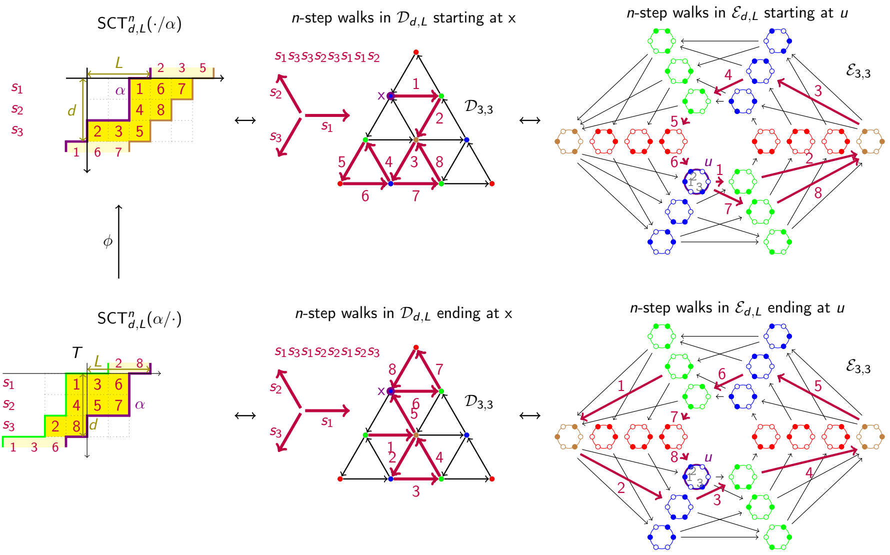


$\longleftrightarrow$

n-step walks in $\mathcal{E}_{d,L}$ starting at u



Translating ϕ to the other settings



$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$

n-step walks in $\mathcal{D}_{d,L}$ starting at x

n-step walks in $\mathcal{E}_{d,L}$ starting at u

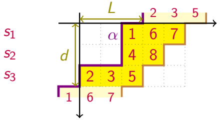
$\mathrm{SCT}^n_{d,L}(\alpha/\cdot)$

n-step walks in $\mathcal{D}_{d,L}$ ending at x

n-step walks in $\mathcal{E}_{d,L}$ ending at u

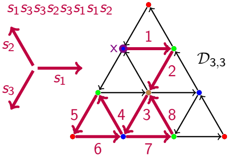
Translating ϕ to the other settings

$\mathrm{SCT}^n_{d,L}(\cdot/\alpha)$



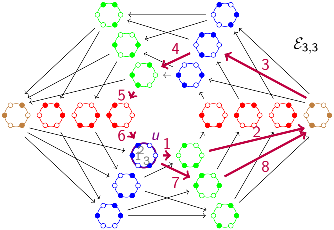
↔

n-step walks in $\mathcal{D}_{d,L}$ starting at x



↔

n-step walks in $\mathcal{E}_{d,L}$ starting at u



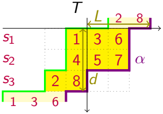
ϕ ↑

new proof of the
Forward-Backward Theorem

↕

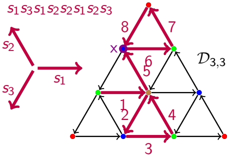
↕

$\mathrm{SCT}^n_{d,L}(\alpha/\cdot)$



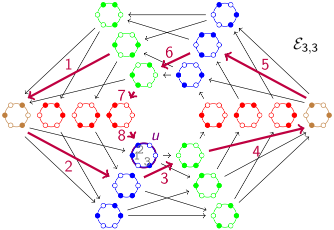
↔

n-step walks in $\mathcal{D}_{d,L}$ ending at x



↔

n-step walks in $\mathcal{E}_{d,L}$ ending at u



Growth diagrams

Growth diagrams were introduced by Fomin as an alternative description of the Robinson–Schensted correspondence, in order to generalize it to differential posets.

correspondence	insertion-based version	growth diagram version
RS	Robinson'38, Schensted'61	Fomin'86

Growth diagrams

Growth diagrams were introduced by Fomin as an alternative description of the Robinson–Schensted correspondence, in order to generalize it to differential posets.

