Descents on quasi-Stirling permutations

Sergi Elizalde

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Let $\pi = \pi_1 \pi_2 \ldots \pi_r$ be a sequence of positive integers. $i$ is a descent of $\pi$ if $\pi_i > \pi_{i+1}$ or $i = r$. 

Definition
Descents

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\( \text{des}(\pi) = \) number of descents of \( \pi \).

**Example**

\( \text{des}(36522131) = 5 \)
Eulerian polynomials

\[ S_n = \text{set of permutations of } \{1, 2, \ldots, n\}. \]
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Eulerian polynomials:

\[ A_n(t) = \sum_{\pi \in S_n} t^{\text{des} (\pi)} \]

These polynomials appear in work of Euler from 1755.
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\[ A_1(t) = t \]
\[ A_2(t) = t + t^2 \]
\[ A_3(t) = t + 4t^2 + t^3 \]
\[ A_4(t) = t + 11t^2 + 11t^3 + t^4 \]

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Eulerian polynomials

\begin{align*}
\alpha &= \frac{1}{1(p-1)} \\
\beta &= \frac{p + 1}{1.2 (p-1)^2} \\
\gamma &= \frac{pp + 4p + 1}{1.2.3 (p-1)^3} \\
\delta &= \frac{p^3 + 11p^2 + 11p + 1}{1.2.3.4 (p-1)^4} \\
\varepsilon &= \frac{p^4 + 26p^3 + 66p^2 + 26p + 1}{1.2.3.4.5 (p-1)^5} \\
\zeta &= \frac{p^5 + 57p^4 + 302p^3 + 302p^2 + 57p + 1}{1.2.3.4.5.6 (p-1)^6} \\
\eta &= \frac{p^6 + 120p^5 + 1191p^4 + 2416p^3 + 1191p^2 + 120p + 1}{1.2.3.4.5.6.7 (p-1)^7}
\end{align*}

L. Euler, 1755.

Eulerian Polynomials

\[ \frac{A_n(p)}{p^{n!}(p-1)^n} \quad (1 \leq n \leq 7) \]
Euler was considering the series

\[
\sum_{m \geq 0} m t^m = \frac{t}{(1-t)^2}
\]

\[
\sum_{m \geq 0} m^2 t^m = \frac{t + t^2}{(1-t)^3}
\]

\[
\sum_{m \geq 0} m^3 t^m = \frac{t + 4t^2 + t^3}{(1-t)^4}
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In general,

\[ \sum_{m \geq 0} m^n t^m = \frac{A_n(t)}{(1 - t)^{n+1}}. \]
Stirling numbers

**Definition**

The Stirling number of the second kind $S(n, k)$ is the number of partitions of the set $\{1, 2, \ldots, n\}$ into $k$ blocks.

What are the polynomials in the numerator?
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In 1978, Gessel and Stanley were interested in the series

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\sum_{m \geq 0} S(m + 1, m) \ t^m = \frac{t}{(1 - t)^3}
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\]

\[
\sum_{m \geq 0} S(m + 3, m) \ t^m = \frac{t + 8t^2 + 6t^3}{(1 - t)^7}
\]

\[
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What are the polynomials in the numerator?
**Definition (Gessel–Stanley ’78)**

A *Stirling permutation* is a permutation of the multiset \(\{1, 1, 2, 2, \ldots, n, n\}\) that avoids the pattern 212.

<table>
<thead>
<tr>
<th>Example</th>
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<tbody>
<tr>
<td>(Q_2 = {1122, 1221, 2211})</td>
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We have \(|Q_n|\) = \((2n-1)!! = (2n-1) \cdot (2n-3) \cdot \ldots \cdot 3 \cdot 1\), since every permutation in \(Q_n\) can be obtained by inserting \(nn\) into one of the \(2n-1\) spaces of a permutation in \(Q_{n-1}\).
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In other words, Stirling permutations \(\pi_1\pi_2\ldots\pi_{2n}\) satisfy that, if \(i < j < k\) and \(\pi_i = \pi_k\), then \(\pi_j > \pi_i\).
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\[ Q_n(t) = \sum_{\pi \in Q_n} t^{\text{des}(\pi)} \]

