# Schur-positive statistics on 321-avoiding permutations and other Catalan objects

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# Integer-valued statistics

Let $\mathcal{A}_n$ be a finite set, and let $\mathrm{st} : \mathcal{A}_n \to \mathbb{Z}_{\geq 0}$ be an integer-valued statistic.

The distribution of st over $\mathcal{A}_n$ can be encoded by a polynomial

$$\sum_{a \in \mathcal{A}_n} x^{\mathrm{st}(a)}.$$

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**Example**

$$F_{3,\{2\}} = \sum_{i_1\le i_2<i_3} x_{i_1}x_{i_2}x_{i_3} = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots + x_1x_2x_3 + x_1x_2x_4 + \ldots$$

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A common case: $\mathcal{A}_n \subseteq \mathcal{S}_n$, $\mathrm{St} = \mathrm{Des}$, where

$$\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}.$$

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$$Q_{\mathrm{Des}}(\mathcal{S}_3) = F_{3,\emptyset} + 2F_{3,\{1\}} + 2F_{3,\{2\}} + F_{3,\{1,2\}}.$$

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## Example ($\lambda = (2,1)$)

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$s_{(2,1)} =$
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$$s_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + \cdots + 2x_1x_2x_3 + 2x_1x_3x_4 + \cdots$$

Schur functions form a basis of the ring of symmetric functions.
A symmetric function is Schur-positive if all coefficients in its Schur expansion are nonnegative.

## Back to the previous example

**Example**

$$\begin{aligned} Q_{\mathrm{Des}}(\mathcal{S}_3) &= F_{3,\emptyset} + 2F_{3,\{1\}} + 2F_{3,\{2\}} + F_{3,\{1,2\}} \\ &= (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + \dots) \\ &\quad + 2(x_1x_2^2 + x_1x_2x_3 + \dots) + 2(x_1^2x_2 + x_1x_2x_3 + \dots) \\ &\quad + (x_1x_2x_3 + \dots) \\ &= s_{(3)} + 2s_{(2,1)} + s_{(1,1,1)}. \end{aligned}$$

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For which pairs $(\mathcal{A}_n, \mathrm{St})$ is $Q_{\mathrm{St}}(\mathcal{A}_n)$ symmetric and Schur-positive?

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Today's goal: describe other Schur-positive pairs $(\mathcal{A}_n, \mathsf{St})$.
We will focus on Catalan objects, i.e., where $|\mathcal{A}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$.

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## Example

$|\mathsf{SYT}(n, n)| = C_n$ and $Q_{\mathsf{Des}}(\mathsf{SYT}(n, n)) = s_{(n,n)}$.

# 321-avoiding permutations

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**Definition**

$\pi \in \mathcal{S}_n$ is indecomposable if $\{\pi_1,\dots,\pi_i\} \neq \{1,\dots,i\}$ for all $1 \le i < n$ (i.e., we can't write $\pi = \sigma \oplus \tau$).

$\mathcal{S}^*_n(321) =$ indecomposable permutations in $\mathcal{S}_n(321)$.

A little less well known: $|\mathcal{S}^*_n(321)| = C_{n-1}$.

# Indecomposable 321-avoiding permutations

**Theorem [Adin–Bagno–Roichman '18]**

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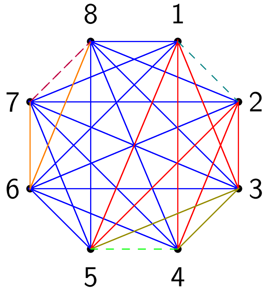
## Example

| $\pi \in \mathcal{S}_4^*(321)$ | $\mathrm{Des}(\pi)$ |
|---|---|
| 2341 | $\{3\}$ |
| 2413 | $\{2\}$ |
| 3142 | $\{1,3\}$ |
| 3412 | $\{2\}$ |
| 4123 | $\{1\}$ |

$$Q_{\mathrm{Des}}(\mathcal{S}_n^*(321)) = F_{4,\{1\}} + 2F_{4,\{2\}} + F_{4,\{3\}} + F_{4,\{1,3\}} = s_{3,1} + s_{2,2}.$$

# Transitive partitions

Let $c : E(K_n) \to \{1, \ldots, m\}$ be an edge-coloring of the complete graph, and consider the resulting partition of the edges of $K_n$.