Roby showed that Sagan and Stanley's analogue of RS for skew tableaux also has a natural description in terms of growth diagrams.

correspondence	insertion-based version	growth diagram version
RS	Robinson'38, Schensted'61	Fomin'86
skew RS	Stanley–Sagan'90	Roby'91

Growth diagrams

Growth diagrams were introduced by Fomin as an alternative description of the Robinson–Schensted correspondence, in order to generalize it to differential posets.

Roby showed that Sagan and Stanley's analogue of RS for skew tableaux also has a natural description in terms of growth diagrams.

correspondence	insertion-based version	growth diagram version
RS	Robinson'38, Schensted'61	Fomin'86
skew RS	Stanley–Sagan'90	Roby'91
cylindric RS	Neyman'15	??

Let's extend growth diagrams to the cylindric case.

Cylindric growth diagrams

Given $T \in \mathrm{SCT}_{d,L}(\alpha/\mu)$ and $U \in \mathrm{SCT}_{d,L}(\beta/\mu)$, where $|\alpha/\mu| = n$ and $|\beta/\mu| = m$, we will draw an $m \times n$ grid whose vertices are labeled by cylindric shapes.

Cylindric growth diagrams

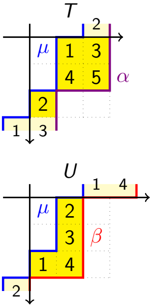
Given $T \in \mathrm{SCT}_{d,L}(\alpha/\mu)$ and $U \in \mathrm{SCT}_{d,L}(\beta/\mu)$, where $|\alpha/\mu| = n$ and $|\beta/\mu| = m$, we will draw an $m \times n$ grid whose vertices are labeled by cylindric shapes.

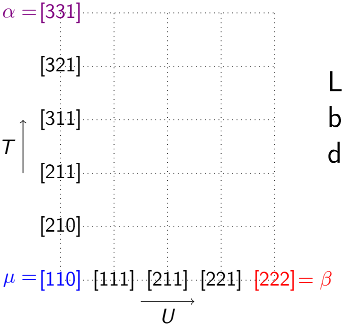
One can view a SCT as a sequence of cylindric shapes, each one obtained from the previous one by adding a cell.

Cylindric growth diagrams

Given $T \in \mathrm{SCT}_{d,L}(\alpha/\mu)$ and $U \in \mathrm{SCT}_{d,L}(\beta/\mu)$, where $|\alpha/\mu| = n$ and $|\beta/\mu| = m$, we will draw an $m \times n$ grid whose vertices are labeled by cylindric shapes.

One can view a SCT as a sequence of cylindric shapes, each one obtained from the previous one by adding a cell.





Label the left and bottom boundaries by the shapes determined by T and U.

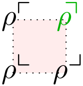
Local rules

For the rest of the labels, we use the **forward local rule**.

Local rules

For the rest of the labels, we use the **forward local rule**.

Given a square

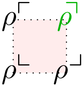


where $\rho^{\llcorner}$, $\rho^{\ulcorner}$, $\rho^{\lrcorner}$ have been computed, we compute $\rho^{\urcorner}$ as follows:

Local rules

For the rest of the labels, we use the **forward local rule**.

Given a square



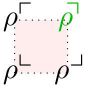
where $\rho^{\llcorner}$, $\rho^{\ulcorner}$, $\rho^{\lrcorner}$ have been computed, we compute $\rho^{\urcorner}$ as follows:

- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.

Local rules

For the rest of the labels, we use the **forward local rule**.

Given a square

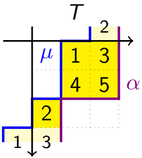


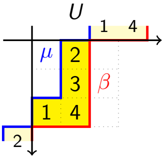
where $\rho^{\llcorner}$, $\rho^{\ulcorner}$, $\rho^{\lrcorner}$ have been computed, we compute $\rho^{\urcorner}$ as follows:

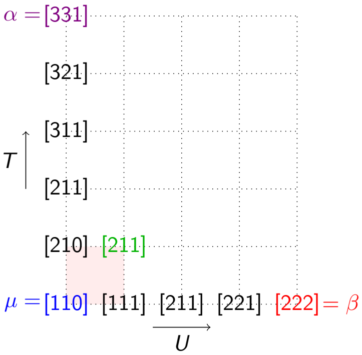
- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.
- If $\rho^{\ulcorner} = \rho^{\lrcorner}$ and this shape is obtained from $\rho^{\llcorner}$ by adding a cell to row i, let $\rho^{\urcorner}$ be obtained from $\rho^{\ulcorner} = \rho^{\lrcorner}$ by adding a cell to row $i + 1 \pmod{d}$.

