Refined enumeration of pattern-avoiding permutations

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Overview

- Pattern avoidance
  - Definitions
  - Exact enumeration
  - Asymptotic enumeration
  - Simultaneous avoidance
  - Consecutive patterns
  - Generalized patterns

- Statistics on pattern-avoiding permutations
  - Equidistribution results of fixed points and excedances
  - Statistics on Dyck paths
  - Bijective proofs
  - Statistics and simultaneous avoidance
  - Statistics and generalized patterns

Basic definitions

$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_k$

π *contains* σ if there exist $i_1 < \ldots < i_k$ s.t. $\pi_{i_1}\cdots\pi_{i_k}$ has type σ (i.e., $\pi_{i_a} < \pi_{i_b} \iff \sigma_a < \sigma_b$)

Otherwise, we say that π *avoids* σ

Example:
<u>2</u>4<u>53</u>1 contains 132
42351 avoids 132

$\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$

Natural questions:

- What can we say about $|\mathcal{S}_n(\sigma)|$? Exact formula? Asymptotic formula?
- For which σ_1 and σ_2 do we have $|\mathcal{S}_n(\sigma_1)| = |\mathcal{S}_n(\sigma_2)|$?

Patterns of length 3

By trivial bijections,

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$$

$$|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$$

Knuth '73.

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

(Catalan number)

Patterns of length 4

West '90, Stankova '94, '96.

Patterns $\sigma \in \mathcal{S}_4$ fall in three different classes:

(a) 1234 $\longrightarrow$ enumerated by I. Gessel

(b) 1342 $\longrightarrow$ enumerated by M. Bóna

(c) 1324 $\longrightarrow$ no formula known for $|\mathcal{S}_n(1324)|$

Bóna '97. For $n \geq 7$,

$$|\mathcal{S}_n(1342)| < |\mathcal{S}_n(1234)| < |\mathcal{S}_n(1324)|$$

Bóna '97.

$$\sum_{n\geq 0} |\mathcal{S}_n(1342)| z^n = \frac{32z}{1 + 20z - 8z^2 - (1-8z)^{3/2}}$$

Proof: bijection between *indecomposable* 1324-avoiding permutations and a certain kind of labelled trees

Patterns of arbitrary length

Gessel '90.

$$\sum_{n\geq 0} \frac{|\mathcal{S}_n(123\cdots k)|}{n!^2} z^{2n} = \det(I_{|r-s|}(2z))_{r,s=1,\ldots,k-1}$$

where $I_j(2z) = \sum_{n\geq 0} \frac{z^{2n+j}}{n!(n-j)!}$ are Bessel functions of imaginary argument

Babson, West '03. For all r, t, n,

$$|\mathcal{S}_n(123\cdots r a_{r+1} a_{r+2}\cdots a_{r+t})| = |\mathcal{S}_n(r\cdots 321 a_{r+1} a_{r+2}\cdots a_{r+t})|$$

Asymptotic enumeration

Regev '81.

$$|S_n(123\cdots k)| \sim c\,\frac{(k-1)^{2n}}{n^{\frac{k^2-2k}{2}}}$$

Stanley-Wilf Conjecture '90.
For every pattern σ, there exists λ s.t.

$$|\mathcal{S}_n(\sigma)| < \lambda^n$$

for all n.

Proved in Nov **'03** by **Marcus** and **Tardos**
Idea of the proof:

- Generalize avoidance to 0-1 matrices
- For a 0-1 matrix P, let $f(n,P) :=$ max # of 1's in an $n \times n$ 0-1 matrix avoiding P
- Main result: $f(n,P) = O(n)$
- The theorem follows from a result of Klazar

Simultaneous avoidance

Defn:

$$\mathcal{S}_n(\sigma_1, \ldots, \sigma_m) = \bigcap_{i=1}^{m} \mathcal{S}_n(\sigma_i)$$

Simion, Schmidt '85. Formula for $|\mathcal{S}_n(\Sigma)|$ for every $\Sigma \subseteq \mathcal{S}_3$. Examples:
$|\mathcal{S}_n(123, 132)| = 2^{n-1}$
$|\mathcal{S}_n(132, 321)| = \binom{n}{2} + 1$
$|\mathcal{S}_n(123, 132, 213)| = F_{n+1}$ (Fibonacci number)
$|\mathcal{S}_n(123, 132, 231)| = n$

