Pattern-avoiding permutations: old results and new developments

Sergi Elizalde

MIT Combinatorics Seminar
November 2003

Slides available at
http://www-math.mit.edu/~sergi

Overview

- Classical pattern avoidance
  - Definitions
  - Exact enumeration
  - Asymptotic enumeration

- Some generalizations
  - Multiple avoidance
  - Enumeration of occurrences
  - Generalized patterns

- Statistics in pattern-avoiding permutations
  - Equidistribution results
  - Bijective proofs
  - Statistics and multiple avoidance
  - Statistics and generalized patterns

Basic definitions

$\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_k$

π *contains* σ if there exist $i_1 < \ldots < i_k$ s.t. $\pi_{i_1}\cdots\pi_{i_k}$ has type σ (i.e., $\pi_{i_a} < \pi_{i_b} \iff \sigma_a < \sigma_b$)

Otherwise, we say that π *avoids* σ

Example:
24531 contains 132
42351 avoids 132

$\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$

Basic questions:

- What can we say about $|\mathcal{S}_n(\sigma)|$? Exact formula? Asymptotic formula?
- For which σ_1 and σ_2 do we have $|\mathcal{S}_n(\sigma_1)| = |\mathcal{S}_n(\sigma_2)|$?

Patterns of length 3

By trivial bijections,

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$$

$$|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$$

Knuth '73.

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

(Catalan number)

Patterns of length 4

West '90, Stankova '94, '96.

Patterns $\sigma \in \mathcal{S}_4$ fall in three different classes:

(a) 1234 $\longrightarrow$ enumerated by I. Gessel

(b) 1342 $\longrightarrow$ enumerated by M. Bóna

(c) 1324 $\longrightarrow$ no formula known for $|\mathcal{S}_n(1324)|$

Bóna '97. For $n \geq 7$,

$$|\mathcal{S}_n(1342)| < |\mathcal{S}_n(1234)| < |\mathcal{S}_n(1324)|$$

Bóna '97.

$$\sum_{n \geq 0} |\mathcal{S}_n(1342)| z^n = \frac{32z}{1 + 20z - 8z^2 - (1-8z)^{3/2}}$$

Proof: bijection between *indecomposable* 1324-avoiding permutations and a certain kind of labelled trees

Patterns of arbitrary length

Gessel '90 (combinatorial proof in '98).

$$\sum_{n\geq 0} \frac{|\mathcal{S}_n(123\cdots k)|}{n!^2} z^{2n} = \det(I_{|r-s|}(2z))_{r,s=1,\ldots,k-1}$$

where $I_j(2z) = \sum_{n\geq 0} \frac{z^{2n+j}}{n!(n-j)!}$ are Bessel functions of imaginary argument

Babson, West '03. For all r, t, n,

$$|\mathcal{S}_n(123\cdots r a_{r+1} a_{r+2} \cdots a_{r+t})| = |\mathcal{S}_n(r\cdots 321 a_{r+1} a_{r+2} \cdots a_{r+t})|$$

Asymptotic enumeration

Regev '81.

$$|S_n(123\cdots k)| \sim c\,\frac{(k-1)^{2n}}{n^{\frac{k^2-2k}{2}}}$$

Stanley-Wilf Conjecture '90.
For every pattern σ, there exists λ s.t.

$$|\mathcal{S}_n(\sigma)| < \lambda^n$$

for all n.

Open for most patterns. Some partial results:

- Regev '81: True for $\sigma = 123\cdots k$
- Bóna '97: True for $\sigma \in \mathcal{S}_4$
- Bóna '99: True for *layered* patterns.
  Ex: $\sigma = 321|54|876|9$
- Alon, Friedgut '00: $\forall\sigma$, $\exists\lambda$ s.t.

  $$|\mathcal{S}_n(\sigma)| < \lambda^{n\,\gamma^*(n)}$$

  where γ^* is a extremely slow growing function

Some generalizations

- Simultaneous avoidance
  Defn:
  $$\mathcal{S}_n(\sigma_1, \ldots, \sigma_m) = \bigcap_{i=1}^{m} \mathcal{S}_n(\sigma_i)$$