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\[ Q_1(t) = t \]
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\[ \sum_{m \geq 0} S(m + n, m) t^m = \frac{Q_n(t)}{(1 - t)^{2n+1}}. \]
There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- **Bóna '08**: \( Q_n(t) \) also gives the enumeration of \( Q_n \) by the number of plateaus, that is, positions \( i \) such that \( \pi_i = \pi_{i+1} \).
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- **Haglund and Visontai ’12**: The multivariable polynomials tracking these 3 statistics are stable (i.e., they don’t vanish when all the variables have a positive imaginary part).
- The coefficients of $Q_n(t)$ are sometimes called second-order Eulerian numbers.
\[
\mathcal{I}_n = \text{set of increasing edge-labeled plane rooted trees with } n \text{ edges.}
\]
Stirling permutations and trees

\[ \mathcal{I}_n = \text{set of increasing edge-labeled plane rooted trees with } n \text{ edges.} \]

\[
\begin{array}{c}
\begin{array}{cccc}
4 & 1 & 2 \\
6 & 7 & 5 & 3
\end{array}
\end{array}
\]

\[ \varphi \quad 4664112775885332 \]

**Theorem (Janson ’08)**

There is a bijection \( \varphi : \mathcal{I}_n \longrightarrow \mathcal{Q}_n \) obtained by traversing the edges of the tree along depth-first walk from left to right, and recording their labels.
\( \mathcal{I}_n \) = set of increasing edge-labeled plane rooted trees with \( n \) edges.

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There is a bijection \( \varphi : \mathcal{I}_n \rightarrow \mathcal{Q}_n \) obtained by traversing the edges of the tree along depth-first walk from left to right, and recording their labels.

If we remove the increasing condition on the trees, what is the image of \( \varphi \)?
\begin{itemize}
    \item \(T_n = \) set of edge-labeled plane rooted trees with \(n\) edges.
\end{itemize}
\( \mathcal{T}_n = \) set of edge-labeled plane rooted trees with \( n \) edges.

\[
\begin{array}{c}
1 & 7 & 5 & 2 \\
   & 4 & 6 & 3 \\
   &     &   & 8 \\
\end{array}
\]

\( \varphi \rightarrow 4114663775885223 \)
Quasi-Stirling permutations and trees

\[ \mathcal{T}_n = \text{set of edge-labeled plane rooted trees with } n \text{ edges.} \]

Definition (Archer–Gregory–Pennington–Slayden ’19)

A quasi-Stirling permutation is a permutation of the multiset \{1, 1, 2, 2, \ldots, n, n\} that avoids the patterns 1212 and 2121.
Quasi-Stirling permutations and trees

\[ T_n = \text{set of edge-labeled plane rooted trees with } n \text{ edges.} \]

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A quasi-Stirling permutation is a permutation of the multiset \{1, 1, 2, 2, \ldots, n, n\} that avoids the patterns 1212 and 2121. In other words, it does not have four positions \( i < j < k < \ell \) with \( \pi_i = \pi_k \) and \( \pi_j = \pi_\ell \) (i.e., it is non-crossing).
Quasi-Stirling permutations

\( \overline{Q}_n \) = set of quasi-Stirling permutations of \( \{1, 1, 2, 2, \ldots, n, n\} \).

**Example**

\( \overline{Q}_2 = \{1122, 1221, 2211, 2112\} \)
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The number of unlabeled plane rooted trees with \( n \) edges is the Catalan number \( C_n \).
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Example

$\overline{Q}_2 = \{1122, 1221, 2211, 2112\}$

Theorem (Archer–Gregory–Pennington–Slayden ’19)

$\varphi$ is a bijection between $T_n$ and $\overline{Q}_n$.

The number of unlabeled plane rooted trees with $n$ edges is the Catalan number $C_n$.

It follows that

$|\overline{Q}_n| = n! C_n = \frac{(2n)!}{(n+1)!}$. 
Conjecture (Archer–Gregory–Pennington–Slayden ’19)

The number of $\pi \in \mathcal{Q}_n$ with des($\pi$) = $n$ is equal to $(n + 1)^{n-1}$. 
Conjecture (Archer–Gregory–Pennington–Slayden ’19)

The number of $\pi \in \overline{Q}_n$ with $\text{des}(\pi) = n$ is equal to $(n + 1)^{n-1}$.