West '96. Formulas for $|\mathcal{S}_n(\sigma_1, \sigma_2)|$ where $\sigma_1 \in \mathcal{S}_3$, $\sigma_2 \in \mathcal{S}_4$. Examples:
$|\mathcal{S}_n(123, 3241)| = 3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$
$|\mathcal{S}_n(123, 3214)| = F_{2n}$

Proof uses generating trees

Gire '93, Kremer '00, West '96.
$|\mathcal{S}_n(\sigma_1, \sigma_2)| = r_{n-1}$ (large Schröder number)
for several pairs $\sigma_1, \sigma_2 \in \mathcal{S}_4$

Consecutive patterns

Defn:

π *contains* the *consecutive pattern* σ if $\exists$ i s.t. $\pi_{i+1} \cdots \pi_{i+k}$ has type σ

Example:

1<u>352</u>4 contains 231

23541 avoids 231

$$c_\sigma(\pi) := \#\text{ occurrences of } \sigma \text{ in } \pi$$

$$P_\sigma(u,z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n} u^{c_\sigma(\pi)} \frac{z^n}{n!}$$

E., Noy '00.

a)

$$\sigma = 12\cdots(m+2)$$

$P_\sigma(u,z) = \dfrac{1}{\omega(u,z)}$, where ω is the solution of

$$\omega^{(m+1)} + (1-u)(\omega^{(m)} + \omega^{(m-1)} + \ldots + \omega' + \omega) = 0$$

$\omega(0) = 1,\ \omega'(0) = -1,\ \omega^{(k)}(0) = 0,\ 2 \le k \le m.$

b)

$$\sigma = 12\cdots(a-1)a\underbrace{\boxed{}}_{\substack{\downarrow \\ \text{any perm. of } \{a+2,\, a+3, \ldots, m+2\}}}(a+1)$$

$P_\sigma(u,z) = \dfrac{1}{\xi(u,z)}$, where ξ is the solution of

$$\xi^{(a+1)} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}\xi' = 0$$

$\xi(0) = 1,\ \xi'(0) = -1,\ \xi^{(k)}(0) = 0,\ 2 \le k \le a.$

Proof uses representations of permutations as increasing binary trees

Generalized patterns

(Babson, Steingrímsson '00)

Dashes between some letters of σ

If no dash, elements have to be adjacent in π

Example:

$\underline{35}42\underline{6}1\underline{7}$ is an occurrence of 12-3-4, but

$\underline{3}5\underline{4}2\underline{6}1\underline{7}$ is not

Claesson '01.

$|\mathcal{S}_n(1\text{-}23)| = |\mathcal{S}_n(1\text{-}32)| = B_n$ (Bell number)

$|\mathcal{S}_n(2\text{-}13)| = C_n$

$|\mathcal{S}_n(1\text{-}23, 12\text{-}3)| = B_n^*$ (Bessel number)

$|\mathcal{S}_n(1\text{-}23, 1\text{-}32)| = I_n$ (involutions in $\mathcal{S}_n$)

$|\mathcal{S}_n(1\text{-}23, 13\text{-}2)| = M_n$ (Motzkin number)

Permutation statistics

i is a *fixed point* of π if $\pi_i = i$
$\mathrm{fp}(\pi) :=$ number of fixed points of π

i is an *excedance* of π if $\pi_i > i$
$\mathrm{exc}(\pi) :=$ number of excedances of π

i is a *descent* of π if $\pi_i > \pi_{i+1}$ (otw. i is a *rise*)
$\mathrm{des}(\pi) :=$ number of descents of π

$\mathrm{lis}(\pi) :=$ length of longest increasing subseq. of π

$\mathrm{lds}(\pi) :=$ length of longest decreasing subseq. of π

Ex: if $\pi = 4{\cdot}2{\cdot}17{\cdot}5{\cdot}36$, then
$\mathrm{fp}(\pi) = 2$, $\mathrm{exc}(\pi) = 2$, $\mathrm{des}(\pi) = 4$,
$\mathrm{lis}(\pi) = 3$, $\mathrm{lds}(\pi) = 3$

Robertson, Saracino, Zeilberger '02.