- Enumeration of occurrences of a pattern
  Defn:
  $\mathcal{S}_n^r(\sigma) = \{\pi \in \mathcal{S}_n : \pi \text{ contains } r \text{ occurrences of } \sigma\}$
  Example:
  $\pi = 35142$ has 3 occurrences of 312:
  $$\underline{3}5\underline{1}4\underline{2},\ 3\underline{51}4\underline{2},\ 3\underline{514}2$$

- Generalized patterns (Babson, Steingrímsson '00)
  Dashes between letters of σ
  If no dash, elements have to be adjacent in π
  Example:
  $\underline{35}42\underline{6}1\underline{7}$ is an occurrence of 12-3-4, but
  $3\underline{5}4\underline{2}6\underline{1}7$ is not

Some results in simultaneous avoidance

Simion, Schmidt '85. Formula for $|\mathcal{S}_n(\Sigma)|$ for every $\Sigma \subseteq \mathcal{S}_3$. Examples:
$|\mathcal{S}_n(123, 132)| = 2^{n-1}$
$|\mathcal{S}_n(132, 321)| = \binom{n}{2} + 1$
$|\mathcal{S}_n(123, 132, 213)| = F_{n+1}$ (Fibonacci number)
$|\mathcal{S}_n(123, 132, 231)| = n$

West '96. Formulas for $|\mathcal{S}_n(\sigma_1, \sigma_2)|$ where $\sigma_1 \in \mathcal{S}_3$, $\sigma_2 \in \mathcal{S}_4$. Examples:
$|\mathcal{S}_n(123, 3241)| = 3 \cdot 2^{n-1} - \binom{n+1}{2} - 1$
$|\mathcal{S}_n(123, 3214)| = F_{2n}$

Proof uses generating trees

Gire '93, Kremer '00, West '96.
$|\mathcal{S}_n(\sigma_1, \sigma_2)| = r_{n-1}$ (large Schröder number)
for several pairs $\sigma_1, \sigma_2 \in \mathcal{S}_4$

Chow, West '99.

$$\sum_{n\geq 0} |\mathcal{S}_n(132, 12\cdots k)|\ z^n = \frac{U_{k-1}(\frac{1}{2\sqrt{z}})}{\sqrt{z}\,U_k(\frac{1}{2\sqrt{z}})}$$

where $U_m(\cos t) = \frac{\sin(m+1)t}{\sin t}$
(Chebyshev polynomials of the second kind)

Mansour, Vainshtein '00 (generalizing Robertson, Wilf, Zeilberger '99).

$$\sum_{r,n} |\mathcal{S}_n(132)\cap\mathcal{S}_n^r(12\cdots k)|\ y^r z^n = \cfrac{1}{1-\cfrac{zy^{\binom{0}{k-1}}}{1-\cfrac{zy^{\binom{1}{k-1}}}{1-\cfrac{zy^{\binom{2}{k-1}}}{\cdots}}}}$$

Mansour '02 (bijective proof by **Reifegerste**).
For all n, k, s,

$$|\mathcal{S}_n(132, 12\cdots k)| = |\mathcal{S}_n(132, s(s+1)\cdots k12\cdots(s-1))|$$

Generalized patterns with no dashes
(consecutive positions)

$c_\sigma(\pi) :=$ # occurrences of σ in π

$$P_\sigma(u,z) := \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n} u^{c_\sigma(\pi)}\frac{z^n}{n!}$$

E., Noy '00.

$\boxed{\sigma = 12\cdots(m+2)}$

$P_\sigma(u,z) = \dfrac{1}{\omega(u,z)}$, where ω is the solution of

$$\omega^{(m+1)} + (1-u)(\omega^{(m)} + \omega^{(m-1)} + \ldots + \omega' + \omega) = 0$$
$$\omega(0) = 1,\ \omega'(0) = -1,\ \omega^{(k)}(0) = 0,\ 2 \leq k \leq m.$$

$\boxed{\sigma = 12\cdots(a-1)a\,\tau\,(a+1)}$

where τ any perm. of $\{a+2, a+3, \ldots, m+2\}$

$P_\sigma(u,z) = \dfrac{1}{\xi(u,z)}$, where ξ is the solution of

$$\xi^{(a+1)} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}\xi' = 0$$
$$\xi(0) = 1,\ \xi'(0) = -1,\ \xi^{(k)}(0) = 0,\ 2 \leq k \leq a.$$

Proof uses representations of permutations as increasing binary trees

Generalized patterns

Claesson '01.