Computing the remaining labels

- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.
- If $\rho^{\ulcorner} = \rho^{\lrcorner}$ and this shape is obtained from $\rho^{\llcorner}$ by adding a cell to row i, let $\rho^{\urcorner}$ be obtained from $\rho^{\ulcorner} = \rho^{\lrcorner}$ by adding a cell to row $i+1$ (mod d).

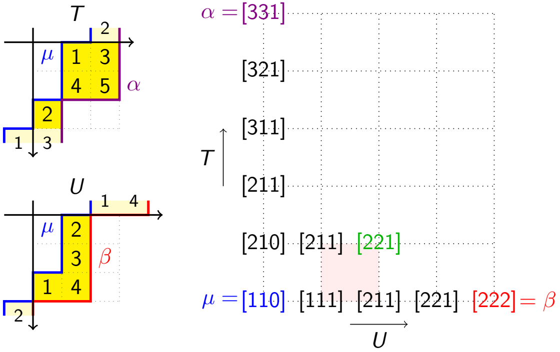






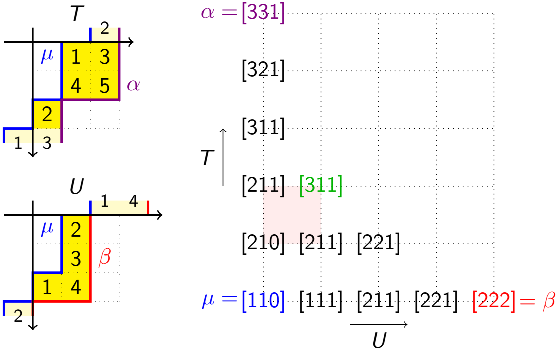
Computing the remaining labels

- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.
- If $\rho^{\ulcorner} = \rho^{\lrcorner}$ and this shape is obtained from $\rho^{\llcorner}$ by adding a cell to row i, let $\rho^{\urcorner}$ be obtained from $\rho^{\ulcorner} = \rho^{\lrcorner}$ by adding a cell to row $i+1$ (mod d).



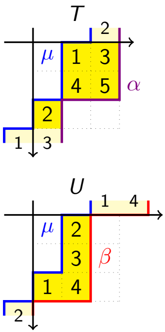
Computing the remaining labels

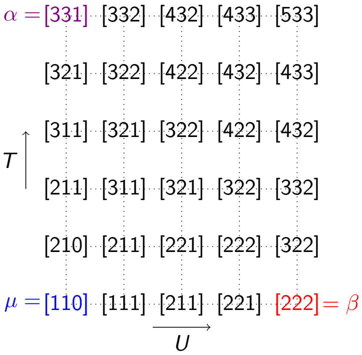
- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.
- If $\rho^{\ulcorner} = \rho^{\lrcorner}$ and this shape is obtained from $\rho^{\llcorner}$ by adding a cell to row i, let $\rho^{\urcorner}$ be obtained from $\rho^{\ulcorner} = \rho^{\lrcorner}$ by adding a cell to row $i + 1$ (mod d).



Computing the remaining labels

- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.
- If $\rho^{\ulcorner} = \rho^{\lrcorner}$ and this shape is obtained from $\rho^{\llcorner}$ by adding a cell to row i, let $\rho^{\urcorner}$ be obtained from $\rho^{\ulcorner} = \rho^{\lrcorner}$ by adding a cell to row $i + 1 \pmod{d}$.

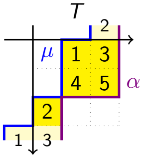


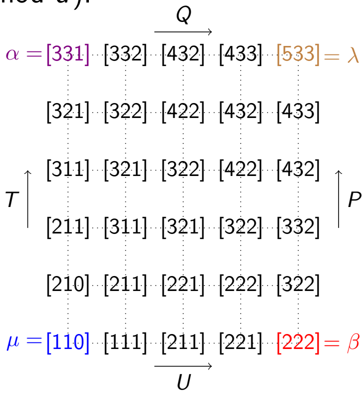


$\alpha =$	[331]	[332]	[432]	[433]	[533]	
	[321]	[322]	[422]	[432]	[433]	
	[311]	[321]	[322]	[422]	[432]	
	[211]	[311]	[321]	[322]	[332]	
	[210]	[211]	[221]	[222]	[322]	
$\mu =$	[110]	[111]	[211]	[221]	[222]	$= \beta$