<table>
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<tr>
<td>Set of $\pi \in \overline{Q}_3$ with $\text{des}(\pi) = 1$: ${112233}$</td>
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<td>with $\text{des}(\pi) = 2$:</td>
</tr>
<tr>
<td>${112332, 113223, 113322, 122133, 122331, 133122, 211233, 221133, 223113, 223311, 233112, 311223, 331122}$</td>
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<tr>
<td>with $\text{des}(\pi) = 3$:</td>
</tr>
<tr>
<td>${123321, 132231, 133221, 211332, 213312, 221331, 231132, 233211, 311322, 312213, 321123, 322113, 322311, 331221, 332112, 332211}$</td>
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Conjecture (Archer–Gregory–Pennington–Slayden ’19)

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One can show that $\text{des}(\pi) \leq n$ for all $\pi \in \overline{Q}_n$. 

Sergi Elizalde  Descents on quasi-Stirling permutations
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One can show that $\text{des}(\pi) \leq n$ for all $\pi \in \overline{Q}_n$.

To prove this conjecture, we look at how descents are transformed by the bijection $\varphi$. 
Lemma

If $T \in \mathcal{T}_n$ and $\pi = \varphi(T) \in \overline{Q}_n$, then

$$\text{des}(\pi) = \text{cdes}(T),$$

where $\text{cdes}(T)$ is obtained by adding the number of cyclic descents of the edge labels counterclockwise around each vertex of $T$. 

Example

\[ \begin{array}{cccccccc}
4 & 6 & 7 & 3 & 2 & 5 & 8 & 1 \\
\end{array} \rightarrow \\
\begin{array}{cccccccc}
4 & 1 & 1 & 4 & 6 & 6 & 3 & 7 & 7 & 5 & 8 & 8 & 5 & 2 & 2 \end{array} \]

$\text{des}(\pi) = 6$

$\text{cdes}(T) = 1 + 2 + 1 + 2 = 6$
Lemma

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Example

![Graph Diagram](image)
Descents on quasi-Stirling permutations

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Example

$$\text{cdes}(T) = 1 + 2 + 1 + 2 = 6$$

$$\pi = 4114663775885223$$

$$\text{des}(\pi) = 6$$
Lemma

If \( T \in T_n \) and \( \pi = \varphi(T) \in Q_n \), then

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\text{des}(\pi) = \text{cdes}(T),
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where \( \text{cdes}(T) \) is obtained by adding the number of cyclic descents of the edge labels counterclockwise around each vertex of \( T \).

Example

\[
\begin{align*}
\text{cdes}(T) &= 1 + 2 + 1 + 2 = 6 \\
\text{des}(\pi) &= 6
\end{align*}
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Descents on quasi-Stirling permutations

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If \( T \in \mathcal{T}_n \) and \( \pi = \varphi(T) \in \overline{Q}_n \), then

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Example

\[
\begin{align*}
\varphi & \quad \rightarrow \\
4 & \rightarrow 1 & 2 \rightarrow 4114663775885223 \\
6 & \rightarrow 3 & \text{des}(\pi) = 6 \\
3 & \rightarrow 5 & cdes(T) = 1 + 2 + 1 + 2 = 6
\end{align*}
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If $T \in T_n$ and $\pi = \varphi(T) \in Q_n$, then

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Example

$$\varphi \quad \rightarrow \quad 4114663775885223$$

cdes$(T) = 1 + 2 + 1 + 2 = 6$

des$(\pi) = 6$
Lemma

If $T \in T_n$ and $\pi = \varphi(T) \in \overline{Q}_n$, then

$$\text{des}(\pi) = \text{cdes}(T),$$

where $\text{cdes}(T)$ is obtained by adding the number of cyclic descents of the edge labels counterclockwise around each vertex of $T$.

Example

![Diagram](image)

$$\text{cdes}(T) = 1 + 2 + 1 + 2 = 6$$

$$\text{des}(\pi) = 6$$
The number of $\pi \in \overline{Q}_n$ with $\text{des}(\pi) = n$ is equal to $(n + 1)^{n-1}$. 