For any k, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k\}| \\ = |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k\}|$$

Their proof is not bijective

Questions:

- Is there a simple bijective proof?
- Can this theorem be generalized, considering other statistics in permutations?

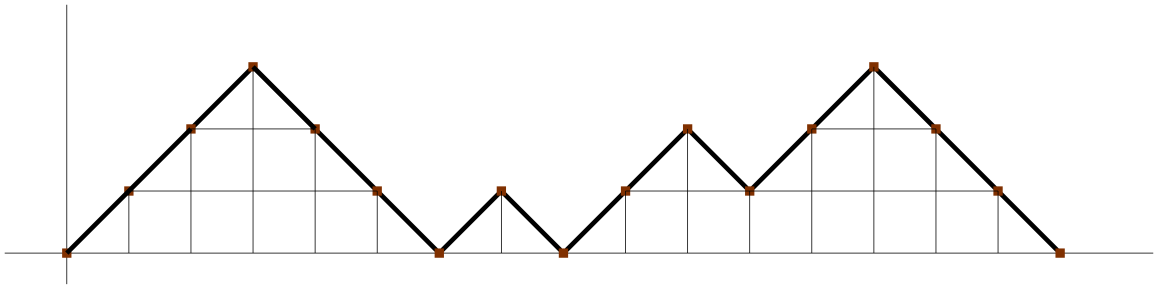
We will see a bijective proof

Idea: bijections between restricted permutations and Dyck paths

Dyck paths and Motzkin paths

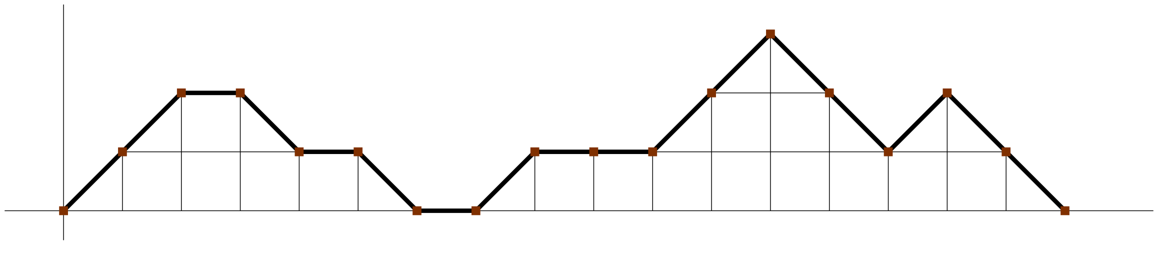
Start and end at the x-axis; never go below it

Dyck path: steps $u = (1,1)$ and $d = (1,-1)$



$\mathcal{D}_n :=$ set of Dyck paths of length $2n$
$|\mathcal{D}_n| = C_n$

Motzkin path: steps $u = (1,1)$, $d = (1,-1)$ and $h = (1,0)$



$\mathcal{M}_n :=$ set of Motzkin paths of length n
$|\mathcal{M}_n| = M_n$

For a Dyck path D, define:

peak: ud (up-step followed by down-step)

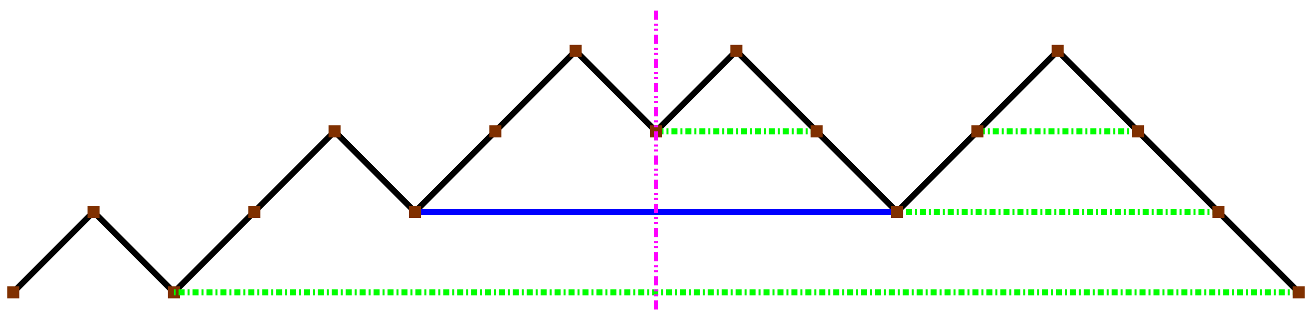
hill: peak at lowest height

tunnel: horizontal segment between two lattice points of D that stays always below D