Computing the remaining labels

- If $\rho^{\ulcorner} \neq \rho^{\lrcorner}$, let $\rho^{\urcorner} = \rho^{\ulcorner} \cup \rho^{\lrcorner}$.
- If $\rho^{\ulcorner} = \rho^{\lrcorner}$ and this shape is obtained from $\rho^{\llcorner}$ by adding a cell to row i, let $\rho^{\urcorner}$ be obtained from $\rho^{\ulcorner} = \rho^{\lrcorner}$ by adding a cell to row $i + 1 \pmod{d}$.





$\alpha =$ [331] [332] [432] [433] [533] $= \lambda$

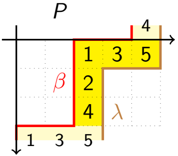
[321] [322] [422] [432] [433]

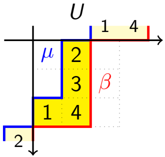
[311] [321] [322] [422] [432]

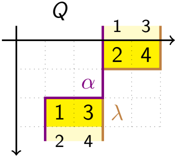
[211] [311] [321] [322] [332]

[210] [211] [221] [222] [322]

$\mu =$ [110] [111] [211] [221] [222] $= \beta$







The right and the top boundaries determine $(P, Q) = \mathrm{CRS}(T, U)$.

A symmetry of CRS

The following symmetry, first proved by Neyman using the insertion-based version, is now immediate.

Corollary

$\mathrm{CRS}(T, U) = (P, Q) \iff \mathrm{CRS}(U, T) = (Q, P).$

A symmetry of CRS

The following symmetry, first proved by Neyman using the insertion-based version, is now immediate.

Corollary

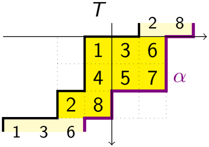
$\mathrm{CRS}(T, U) = (P, Q) \Longleftrightarrow \mathrm{CRS}(U, T) = (Q, P)$.

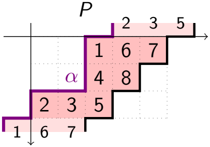
In particular, $\mathrm{CRS}(T, T) = (P, P)$, which defines the bijection

$$\begin{array}{rccc} \phi: & \mathrm{SCT}^n_{d,L}(\alpha/\cdot) & \to & \mathrm{SCT}^n_{d,L}(\cdot/\alpha) \\ & T & \mapsto & P \end{array}$$

from before.

An example of $\phi(T) = P$ using cylindric growth diagrams





notation: $\bar{k} = -k$

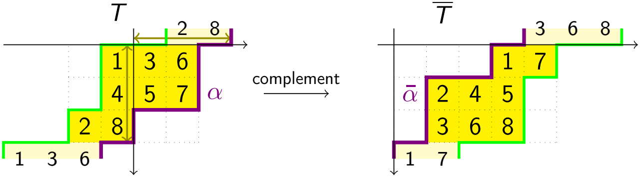
$\xrightarrow{P}$

$\alpha = [220]$	$[320]$	$[321]$	$[322]$	$[332]$	$[333]$	$[433]$	$[533]$	$[543]$
$[22\bar{1}]$	$[220]$	$[221]$	$[222]$	$[322]$	$[332]$	$[432]$	$[433]$	$[533]$
$[21\bar{1}]$	$[210]$	$[211]$	$[221]$	$[222]$	$[322]$	$[422]$	$[432]$	$[433]$
$[11\bar{1}]$	$[110]$	$[111]$	$[211]$	$[221]$	$[321]$	$[322]$	$[332]$	$[333]$
$T \uparrow$ $[10\bar{1}]$	$[100]$	$[110]$	$[210]$	$[220]$	$[221]$	$[222]$	$[322]$	$[332]$ $\uparrow P$
$[1\bar{1}\bar{1}]$	$[10\bar{1}]$	$[11\bar{1}]$	$[110]$	$[210]$	$[211]$	$[221]$	$[222]$	$[322]$
$[0\bar{1}\bar{1}]$	$[00\bar{1}]$	$[10\bar{1}]$	$[11\bar{1}]$	$[110]$	$[111]$	$[211]$	$[221]$	$[321]$
$[0\bar{1}\bar{2}]$	$[00\bar{2}]$	$[00\bar{1}]$	$[10\bar{1}]$	$[100]$	$[110]$	$[210]$	$[220]$	$[320]$
$[\bar{1}\bar{1}\bar{2}]$	$[0\bar{1}\bar{2}]$	$[0\bar{1}\bar{1}]$	$[1\bar{1}\bar{1}]$	$[10\bar{1}]$	$[11\bar{1}]$	$[21\bar{1}]$	$[22\bar{1}]$	$[220] = \alpha$