$|\mathcal{S}_n(1\text{-}23)| = |\mathcal{S}_n(1\text{-}32)| = B_n$ (Bell number)

$|\mathcal{S}_n(2\text{-}13)| = C_n$

$|\mathcal{S}_n(1\text{-}23, 12\text{-}3)| = B_n^*$ (Bessel number)

$|\mathcal{S}_n(1\text{-}23, 1\text{-}32)| = I_n$ (involutions in $\mathcal{S}_n$)

$|\mathcal{S}_n(1\text{-}23, 13\text{-}2)| = M_n$ (Motzkin number)

Kitaev '03 solves several cases of simultaneous avoidance of patterns of length 3 with no dashes

Permutation statistics

i is a *fixed point* of π if $\pi_i = i$
$\mathrm{fp}(\pi)$:= number of fixed points of π

i is an *excedance* of π if $\pi_i > i$
$\mathrm{exc}(\pi)$:= number of excedances of π

i is a *descent* of π if $\pi_i > \pi_{i+1}$ (otw. i is a *rise*)
$\mathrm{des}(\pi)$:= number of descents of π

$\mathrm{lis}(\pi)$:= length of longest increasing subseq. of π

$\mathrm{lds}(\pi)$:= length of longest decreasing subseq. of π

Ex: if $\pi = 4{\cdot}2{\cdot}17{\cdot}5{\cdot}36$, then
$\mathrm{fp}(\pi) = 2$, $\mathrm{exc}(\pi) = 2$, $\mathrm{des}(\pi) = 4$,
$\mathrm{lis}(\pi) = 3$, $\mathrm{lds}(\pi) = 3$

Robertson, Saracino, Zeilberger '02.

For any k, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathrm{fp}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(132) : \mathrm{fp}(\pi) = k\}|$$

E. '02.

For any k, l, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathrm{fp}(\pi) = k,\ \mathrm{exc}(\pi) = l\}| = |\{\pi \in \mathcal{S}_n(132) : \mathrm{fp}(\pi) = k,\ \mathrm{exc}(\pi) = l\}|$$