Proof sketch. Equivalent to counting $T \in T_n$ with $\text{cdes}(T) = n$, i.e., trees where the number of cyclic descents around each vertex equals its number of children. Such trees are in bijection with unordered trees: 

\[
\begin{array}{cccccccc}
4 & 6 & 2 & 3 & 7 & 5 & 8 & 1 \\
\end{array}
\]

By Cayley’s formula, there are $(n + 1)^{n-1}$ such trees.
Theorem

The number of $\pi \in \overline{Q}_n$ with $\text{des}(\pi) = n$ is equal to $(n + 1)^{n-1}$.

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Equivalent to counting $T \in \mathcal{T}_n$ with $c\text{des}(T) = n$,
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$$\begin{align*}
\text{unordered} & \quad \leftrightarrow \\
\text{having } \text{cdes}(T) = n
\end{align*}$$
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Proof sketch.

Equivalent to counting $T \in \mathcal{T}_n$ with $\text{cdes}(T) = n$, i.e., trees where the number of cyclic descents around each vertex equals its number of children. Such trees are in bijection with unordered trees:

By Cayley’s formula, there are $(n + 1)^{n-1}$ such trees.
More generally, we are interested in the distribution of des on $\overline{Q}_n$. 

Define the quasi-Stirling polynomials $Q_n(t) = \sum_{\pi \in Q_n} t^{\text{des}(\pi)}$.

Example

$Q_1(t) = t$

$Q_2(t) = t + 3t^2$

$Q_3(t) = t + 13t^2 + 16t^3$
More generally, we are interested in the distribution of des on $\overline{Q}_n$. Define the \textbf{quasi-Stirling polynomials}

$$\overline{Q}_n(t) = \sum_{\pi \in \overline{Q}_n} t^{\text{des}(\pi)}.$$
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Define their exponential generating function (EGF):

$$\overline{Q}(t, z) = \sum_{n \geq 0} \overline{Q}_n(t) \frac{z^n}{n!}.$$
Recall the Eulerian polynomials

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does has a well-known closed form

$$A(t, z) = \frac{1 - t}{1 - te^{(1-t)z}}.$$ 

Now we are ready to give an expression for $\overline{Q}(t, z)$. 
The EGF \( \overline{Q}(t, z) \) for quasi-Stirling permutations by the number of descents satisfies the implicit equation

\[
\overline{Q}(t, z) = A(t, z \overline{Q}(t, z)),
\]

that is,

\[
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The EGF $Q(t, z)$ for quasi-Stirling permutations by the number of descents satisfies the implicit equation

$$Q(t, z) = A(t, zQ(t, z)),$$

that is,

$$Q(t, z) = \frac{1 - t}{1 - te^{(1-t)zQ(t,z)}}.$$

Its coefficients satisfy

$$Q_n(t) = \frac{n!}{n+1} [z^n]A(t, z)^{n+1}.$$

Here $[z^n]F(z)$ denotes the coefficient of $z^n$ in $F(z)$. 
Proof ideas

By the bijection \( \varphi \),

\[
\overline{Q}(t, z) = \sum_{n \geq 0} \sum_{T \in \mathcal{T}_n} t^{\text{cdes}(T)} \frac{z^n}{n!}.
\]
By the bijection $\varphi$,

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\overline{Q}(t, z) = \sum_{n \geq 0} \sum_{T \in T_n} t^{\text{cdes}(T)} \frac{z^n}{n!}.
$$

Decompose trees in $T_n$ as

$$
T = \begin{array}{c}
\begin{array}{c}
T_1 \\
T_2 \\
\vdots \\
T_r
\end{array}
\end{array}
$$

$$
a_1 \quad a_2 \quad a_r
$$
Proof ideas

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$$

Decompose trees in $\mathcal{T}_n$ as

$T = a_1 a_2 \ldots a_r$

and use that

$$
\text{cdes}(T) = \sum_{i=1}^{r} (\text{cdes}(T_i) - 1) + \text{des}(a_1 a_2 \ldots a_r).
$$
Proof ideas

The EGF for each piece $T_i$ is $zQ(t, z)$.
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The EGF for each piece $T_i$ is $z\overline{Q}(t, z)$.