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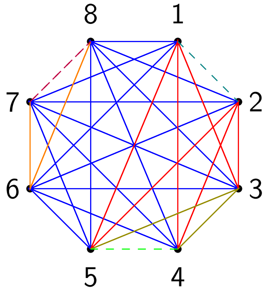


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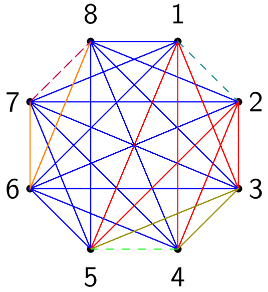


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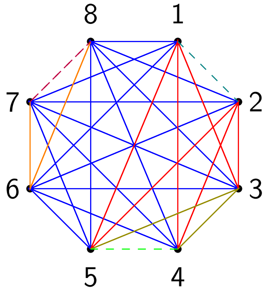


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Let $\mathcal{P}_n$ = maximal transitive partitions of $K_n$.

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**Theorem [Adin et al. '25]**

$|\mathcal{P}_n| = C_{n-1}$, and

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Problem (Adin et al.): Find a bijective proof.

We provide one, and we find other families of equidistributed statistics.

# Main result: nine set-valued statistics

## Theorem [E.–Roichman '26+]

There are explicit bijections showing that the following set-valued statistics on Catalan objects are equidistributed:

a. Des on indecomposable 321-avoiding permutations $\mathcal{S}_n^*(321)$
b. Sng on maximal transitive partitions $\mathcal{P}_n$
c. Che on binary trees $\mathcal{B}_n$
d. $\Gamma$ on elevated Dyck paths $\mathcal{D}_n^*$
e. $\overline{\Gamma}$ on Dyck paths $\mathcal{D}_{n-1}$
f. $\overline{\mathrm{Des}}$ on 321-avoiding permutations $\mathcal{S}_{n-1}(321)$
g. $\mathrm{Pea}_{\leq n-1}$ on Dyck paths $\mathcal{D}_{n-1}$
h. Ear on triangulations $\mathcal{T}_{n+1}$
i. Sho on noncrossing alternating trees $\mathcal{N}_n$

# Corollary: Schur-positivity

## Corollary [E.–Roichman, 26+]

For each of the nine pairs $(\mathcal{A}_n, \mathrm{St})$ from the theorem, we have

$$Q_{\mathrm{St}}(\mathcal{A}_n) = \sum_{k=1}^{\lfloor n/2 \rfloor} |\mathrm{SYT}(n-k-1, k-1)| \; s_{(n-k,k)}.$$

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## Corollary [E.–Roichman, 26+]

For each of the nine pairs $(\mathcal{A}_n, \mathrm{St})$, the integer-valued statistics $\mathrm{st}(a) = |\mathrm{St}(a)|$ are equidistributed, and

$$\sum_{n \geq 0} \sum_{a \in \mathcal{A}_n} t^{\mathrm{st}(a)} z^n = \frac{1 - \sqrt{1 - 4z + 4(1-t)z^2}}{2z}.$$

# Cherries in binary trees

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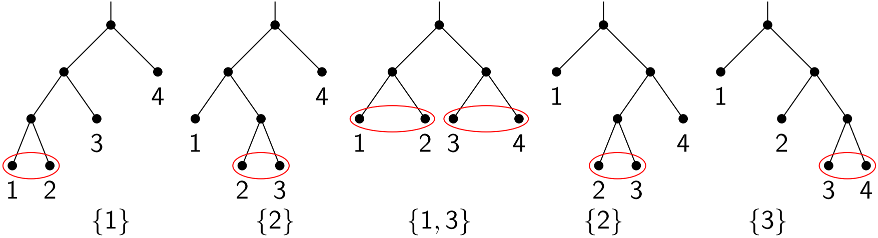
Label the leaves of $B \in \mathcal{B}_n$ with $1, \dots, n$ from left to right.

$$\mathrm{Che}(B) = \{i : \text{leaves } i, i+1 \text{ have a common parent}\}.$$



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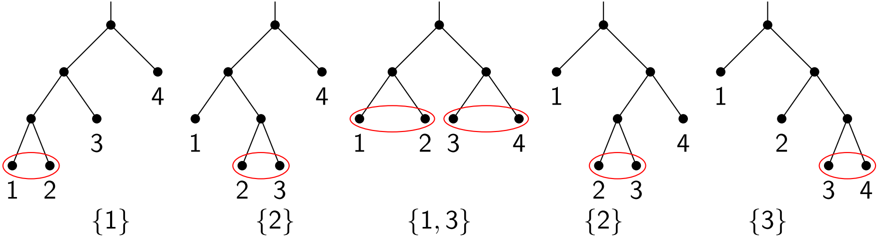