$\xrightarrow[T]{}$

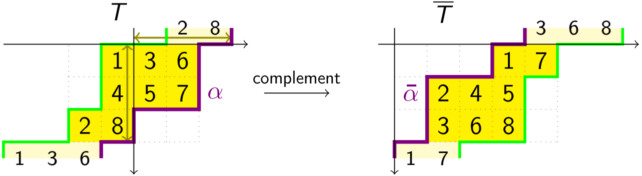
Another symmetry of CRS

The **complement** of a SCT with n cells is obtained by performing a 180° rotation and replacing each entry k with $n+1-k$.



Another symmetry of CRS

The **complement** of a SCT with n cells is obtained by performing a 180° rotation and replacing each entry k with $n+1-k$.



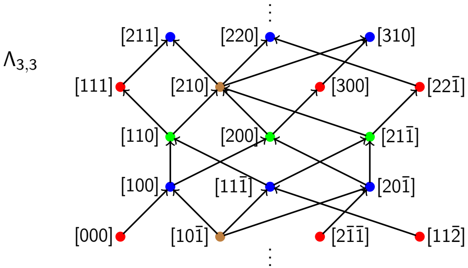
Corollary

$\mathrm{CRS}(T,U) = (P,Q) \iff \mathrm{CRS}(\overline{P},\overline{Q}) = (\overline{T},\overline{U})$.

In particular, $\phi(T) = P \iff \phi(\overline{P}) = \overline{T}$.

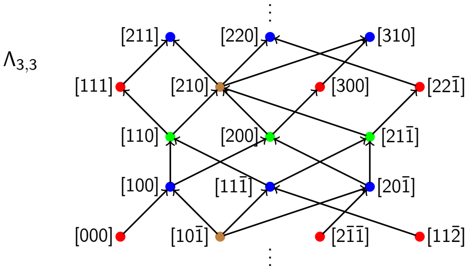
Why are there no crosses in the growth diagram?

Unlike Young's lattice, the poset $(\Lambda_{d,L}, \subseteq)$ of cylindric shapes ordered by containment is technically not an ***r*-differential poset**, since it does not have a minimal element.



Why are there no crosses in the growth diagram?

Unlike Young's lattice, the poset $(\Lambda_{d,L}, \subseteq)$ of cylindric shapes ordered by containment is technically not an **r-differential poset**, since it does not have a minimal element.



However, disregarding this condition and relaxing the usual requirement that $r \geq 1$, $(\Lambda_{d,L}, \subseteq)$ could be considered a **"0-differential poset"**.

Oscillating cylindric tableaux

Oscillating cylindric tableaux (OCT) are sequences of cylindric shapes where each one is obtained from the previous one by either adding or removing a cell, e.g.

$$[1, 0, 0], [2, 0, 0], [2, 0, -1], [1, 0, -1]$$

Oscillating cylindric tableaux

Oscillating cylindric tableaux (OCT) are sequences of cylindric shapes where each one is obtained from the previous one by either adding or removing a cell, e.g.

$$[1,0,0],[2,0,0],[2,0,-1],[1,0,-1]$$

Note: it is always possible to remove a cell from a cylindric shape.

Oscillating cylindric tableaux

Oscillating cylindric tableaux (OCT) are sequences of cylindric shapes where each one is obtained from the previous one by either adding or removing a cell, e.g.

$$[1,0,0],[2,0,0],[2,0,-1],[1,0,-1]$$

Note: it is always possible to remove a cell from a cylindric shape.

