A bijection for descent sets of permutations with only even and only odd cycles

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Two ways to write permutations

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- in *one-line notation*, e.g. $\pi = 3\,1\,7\,6\,5\,4\,2$,
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Example

$\mathcal{S}_3^o = \{(1,2,3),(1,3,2),(1)(2)(3)\}$,
$\mathcal{S}_3^e = \{(1,2)(3),(1,3)(2),(2,3)(1)\}$.

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$$|\mathcal{S}_n^o| = |\mathcal{S}_n^e| = \begin{cases} (n-1)!!^2 = (n-1)^2(n-3)^2\dots 1^2 & \text{if } n \text{ is even,} \\ n\,(n-2)!!^2 = n(n-2)^2(n-4)^2\dots 2^2 & \text{if } n \text{ is odd.} \end{cases}$$

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1. Write each cycle of $\pi \in \mathcal{S}_n^o$ starting with its largest element, and order the cycles by increasing first element, e.g. $\pi = (4)(5,1,3)(7,2,6)(8)$.

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2. Move the last element of the 1st cycle to the end of the 2nd cycle, the last element of the 3rd cycle to the end of the 4th cycle, etc., e.g.

$$(4)(5,1,3)(7,2,6)(8) \mapsto (5,1,3,4)(7,2)(8,6) \in \mathcal{S}_n^e.$$

A surprising refinement

Definition

For $\pi = \pi_1\pi_2\dots\pi_n \in \mathcal{S}_n$,

- its descent set is $\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$,
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Theorem (Adin, Hegedűs, Roichman '25)

For any n and any subset $J \subseteq [n-1]$,

$$|\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) = J\}| = |\{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) = J\}|.$$

A surprising refinement

$$|\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) = J\}| = |\{\pi' \in \mathcal{S}_n^e : \mathrm{Des}(\pi') = J\}|$$

Example ($n = 4$)

$\pi \in \mathcal{S}_4^o$	$\mathrm{Asc}(\pi) = \mathrm{Des}(\pi')$	$\pi' \in \mathcal{S}_4^e$
$(1,2,4)(3) = 2431$	$\{1\}$	$(1,4,3,2) = 4123$
$(1,4,2)(3) = 4132$	$\{2\}$	$(1,2,4,3) = 2413$
$(1,3,4)(2) = 3241$	$\{2\}$	$(1,3)(2,4) = 3412$
$(1,4,3)(2) = 4213$	$\{3\}$	$(1,2,3,4) = 2341$
$(2,3,4)(1) = 1342$	$\{1,2\}$	$(1,4,2,3) = 4312$
$(2,4,3)(1) = 1423$	$\{1,3\}$	$(1,3,4,2) = 3142$
$(1,2,3)(4) = 2314$	$\{1,3\}$	$(1,2)(3,4) = 2143$
$(1,3,2)(4) = 3124$	$\{2,3\}$	$(1,3,2,4) = 3421$
$(1)(2)(3)(4) = 1234$	$\{1,2,3\}$	$(1,4)(2,3) = 4321$

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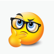
Unfortunately, Bóna's bijection $\mathcal{S}_n^o \to \mathcal{S}_n^e$ does not behave well with respect to Asc and Des.

For any n and any subset $S \subseteq [n-1]$, we will construct an explicit bijection

$$\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \leftrightsquigarrow \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\}.$$

Structure of the bijection



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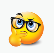

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Multisets of odd, distinct necklaces — Multisets of even necklaces

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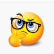

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$\downarrow$		$\downarrow$
Multisets of odd, distinct necklaces		Multisets of even necklaces
$\|\|\|$		$\|\|\|$
Words whose Lyndon factors are odd and distinct	$\longleftrightarrow$	Words whose Lyndon factors are even

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For a set $S = \{s_1, s_2, \dots, s_k\} \subseteq [n-1]$ with $s_1 < \dots < s_k$, let $\alpha = (s_1, s_2 - s_1, \dots, n - s_k)$ and $\operatorname{wt}(S) = \mathtt{a}^{\alpha_1}\mathtt{b}^{\alpha_2}\mathtt{c}^{\alpha_3}\dots$.