(D has n tunnels, one for each step u)

centered tunnel: x-coordinate of midpoint is at the middle of D

right tunnel: x-coordinate of midpoint is in the right half of D



A simple bijective proof of

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k\}| \\ = |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k\}|$$

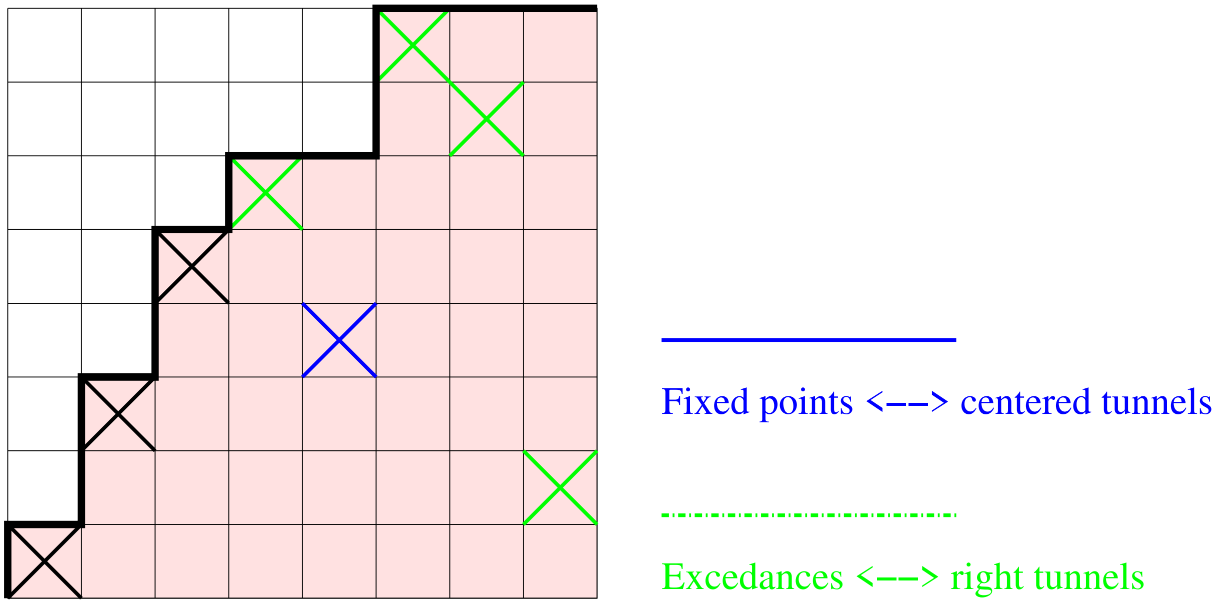
Composition of bijections:

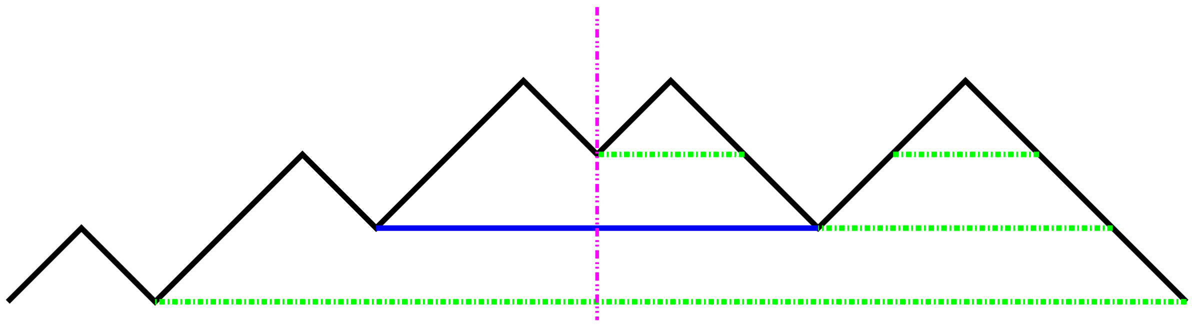
$$\mathcal{S}_n(132) \overset{\varphi_{\mathsf{Krat}}}{\longleftrightarrow} \mathcal{D}_n \overset{\Phi_{\mathsf{ED}}}{\longleftrightarrow} \mathcal{D}_n \overset{\psi_{\mathsf{Stan}}}{\longleftrightarrow} S_n(321)$$

fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills	$\leftrightarrow$	fixed points