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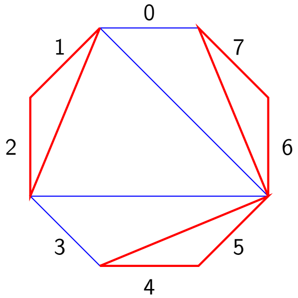
Label the edges counterclockwise with $0, 1, \ldots, n$ from left to right.

$$\text{Ear}(T) = \{i \in [n-1] : \text{edges } i, i+1 \text{ are in the same triangle}\}.$$




$\text{Ear}(T) = \{1, 4, 6\}$

# Occurrences of nee on Dyck paths

$\mathcal{D}_n =$ Dyck paths from $(0,0)$ to $(n,n)$ with $\mathtt{n}$ and $\mathtt{e}$ steps.
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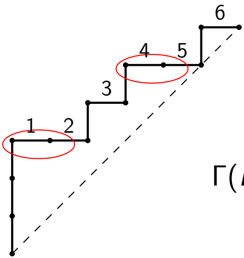
$\Gamma(D) = \{i :$ the $i$th $\mathtt{e}$ step of $D$ is in the middle of an occurrence of $\mathtt{nee}\}$,

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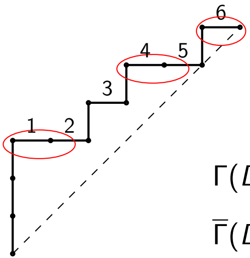
$$\overline{\Gamma}(D) = \begin{cases} \Gamma(D) \cup \{n\} & \text{if } D \text{ ends with } \texttt{ne}, \\ \Gamma(D) & \text{otherwise.} \end{cases}$$

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# Bijection $(\mathcal{S}_n(321), \overline{\mathrm{Des}}) \leftrightarrow (\mathcal{D}_n, \overline{\Gamma})$

For $\pi \in \mathcal{S}_n$, define

$$\overline{\mathrm{Des}}(\pi) = \begin{cases} \mathrm{Des}(\pi) \cup \{n\} & \text{if } \pi_n = n, \\ \mathrm{Des}(\pi) & \text{otherwise.} \end{cases}$$

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For $\pi \in \mathcal{S}_n$, define

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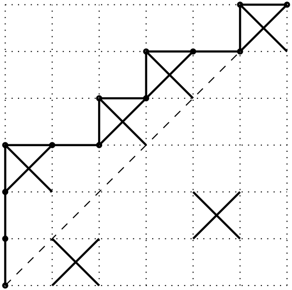
Krattenthaler's bijection $\kappa : \mathcal{S}_n(321) \to \mathcal{D}_n$ satisfies



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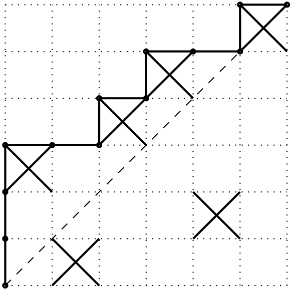

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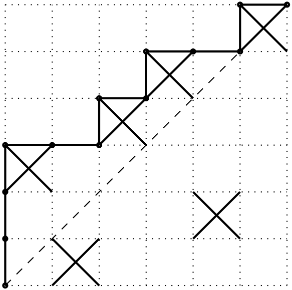

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$$\mathrm{Des}(\pi) = \Gamma(\kappa(\pi)),$$
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Additionally,

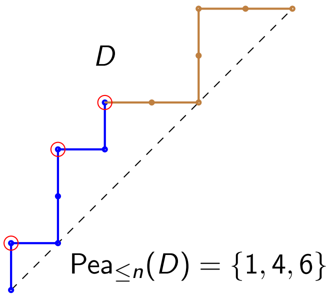
$$\pi \text{ indecomposable} \iff \kappa(\pi) \text{ elevated},$$

so we also get $(\mathcal{S}_n^*(321), \mathrm{Des}) \leftrightarrow (\mathcal{D}_n^*, \Gamma)$.