Example: If $S = \{2, 3\} \subseteq [5]$, then $\alpha = (2, 1, 3)$ and $\operatorname{wt}(S) = \mathtt{a}^2\mathtt{b}\mathtt{c}^3$.

From permutations to multisets of necklaces: $\mathrm{Des} \subseteq S$

In 1993, Gessel and Reutenauer described a bijection

$$\Phi_S : \{\pi \in \mathcal{S}_n : \mathrm{Des}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n : \mathrm{wt}(M) = \mathrm{wt}(S)\}$$

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Let $n = 8$ and $S = \{4, 7\}$, so $\mathrm{wt}(S) = \mathtt{a}^4\mathtt{b}^3\mathtt{c}$.

Take $\pi = 4567{\cdot}238{\cdot}1 \in \mathcal{S}_n$, which has $\mathrm{Des}(\pi) = S$.

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The multiset of necklaces $(\mathtt{a},\mathtt{b})(\mathtt{a},\mathtt{b})(\mathtt{a},\mathtt{a},\mathtt{b},\mathtt{c})$ gives periodic sequences $(\mathtt{abab}\dots,\mathtt{baba}\dots)(\mathtt{abab}\dots,\mathtt{baba}\dots)(\mathtt{aabc}\dots,\mathtt{abca}\dots,\mathtt{bcaa}\dots,\mathtt{caab}\dots)$.

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To recover π from the multiset of necklaces:

- Replace each entry with the periodic sequence obtained by reading around the necklace.
- Order these sequences lexicographically (breaking tikes consistently).

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We get $\pi = (3, 6)(2, 5)(1, 4, 7, 8)$.

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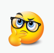

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However, those of *odd length* are primitive 😀

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In this bijection, the role of descents is played by ascents, and the role of the lexicographic order is played by the *alternating lexicographic order*.

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However, those of *odd length* are primitive 😀 and **distinct**! 😮

$\mathcal{M}_n^o =$ multisets of distinct necklaces of odd length.

We get a bijection

$$\Xi_S : \{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} \to \{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathrm{wt}(S)\}.$$

The bijections so far

$\mathcal{M}_n^e =$ multisets of necklaces of even length,
except possibly for one necklace of length one.

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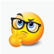

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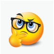

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We will interpret multisets of necklaces as words.

Lyndon words

A Lyndon word is a primitive word that is lexicographically smaller than all of its cyclic rotations. Denote the set of Lyndon words by $\mathcal{L} \subseteq \mathcal{W}$.

Example: $\texttt{aabab} \in \mathcal{L}$, but $\texttt{ababa} \notin \mathcal{L}$.

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We identify necklaces with Lyndon words, e.g., $(\mathtt{a}, \mathtt{b}, \mathtt{a}, \mathtt{b}, \mathtt{a}) \leftrightarrow \mathtt{aabab}$.

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Theorem (Lyndon '58)

Every $w \in \mathcal{W}$ has a unique Lyndon factorization $w = \ell_1|\ell_2|\dots|\ell_m$ where $\ell_i \in \mathcal{L}$ for all i, and $\ell_1 \geq \ell_2 \geq \dots \geq \ell_m$ lexicographically.

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$w = \mathtt{dedccedcdbdbdaabd}$

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We identify multisets of necklaces with words.

The bijections so far

Define the following sets of length-n words:

$\mathcal{W}_n^e =$ words all of whose Lyndon factors have even length,
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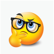

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We want a weight-preserving bijection between $\mathcal{W}_n^o$ and $\mathcal{W}_n^e$.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Given $w \in \mathcal{W}_n^o$ (suppose n is even), initially set $O = w$ and $E = -$.

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Set $\Psi(w) = E$.