$\varphi_{\mathsf{Krat}} : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ (Krattenthaler '01)

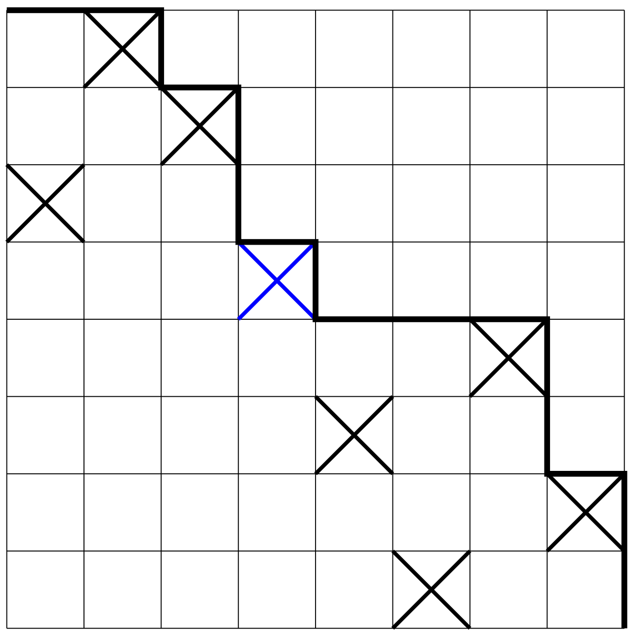
Example: $\pi = 67435281 \in \mathcal{S}_8(132)$





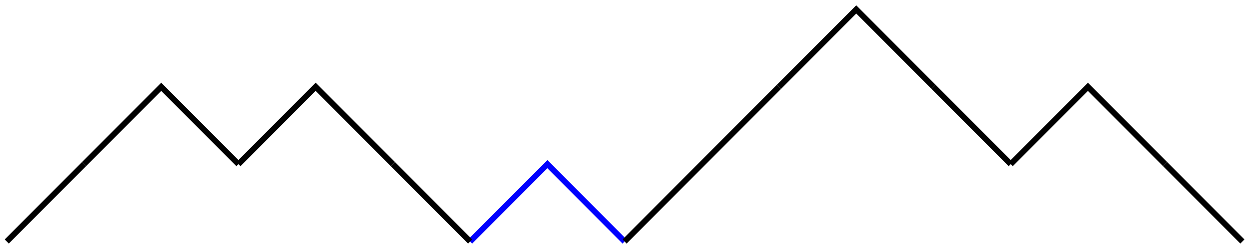
$\psi_{\mathsf{Stan}} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ (Stanley '03)

Example: $\pi = 23147586 \in \mathcal{S}_8(321)$



Fixed points <-–>

hills (= peaks of height 1)