Original proofs are analytical and long

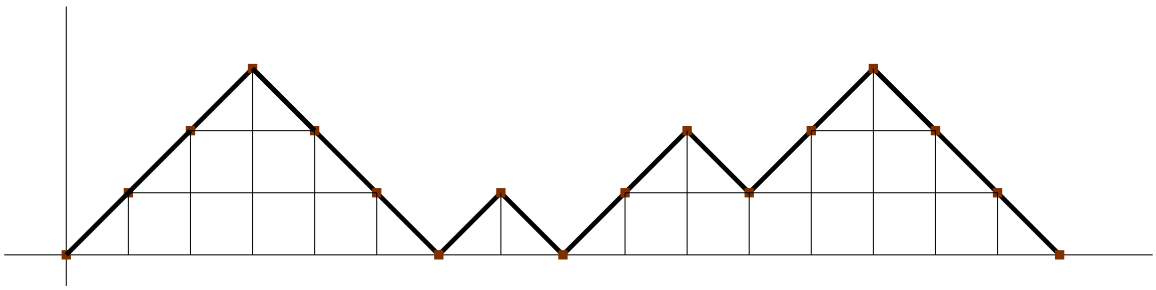
We will see bijective proofs

Idea: bijections between restricted permutations and Dyck paths

Dyck paths and Motzkin paths

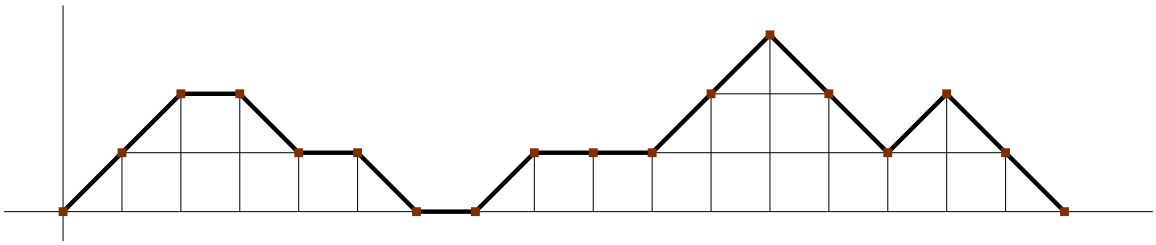
Start and end at the x-axis; never go below it

Dyck path: steps $u = (1, 1)$ and $d = (1, -1)$



$\mathcal{D}_n :=$ set of Dyck paths of length $2n$
$|\mathcal{D}_n| = C_n$

Motzkin path: steps $u = (1, 1)$, $d = (1, -1)$ and $h = (1, 0)$



$\mathcal{M}_n :=$ set of Motzkin paths of length n
$|\mathcal{M}_n| = M_n$

For $D \in \mathcal{D}_n$, define:

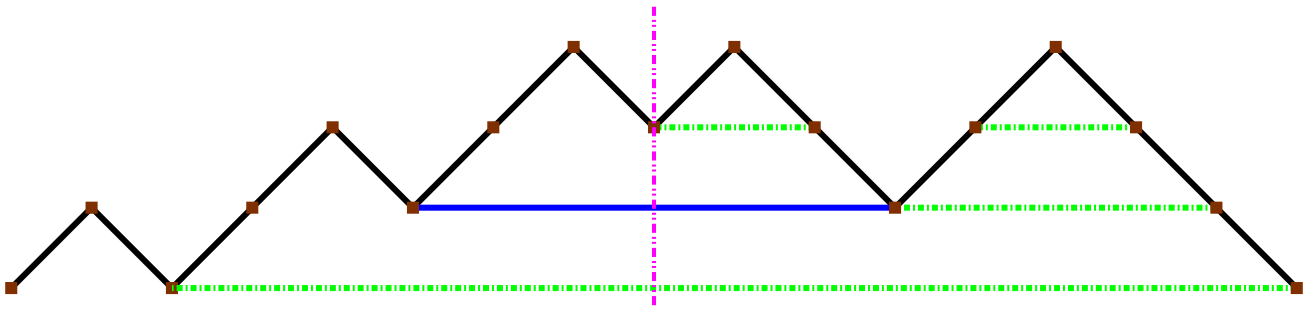
peak: ud (up-step followed by down-step)

hill: peak at lowest height

tunnel: horizontal segment between two lattice points of D that stays always below D
(D has n tunnels, one for each step u)

centered tunnel: x-coordinate of midpoint is n

right tunnel: x-coordinate of midpoint is $> n$



A simple bijective proof of

$$|\{\pi \in \mathcal{S}_n(321) : \mathrm{fp}(\pi) = k\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathrm{fp}(\pi) = k\}|$$

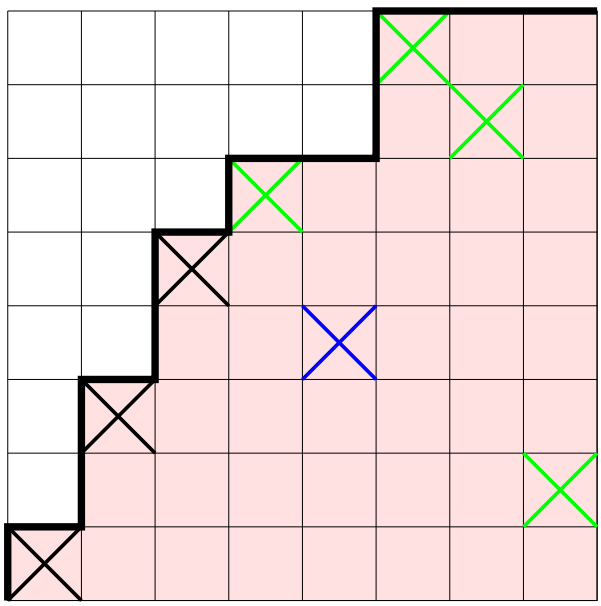
Composition of bijections:

$$\mathcal{S}_n(132) \overset{\varphi_{\mathrm{Krat}}}{\longleftrightarrow} \mathcal{D}_n \overset{\Phi_{\mathrm{ED}}}{\longleftrightarrow} \mathcal{D}_n \overset{\psi_{\mathrm{Stan}}}{\longleftrightarrow} \mathcal{S}_n(321)$$

fixed points $\leftrightarrow$ centered tunnels $\leftrightarrow$ hills $\leftrightarrow$ fixed points

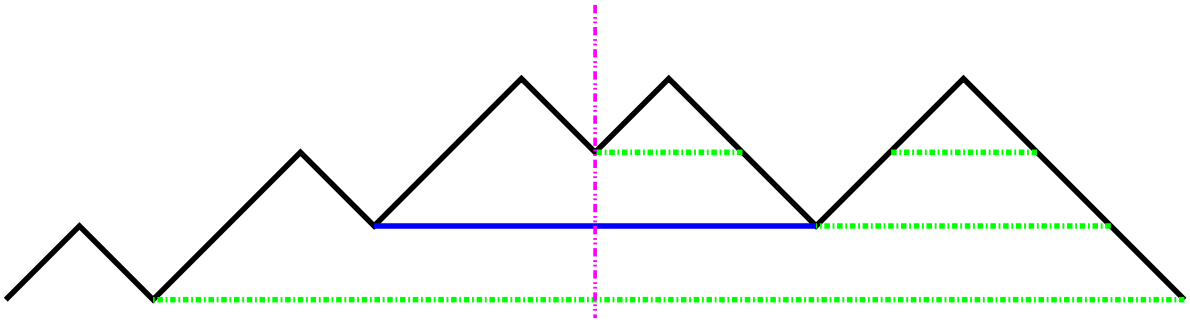
$\varphi_{\text{Krat}} : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ (Krattenthaler '01)

Example: $\pi = 67435281 \in \mathcal{S}_8(132)$



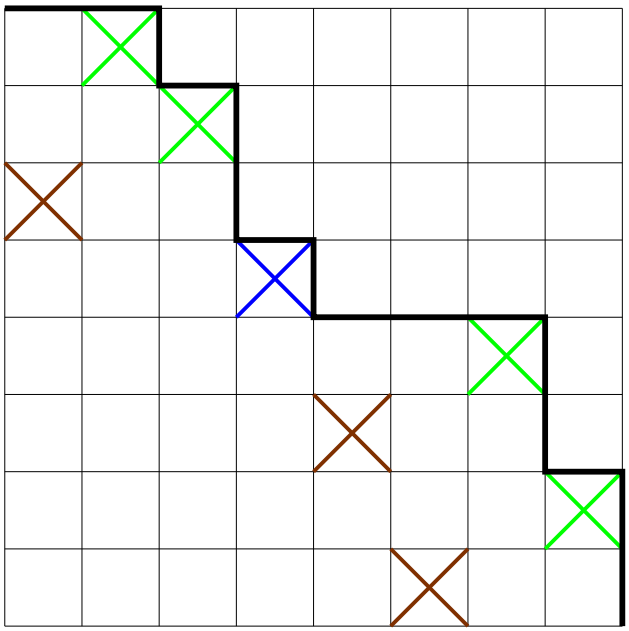
Fixed points <--> centered tunnels

Excedances <--> right tunnels



$\psi_{\mathsf{Stan}} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ (Stanley '03)