The above bijections between
ⓐ SCT, ⓑ walks in simplices, ⓒ sequences of states in TASEP
generalize to bijections between

Oscillating cylindric tableaux

Oscillating cylindric tableaux (OCT) are sequences of cylindric shapes where each one is obtained from the previous one by either adding or removing a cell, e.g.

$$[1,0,0],[2,0,0],[2,0,-1],[1,0,-1]$$

Note: it is always possible to remove a cell from a cylindric shape.

The above bijections between
ⓐ SCT, ⓑ walks in simplices, ⓒ sequences of states in TASEP
generalize to bijections between

- ⓐ **oscillating** cylindric tableaux,

Oscillating cylindric tableaux

Oscillating cylindric tableaux (OCT) are sequences of cylindric shapes where each one is obtained from the previous one by either adding or removing a cell, e.g.

$$[1,0,0],[2,0,0],[2,0,-1],[1,0,-1]$$

Note: it is always possible to remove a cell from a cylindric shape.

The above bijections between
ⓐ SCT, ⓑ walks in simplices, ⓒ sequences of states in TASEP
generalize to bijections between

- ⓐ **oscillating** cylindric tableaux,
- ⓑ walks in simplices where steps can be taken in the **forward** or **backward** direction;

Oscillating cylindric tableaux

Oscillating cylindric tableaux (OCT) are sequences of cylindric shapes where each one is obtained from the previous one by either adding or removing a cell, e.g.

$$[1,0,0],[2,0,0],[2,0,-1],[1,0,-1]$$

Note: it is always possible to remove a cell from a cylindric shape.

The above bijections between
ⓐ SCT, ⓑ walks in simplices, ⓒ sequences of states in TASEP
generalize to bijections between

- ⓐ **oscillating** cylindric tableaux,
- ⓑ walks in simplices where steps can be taken in the **forward** or **backward** direction;
- ⓒ sequences of states in the **symmetric** simple exclusion process (where particles can jump in either direction).

Bijections for oscillating walks

The Forward-Backward Theorem for walks in the simplex has the following generalization.

Theorem (Courtiel–Elvey Price–Marcovici '21)

For any $\mathsf{x} \in \Delta_{d,L}$ *and any two binary words* w *and* w' *of length* n*, there is a bijection between oscillating walks in* $\mathcal{D}_{d,L}$ *starting at* x *of "type"* w *and those of "type"* w'*.*

Bijections for oscillating walks

The Forward-Backward Theorem for walks in the simplex has the following generalization.

Theorem (Courtiel–Elvey Price–Marcovici '21)

For any $x \in \Delta_{d,L}$ *and any two binary words* w *and* w' *of length* n*, there is a bijection between oscillating walks in* $\mathcal{D}_{d,L}$ *starting at* x *of "type"* w *and those of "type"* w'*.*

Their proof repeatedly applies certain flips to adjacent steps and to the last step of the walk.

Bijections for oscillating walks

The Forward-Backward Theorem for walks in the simplex has the following generalization.

Theorem (Courtiel–Elvey Price–Marcovici '21)

For any $\mathrm{x} \in \Delta_{d,L}$ *and any two binary words* w *and* w' *of length* n*, there is a bijection between oscillating walks in* $\mathcal{D}_{d,L}$ *starting at* x *of "type"* w *and those of "type"* w'*.*

Their proof repeatedly applies certain flips to adjacent steps and to the last step of the walk.

We can give a new proof by first translating it into a theorem about OCT, and then using cylindric growth diagrams.

Bijections for oscillating cylindric tableaux

Theorem

For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

Bijections for oscillating cylindric tableaux

Theorem

For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

Proof idea: Both sets are in bijection with the set of **symmetric** cylindric growth diagrams on an $n \times n$ grid where the upper-left and lower-right corners have label α.

Bijections for oscillating cylindric tableaux

Theorem

For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

Proof idea: Both sets are in bijection with the set of **symmetric** cylindric growth diagrams on an $n \times n$ grid where the upper-left and lower-right corners have label α.

Example:

OCT starting at $\alpha = [100]$ of type $+--$:

$[100] \xrightarrow{+} [200] \xrightarrow{-} [20\bar{1}] \xrightarrow{-} [10\bar{1}]$

Bijections for oscillating cylindric tableaux

Theorem

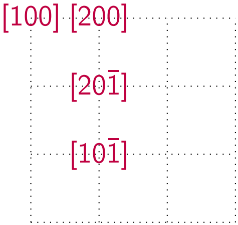
For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

Proof idea: Both sets are in bijection with the set of **symmetric** cylindric growth diagrams on an $n \times n$ grid where the upper-left and lower-right corners have label α.