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Example

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A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

- Let $O = o_1|o_2|\dots|o_m$ be the Lyndon factorization of O.
- If $|o_m| \geq 2$, write $o_m = r{\mid}s$ where s is its smallest proper suffix.
- Say that o_m is *splittable* if $s < o_{m-1}$ (where $o_{m-1} := \infty$ if $m = 1$).
- $$(O, E) \mapsto \begin{cases} (o_1 o_2 \dots o_{m-1} r,\ sE) & \text{if } o_m \text{ splittable and } |s| \text{ even,} \\ (o_1 o_2 \dots o_{m-1} s,\ rE) & \text{if } o_m \text{ splittable and } |r| \text{ even,} \\ (o_1 o_2 \dots o_{m-2},\ o_m o_{m-1} E) & \text{if } o_m \text{ not splittable.} \end{cases}$$

Example

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	–	cdedadbccc	$= \Psi(w)$

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A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ is a weight-preserving bijection.

A weight preserving bijection $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$

Theorem

The map $\Psi : \mathcal{W}_n^o \to \mathcal{W}_n^e$ is a weight-preserving bijection.

Composing the three maps, we obtain the desired bijection:

$$\begin{array}{ccc}
\{\pi \in \mathcal{S}_n^o : \mathrm{Asc}(\pi) \subseteq S\} & \overset{\Phi_S^{-1}\circ\Psi\circ\Xi_S}{\rightsquigarrow} & \{\pi \in \mathcal{S}_n^e : \mathrm{Des}(\pi) \subseteq S\} \\
\Xi_S \downarrow & & \downarrow \Phi_S \\
\{M \in \mathcal{M}_n^o : \mathrm{wt}(M) = \mathrm{wt}(S)\} & & \{M \in \mathcal{M}_n^e : \mathrm{wt}(M) = \mathrm{wt}(S)\} \\
||| & & ||| \\
\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = \mathrm{wt}(S)\} & \xrightarrow{\Psi} & \{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = \mathrm{wt}(S)\}
\end{array}$$

Bonus: a short generating function proof

$$|\{w \in \mathcal{W}_n^o : \mathrm{wt}(w) = x_1^{\alpha_1} x_2^{\alpha_2} \dots\}| = |\{w \in \mathcal{W}_n^e : \mathrm{wt}(w) = x_1^{\alpha_1} x_2^{\alpha_2} \dots\}|$$

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Proof:

- Write $\mathcal{L} = \mathcal{L}^o \sqcup \mathcal{L}^e$, separating Lyndon words of odd and even length.

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$$\sum_{w \in \mathcal{W}^o} \mathrm{wt}(w) = \prod_{\ell \in \mathcal{L}^o} (1 + \mathrm{wt}(\ell))\,.$$

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$$\sum_{w \in \mathcal{W}^e} \mathrm{wt}(w) = (1 + x_1 + x_2 + \dots) \prod_{\ell \in \mathcal{L}^e} \frac{1}{1 - \mathrm{wt}(\ell)}.$$

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- Thus, what we want to prove is

$$\prod_{\ell \in \mathcal{L}^o} (1 + \mathrm{wt}(\ell)) = (1 + x_1 + x_2 + \dots) \prod_{\ell \in \mathcal{L}^e} \frac{1}{1 - \mathrm{wt}(\ell)},$$

Bonus: a short proof using generating functions

or equivalently,

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- Substituting $x_i \mapsto -x_i$ for all i, this is equivalent to

$$\prod_{\ell \in \mathcal{L}^o} (1 - \mathrm{wt}(\ell)) \prod_{\ell \in \mathcal{L}^e} (1 - \mathrm{wt}(\ell)) = 1 - x_1 - x_2 - \dots ,$$

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$$\prod_{\ell \in \mathcal{L}} \frac{1}{1 - \mathrm{wt}(\ell)} = \frac{1}{1 - x_1 - x_2 - \dots}.$$

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- But this holds because every word has a unique Lyndon factorization! $\square$