## Bijection $(\mathcal{S}_n(321), \overline{\mathrm{Des}}) \leftrightarrow (\mathcal{D}_n, \mathrm{Pea}_{\le n})$

For $D \in \mathcal{D}_n$, let

$$\mathrm{Pea}(D) = \{i \in [2n-1] : \text{steps } i \text{ and } i+1 \text{ of } D \text{ form a peak } \mathtt{ne}\},$$
$$\mathrm{Pea}_{\le n}(D) = \mathrm{Pea}(D) \cap [n].$$



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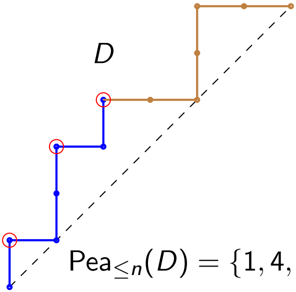
$$\pi \quad \xrightarrow{\text{RSK}} \quad (P, Q)$$

$$3{\cdot}145{\cdot}26{\cdot} \;\rightarrow\; \left( \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \right) \;\rightarrow$$




$\mathrm{Pea}_{\leq n}(D) = \{1, 4, 6\}$

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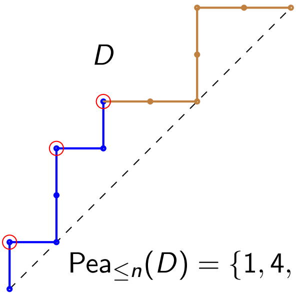


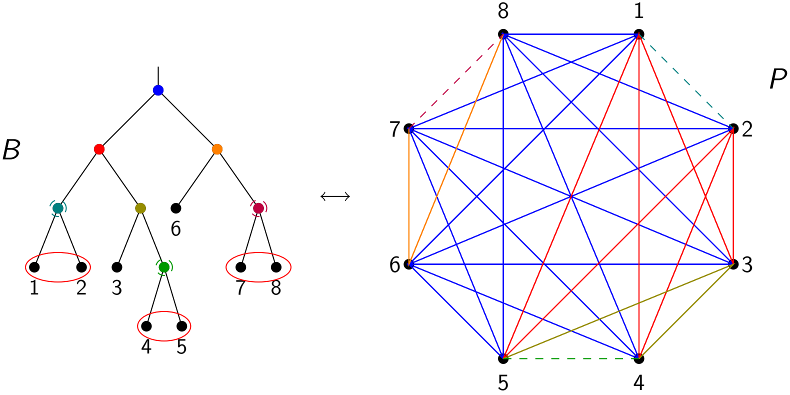

$\mathrm{Pea}_{\leq n}(D) = \{1, 4, 6\}$

One can prove that

$$\overline{\mathrm{Des}}(\pi) = \mathrm{Pea}_{\leq n}(D).$$

# Bijection $(\mathcal{B}_n, \mathrm{Che}) \leftrightarrow (\mathcal{P}_n, \mathrm{Sng})$

Given $B \in \mathcal{B}_n$, build a maximal transitive partition $P$ of $E(K_n)$: each internal node $v$ of $B$ determines a block of $P$ consisting of the edges $\{(i,j) : i \in \text{left subtree of } v,\ j \in \text{right subtree of } v\}$.




$$\mathrm{Che}(B) = \mathrm{Sng}(P)$$

# Generalization to Schröder objects

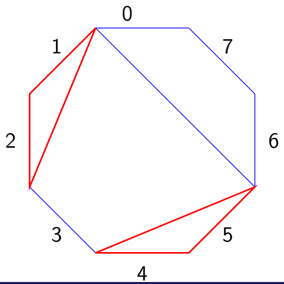
## Theorem [E.–Roichman, 26+]

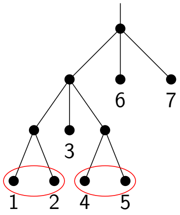
The following set-valued statistics on Schröder objects are equidistributed:

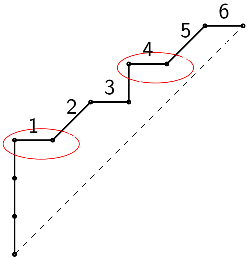
(a) Che on *Schröder trees* on $n$ leaves (rooted plane trees where no vertex has exactly one child)

(b) Ear on polygon dissections of a regular $(n+1)$-gon

(c) $\overline{\Gamma}$ on little Schröder paths from $(0,0)$ to $(n-1,n-1)$ (similar to Dyck paths but allowing $\mathtt{d}=(1,1)$ steps not on the diagonal)



Thank you!