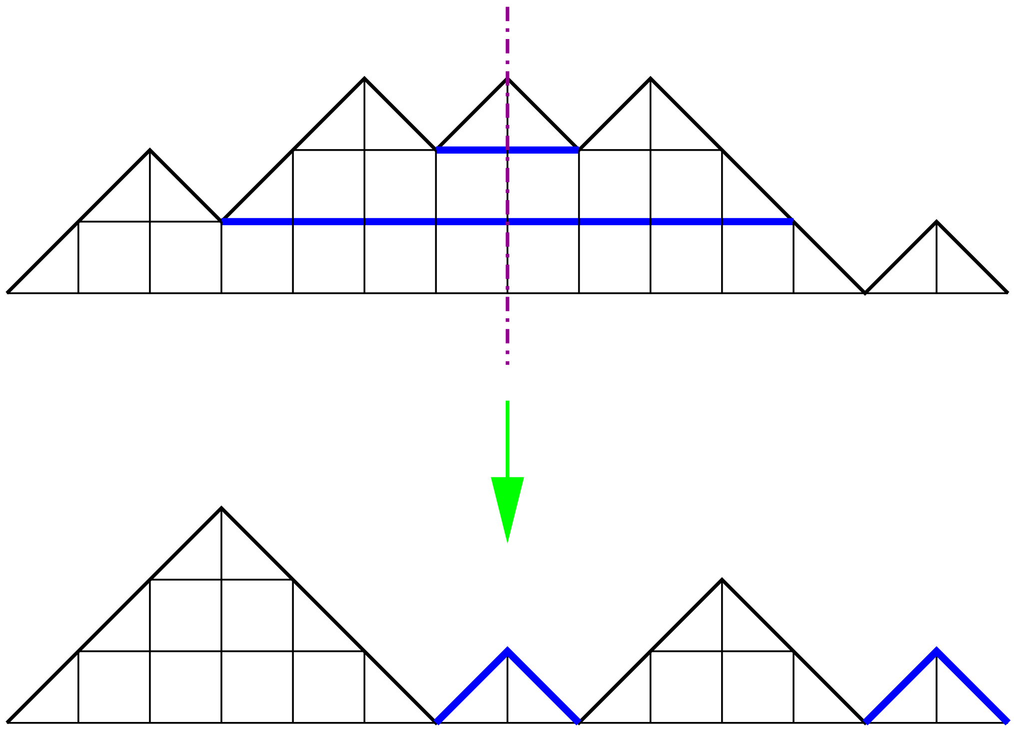


$\Phi_{\text{ED}} : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ (Deutsch, E. '03)

1. Each u in D has a matching d (together they determine a tunnel)

2. Read the steps of D in zigzag:
$1, 2n, 2, 2n-1, \ldots$

3. For each step, if its corresponding matching step has not yet been read, draw an up-step in $\Phi_{\text{ED}}(D)$. Otherwise, draw a down-step



Φ_{ED} maps centered tunnels to hills

$\square$

More generally:

Deutsch, E. '03. Let

$$\alpha_r(\pi) := |\{i : \pi_i = i + r\}|$$

$$\beta_r(\pi) := |\{i : i > r, \pi_i = i\}|$$

Then, for any k, r, n,

$$|\{\pi \in \mathcal{S}_n(132) : \alpha_r(\pi) = k\}| \\ = |\{\pi \in \mathcal{S}_n(321) : \beta_r(\pi) = k\}|$$

Proof uses a generalization of Φ_{ED}

Another application of the bijection

Deutsch, E. '03.

$$1+\sum_{n\ge 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)}q^{\mathsf{exc}(\pi)}p^{\mathsf{des}(\pi)+1}z^n$$
$$=\frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{\spadesuit}}$$

where

$$\spadesuit = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2 - 2(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$$

Proof:

$\mathcal{S}_n(132)$	$\overset{\varphi_{\mathsf{Krat}}}{\longleftrightarrow}$	$\mathcal{D}_n$	$\overset{\Phi_{\mathsf{ED}}}{\longleftrightarrow}$	$\mathcal{D}_n$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	even up-steps
descents+1	$\leftrightarrow$	peaks	$\leftrightarrow$	$\underbrace{\text{"certain statistic"}}_{\text{easier to enumerate}}$

Generalization to excedances

E. '02.
For any k, l, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$

Original proof is analytical and uses nonstandard techniques in generating functions

We will see a bijective proof

A bijective proof (E., Pak '03) of

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$

Composition of bijections:

$$\mathcal{S}_n(132) \xrightarrow{\varphi_{\mathsf{Krat}}} \mathcal{D}_n \xleftarrow{\Psi_{\mathsf{RSK}}} \mathcal{S}_n(321)$$

fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	fixed points
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	excedances

$$\Psi_{\mathsf{RSK}} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$$

Step 1: RSK correspondence

$$\pi \mapsto (P, Q)$$

Example: $\pi = 23514687$

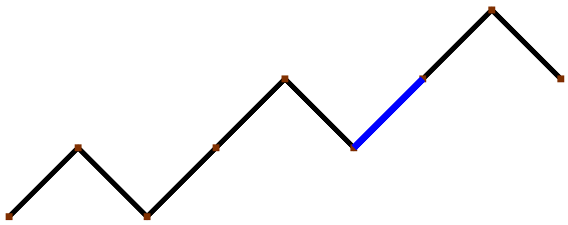
$$\begin{array}{|c|}\hline 2\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1&3&4&6&8\\\hline 2&5\\\cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1&3&4&6&7\\\hline 2&5&8\\\cline{1-3}\end{array}$$

$$P = \begin{array}{|c|c|c|c|c|}\hline 1&3&4&6&7\\\hline 2&5&8\\\cline{1-3}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&6&7\\\hline 4&5&8\\\cline{1-3}\end{array}$$