Example:

OCT starting at $\alpha = [100]$ of type $+--$:

$[100] \xrightarrow{+} [200] \xrightarrow{-} [20\bar{1}] \xrightarrow{-} [10\bar{1}]$



Bijections for oscillating cylindric tableaux

Theorem

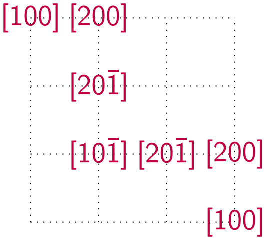
For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

Proof idea: Both sets are in bijection with the set of **symmetric** cylindric growth diagrams on an $n \times n$ grid where the upper-left and lower-right corners have label α.

Example:

OCT starting at $\alpha = [100]$ of type $+--$:

$[100] \xrightarrow{+} [200] \xrightarrow{-} [20\bar{1}] \xrightarrow{-} [10\bar{1}]$



Bijections for oscillating cylindric tableaux

Theorem

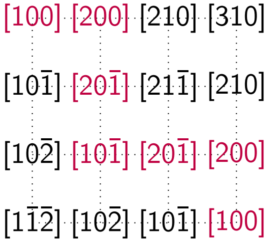
For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

Proof idea: Both sets are in bijection with the set of **symmetric** cylindric growth diagrams on an $n \times n$ grid where the upper-left and lower-right corners have label α.

Example:

OCT starting at $\alpha = [100]$ of type $+--$:

$[100] \xrightarrow{+} [200] \xrightarrow{-} [20\bar{1}] \xrightarrow{-} [10\bar{1}]$



$[100]$	$[200]$	$[210]$	$[310]$
$[10\bar{1}]$	$[20\bar{1}]$	$[21\bar{1}]$	$[210]$
$[10\bar{2}]$	$[10\bar{1}]$	$[20\bar{1}]$	$[200]$
$[1\bar{1}\bar{2}]$	$[10\bar{2}]$	$[10\bar{1}]$	$[100]$

Bijections for oscillating cylindric tableaux

Theorem

For any $\alpha \in \Lambda_{d,L}$ and any two binary words w and w' of length n, there is a bijection between OCT starting at α of type w and those of type w'.

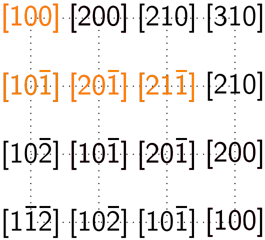
Proof idea: Both sets are in bijection with the set of **symmetric** cylindric growth diagrams on an $n \times n$ grid where the upper-left and lower-right corners have label α.

Example:

OCT starting at $\alpha = [100]$ of type $+--$:

$[100] \xrightarrow{+} [200] \xrightarrow{-} [20\bar{1}] \xrightarrow{-} [10\bar{1}]$

$[100] \xrightarrow{-} [10\bar{1}] \xrightarrow{+} [20\bar{1}] \xrightarrow{+} [21\bar{1}]$



$[100]$	$[200]$	$[210]$	$[310]$
$[10\bar{1}]$	$[20\bar{1}]$	$[21\bar{1}]$	$[210]$
$[10\bar{2}]$	$[10\bar{1}]$	$[20\bar{1}]$	$[200]$
$[1\bar{1}\bar{2}]$	$[10\bar{2}]$	$[10\bar{1}]$	$[100]$

Open problem 1: crossings and nestings in partial matchings

Theorem (Huh–Kim–Krattenthaler–Okada '23)

$\left|\mathrm{SCT}^n_{2h+1,2w+1}(\cdot/[0^d])\right|$ *equals the number of partial matchings on* n *points with no* $(h+1)$*-crossing and no* $(w+1)$*-nesting.*

Open problem 1: crossings and nestings in partial matchings

Theorem (Huh–Kim–Krattenthaler–Okada '23)

$\left|\mathrm{SCT}^{n}_{2h+1,2w+1}(\cdot/[0^d])\right|$ *equals the number of partial matchings on n points with no* $(h+1)$*-crossing and no* $(w+1)$*-nesting.*

Open problem: Give a bijective proof.