Combining the pieces while keeping track of cdes and using the Compositional Formula, we get

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Combining the pieces while keeping track of cdes and using the Compositional Formula, we get

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Finally, extracting its coefficients using Lagrange inversion gives

$$\overline{Q}_n(t) = \frac{n!}{n+1} [z^n] A(t, z)^{n+1}.$$
Consequences

Recall the formulas:

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\sum_{m \geq 0} m^n t^m = \frac{A_n(t)}{(1 - t)^{n+1}} \quad \text{(Eulerian)}
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\[ \sum_{m \geq 0} ??? t^m = \frac{\overline{Q}_n(t)}{(1 - t)^{2n+1}} \quad \text{(quasi-Stirling)} \]
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**Open**: Find a combinatorial proof.
Recall: $i$ is a plateau of $\pi$ if $\pi_i = \pi_{i+1}$,
$i$ is an ascent of $\pi$ if $\pi_i < \pi_{i+1}$ or $i = 0$. 
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**Theorem (Bóna ’08)**

*On average, Stirling permutations in $Q_n$ have $\frac{2n+1}{3}$ ascents, $\frac{2n+1}{3}$ descents, and $\frac{2n+1}{3}$ plateaus.*
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**Theorem**

On average, quasi-Stirling permutations in $\overline{Q}_n$ have $(3n + 1)/4$ ascents, $(3n + 1)/4$ descents, and $(n + 1)/2$ plateaus.
Theorem (Frobenius)

*The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.*
### Properties of quasi-Stirling polynomials

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Corollary: The coefficients of $Q_n(t)$ are unimodal and log-concave.

The distribution of the number of descents on $Q_n$ converges to a normal distribution as $n \to \infty$. 
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The same holds for the quasi-Stirling polynomials $\overline{Q}_n(t)$.

Corollary

- The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.
- The distribution of the number of descents on $\overline{Q}_n$ converges to a normal distribution as $n \to \infty$. 
Proving real-rootedness of $\overline{Q}_n(t)$ is more complicated than for $A_n(t)$ or $Q_n(t)$, because for quasi-Stirling permutations there is no simple recursive description relating $\overline{Q}_n$ and $\overline{Q}_{n-1}$.
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Our proof expresses $\overline{Q}_n(t)$ in terms of $r$-Eulerian polynomials, defined by Riordan and Foata–Schützenberger.
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In the process, we show that

$$\#\{\pi \in Q_n \text{ with } m + 1 \text{ descents}\} = \#\{\text{injections } [n-1] \to [2n] \text{ with } m \text{ excedances}\}.$$
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**Open:** Find a bijective proof.
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**Definition (Gessel–Stanley ’78)**

A **\( k \)-Stirling permutation** is a permutation of the multiset \( \{1^k, 2^k, \ldots, n^k\} \) that avoids the pattern 212.
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Also studied by Brenti, Park, Janson, Kuba, Panholzer, etc.
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**$k$-Stirling and $k$-quasi-$k$-Stirling permutations**

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A **$k$-quasi-$k$-Stirling permutation** is a permutation of the multiset \( \{1^k, 2^k, \ldots, n^k\} \) that avoids the patterns 1212 and 2121.

\[ \overline{Q}_n^k = \text{set of } k\text{-quasi-$k$-Stirling permutations}. \]

For $k = 1$, \( Q_n^1 = \overline{Q}_n^1 = S_n \). For $k = 2$, \( Q_n^2 = Q_n \) and \( \overline{Q}_n^2 = \overline{Q}_n \).
Enumeration of $k$-Stirling and $k$-quasi-Stirling permutations

Counting $k$-Stirling permutations is easy, since every permutation in $Q^k_n$ can be obtained by inserting the string $n^k = nn\ldots n$ into one of the $(n - 1)k + 1$ spaces of a permutation in $Q^k_{n-1}$, so

$$|Q^k_n| = (k + 1)(2k + 1)\cdot\cdot\cdot((n - 1)k + 1).$$
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$$|Q_n^k| = (k + 1)(2k + 1) \cdot \cdots \cdot ((n - 1)k + 1).$$

**Theorem**

For $n \geq 1$ and $k \geq 1$,

$$|Q_n^k| = \frac{(kn)!}{((k - 1)n + 1)!} = n! \ C_{n,k},$$

where

$$C_{n,k} = \frac{1}{(k - 1)n + 1} \binom{kn}{n}$$

is the $n$th $k$-Catalan number.
Gessel’94 & Janson–Kuba–Panholzer’11 describe bijections between $k$-Stirling permutations and two kinds of decorated increasing trees.