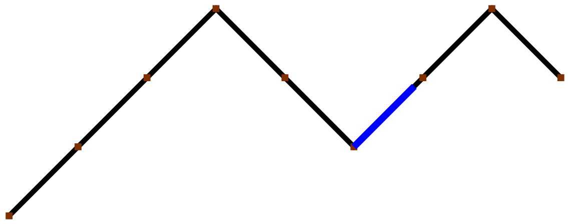
Step 2:

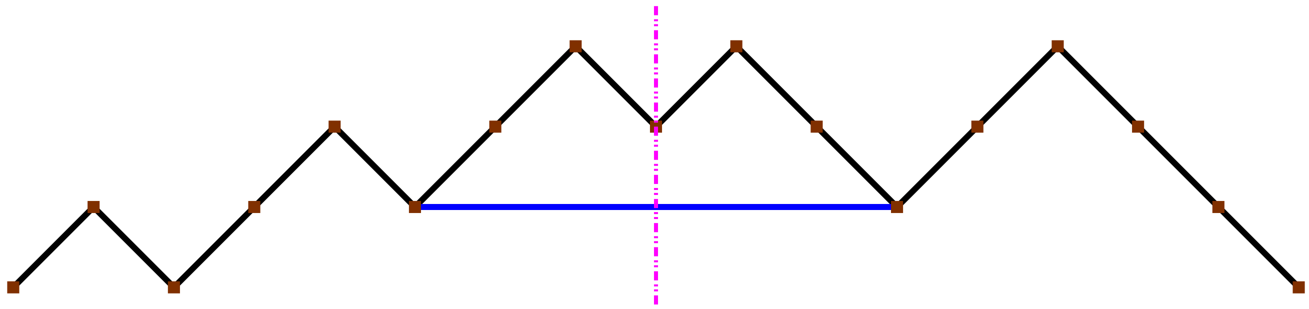
$$(P, Q) \mapsto \Psi_{\mathsf{RSK}}(\pi)$$

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \quad \longrightarrow$$



$$Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array} \quad \longrightarrow$$

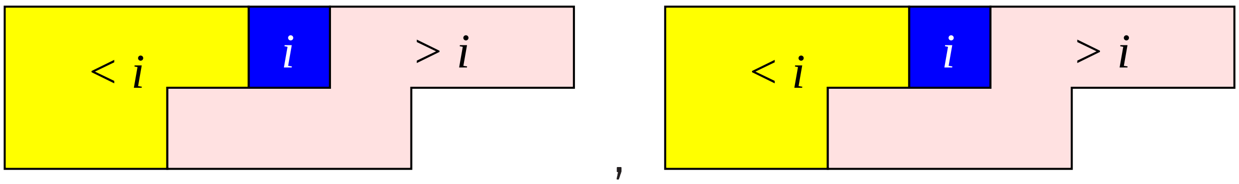




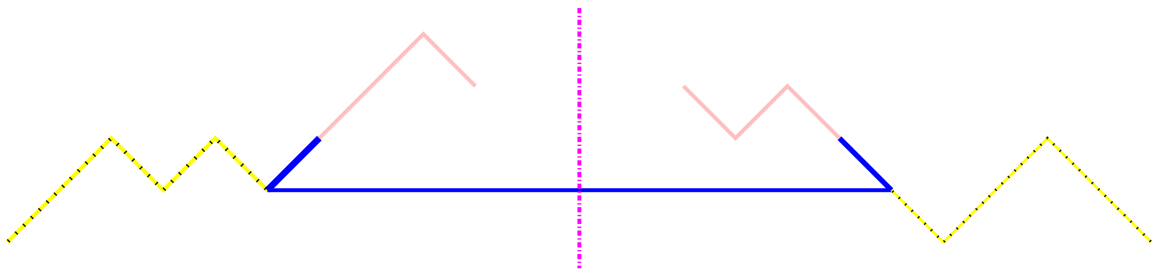
Ψ_{RSK} maps fixed points to centered tunnels:

$$\left.\begin{array}{c} \pi \ 321\text{-avoiding} \\ \pi_i = i \end{array}\right\} \Rightarrow \pi = \underbrace{\cdots\cdots}_{<i} i \underbrace{\cdots\cdots\cdots}_{>i}$$