Open problem 1: crossings and nestings in partial matchings

Theorem (Huh–Kim–Krattenthaler–Okada '23)

$\left|\mathrm{SCT}^n_{2h+1,2w+1}(\cdot/[0^d])\right|$ *equals the number of partial matchings on* n *points with no* $(h+1)$*-crossing and no* $(w+1)$*-nesting.*

Open problem: Give a bijective proof.

When $h = 1$, this is equivalent to the Mortimer–Prellberg result about walks in $\mathcal{D}_{3,L}$ and bounded Motkzin paths. A complicated bijective proof was given by Courtiel, Elvey Price and Marcovici.

Open problem 1: crossings and nestings in partial matchings

Theorem (Huh–Kim–Krattenthaler–Okada '23)

$\left|\mathrm{SCT}^n_{2h+1,2w+1}(\cdot/[0^d])\right|$ *equals the number of partial matchings on n points with no $(h+1)$-crossing and no $(w+1)$-nesting.*

Open problem: Give a bijective proof.

When $h=1$, this is equivalent to the Mortimer–Prellberg result about walks in $\mathcal{D}_{3,L}$ and bounded Motkzin paths. A complicated bijective proof was given by Courtiel, Elvey Price and Marcovici.

When $h \to \infty$, there is a bijective proof using the RS correspondence.

Open problem 2: evacuation in terms of CRS

For $d, L \geq n$, we have $\mathrm{SCT}^n_{d,L}(\cdot/[0^d]) \cong \mathrm{SYT}^n$ (standard Young tableaux of straight shape).

Open problem 2: evacuation in terms of CRS

For $d, L \geq n$, we have $\mathrm{SCT}^n_{d,L}(\cdot/[0^d]) \cong \mathrm{SYT}^n$ (standard Young tableaux of straight shape).

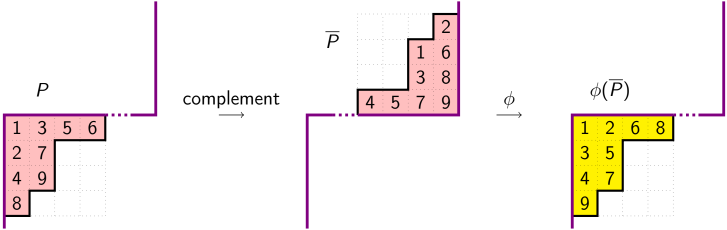
By complementation, $\mathrm{SCT}^n_{d,L}([0^d]/\cdot) \cong \mathrm{SYT}^n$ as well.

Open problem 2: evacuation in terms of CRS

For $d, L \geq n$, we have $\mathrm{SCT}^n_{d,L}(\cdot/[0^d]) \cong \mathrm{SYT}^n$ (standard Young tableaux of straight shape).

By complementation, $\mathrm{SCT}^n_{d,L}([0^d]/\cdot) \cong \mathrm{SYT}^n$ as well.

Thus, the map $\phi : \mathrm{SCT}^n_{d,L}([0^d]/\cdot) \rightarrow \mathrm{SCT}^n_{d,L}(\cdot/[0^d])$ can be viewed as an involution on SYT^n.

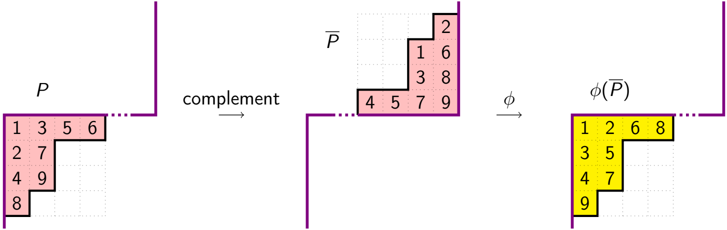


Open problem 2: evacuation in terms of CRS

For $d, L \geq n$, we have $\mathrm{SCT}^n_{d,L}(\cdot/[0^d]) \cong \mathrm{SYT}^n$ (standard Young tableaux of straight shape).

By complementation, $\mathrm{SCT}^n_{d,L}([0^d]/\cdot) \cong \mathrm{SYT}^n$ as well.

Thus, the map $\phi : \mathrm{SCT}^n_{d,L}([0^d]/\cdot) \to \mathrm{SCT}^n_{d,L}(\cdot/[0^d])$ can be viewed as an involution on SYT^n.



Conjecture

This map coincides with Schützenberger's evacuation.

Thank you