Example

A bijection between compartmented trees and $3$-quasi-Stirling permutations:

\[
\begin{array}{cccccccccccccc}
5 & \rightarrow & 622266355537744471113
\end{array}
\]
$k$-quasi-Stirling permutations and trees

Gessel’94 & Janson–Kuba–Panholzer’11 describe bijections between $k$-Stirling permutations and two kinds of decorated increasing trees. We have extended them to bijections between $k$-quasi-Stirling permutations and certain trees.
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Let asc(\(\pi\)) and plat(\(\pi\)) be the number of ascents and plateaus of \(\pi\).
Ascents, descents and plateaus on $k$-quasi-Stirling permutations

Let $\text{asc}(\pi)$ and $\text{plat}(\pi)$ be the number of ascents and plateaus of $\pi$. Consider the homogenization of the Eulerian polynomials

$$\hat{A}_n(q, t) = \sum_{\pi \in S_n} q^{\text{asc}(\pi)} t^{\text{des}(\pi)},$$

and their EGF

$$\hat{A}(q, t) = \sum_{n \geq 0} \hat{A}_n(q, t) z^n = 1 - q + q(q - t) q^{-te(q - t)} z.$$
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Define the multivariate $k$-quasi-Stirling polynomials

$$\overline{P}^{(k)}_n(q, t, u) = \sum_{\pi \in \overline{Q}^k_n} q^{\text{asc}(\pi)} t^{\text{des}(\pi)} u^{\text{plat}(\pi)},$$

and their EGF

$$\overline{P}^{(k)}(q, t, u; z) = \sum_{n \geq 0} \overline{P}^{(k)}_n(q, t, u) \frac{z^n}{n!}.$$
This is the most general version of our main result:

**Theorem**

\[
\overline{P}^{(k)}(q, t, u; z) \text{ satisfies the implicit equation }
\]

\[
\overline{P}^{(k)}(q, t, u; z) = \hat{A}(q, t; z(\overline{P}^{(k)}(q, t, u; z) - 1 + u)^{k-1}).
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Extracting its coefficients using Lagrange inversion,

$$\overline{P}^{(k)}_n(q, t, u) = \frac{n!}{(k - 1)n + 1} [z^n] \left( \hat{A}(q, t; z) - 1 + u \right)^{(k-1)n+1}.$$
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*Extracting its coefficients using Lagrange inversion,*

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The proof follows ascents, descents and plateaus through the bijection \( \phi \), and it uses a decomposition of compartmented trees.
For $k$-Stirling permutations, similar ideas give a nice differential equation for the EGF

$$P^{(k)}(q, t, u; z) = \sum_{n \geq 0} \sum_{\pi \in Q_n^k} q^{\text{asc}(\pi)} t^{\text{des}(\pi)} u^{\text{plat}(\pi)} \frac{z^n}{n!}.$$
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**Theorem**

$P(z) := P^{(k)}(q, t, u; z)$ satisfies the differential equation

$$P'(z) = (P(z) - 1 + q)(P(z) - 1 + t)(P(z) - 1 + u)^{k-1},$$

with initial condition $P(0) = 1$. 
Proof idea:

- $\phi$ restricts to a bijection between $k$-Stirling permutations and \textit{increasing} compartmented trees.
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- $\phi$ restricts to a bijection between $k$-Stirling permutations and increasing compartmented trees.
- These trees can be decomposed as

\[
\begin{align*}
T_0 & \quad P - 1 + q \\
T_1 & \quad P - 1 + u \\
\ldots & \quad \ldots \\
T_k & \quad P - 1 + u \\
1 & \quad \ldots \\
T_{k+1} & \quad P - 1 + t
\end{align*}
\]
Ascents, descents and plateaus on $k$-Stirling permutations

Proof idea: $\phi$ restricts to a bijection between $k$-Stirling permutations and increasing compartmented trees. These trees can be decomposed as:

$$T_0 + T_1 + \cdots + T_k + u$$

Sergi Elizalde

Descents on quasi-Stirling permutations