Then, both P and Q have shape



and i produces a centered tunnel



It can also be checked that Ψ_{RSK} maps excedances to right tunnels

$\square$

Statistics and simultaneous avoidance

$$F_{\sigma_1,\sigma_2} := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\sigma_1,\sigma_2)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} z^n$$

(define $F_{\sigma_1,\sigma_2,\sigma_3}$ similarly)

E. '03. Explicit expressions for F_{σ_1,σ_2} and $F_{\sigma_1,\sigma_2,\sigma_3}$, for any $\sigma_1,\sigma_2,\sigma_3 \in \mathcal{S}_3$.

Examples:

$$F_{132,231} = \frac{1 - z - qz^2 + xqz^3}{(1-xz)(1-z-2qz^2)}$$

$$F_{132,213} = \frac{1-(1+q)z-2qz^2+4q(1+q)z^3-(xq^2+xq+5q^2)z^4+2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}$$

$$F_{123,132,213} = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

Statistics and generalized patterns

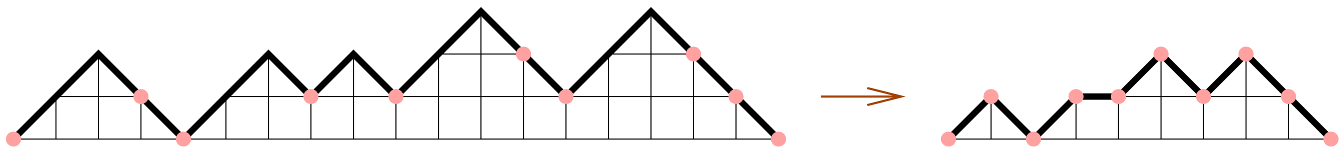
E., Mansour '03.
Bijection between $\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)$ and $\mathcal{M}_n$

$\widehat{\mathcal{D}}_n := \{D \in \mathcal{D}_n$ with no uuu (3 consec. up-steps)$\}$

$$\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23) \xrightarrow{\varphi_{\text{Krat}}} \widehat{\mathcal{D}}_n \xrightarrow{\theta_{\text{EM}}} \mathcal{M}_n$$

lds	$\leftrightarrow$	steps u and h
lis	$\leftrightarrow$	height+1
rises	$\leftrightarrow$	steps u

$$\theta_{\text{EM}} : \widehat{\mathcal{D}}_n \longrightarrow \mathcal{M}_n$$



$$\begin{array}{rcl} uud & \to & u \\ ud & \to & h \\ d & \to & d \end{array}$$

From the bijection,

E., Mansour '03.

$$\sum_{n\geq 0} \sum_{\substack{\pi\in\mathcal{S}_n(132,1\text{-}23,\\ 12\cdots(k+1))}} v^{\mathrm{lds}(\pi)} y^{\#\{\text{rises of } \pi\}} z^n$$

$$= \frac{U_{k-1}\left(\frac{1-vz}{2z\sqrt{vy}}\right)}{z\sqrt{vy}\,U_k\left(\frac{1-vz}{2z\sqrt{vy}}\right)}$$

where $U_m(\cos t) = \frac{\sin(m+1)t}{\sin t}$
(Chebyshev polynomials of the second kind)

Open questions

- Find $|\mathcal{S}_n(\sigma)|$ for other patterns (e.g. $\sigma = 1324$)

- Conjecture (Noonan, Zeilberger '96).
  For all σ, $|\mathcal{S}_n(\sigma)|$ is a P-recursive function of n

- Are there other statistics in restricted permutations having the same distribution for different patterns?

- Find distribution of statistics in permutations avoiding longer patterns (e.g., of length 4)