Example: $\pi = 23147586 \in \mathcal{S}_8(321)$

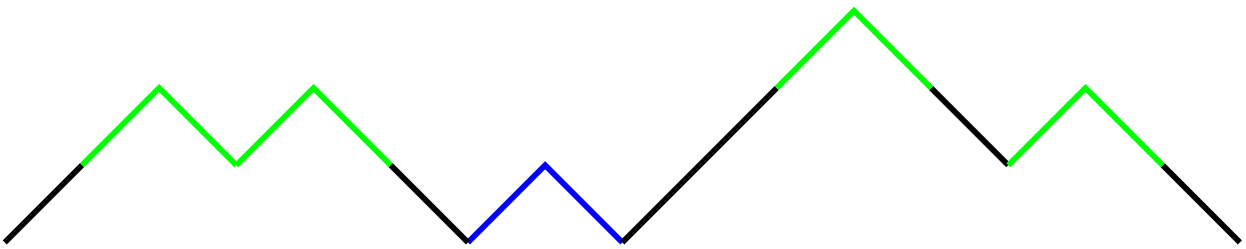


Fixed points <–––>

hills (= peaks of height 1)

Excedances <–––>

peaks of height >1

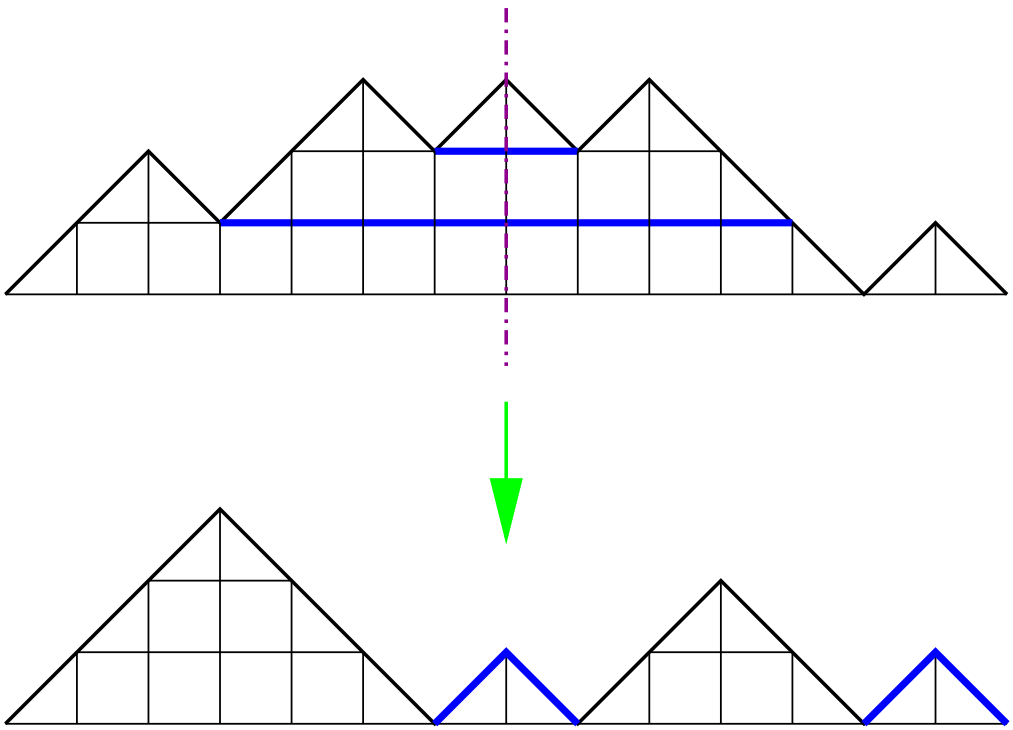


$\Phi_{\mathrm{ED}} : \mathcal{D}_n \longrightarrow \mathcal{D}_n$ (Deutsch, E. '03)

1. Each u in D has a matching d (together they determine a tunnel)

2. Read the steps of D in zigzag: $1, 2n, 2, 2n-1, \ldots$

3. For each step, if its corresponding matching step has not yet been read, draw an up-step in $\Phi_{\mathrm{ED}}(D)$. Otherwise, draw a down-step



Φ_{ED} maps centered tunnels to hills

□

More generally:

Deutsch, E. '03. Let

$$\alpha_r(\pi) := |\{i : \pi_i = i + r\}|$$

$$\beta_r(\pi) := |\{i : i > r, \pi_i = i\}|$$

Then, for any k, r, n,

$$|\{\pi \in \mathcal{S}_n(132) : \alpha_r(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(321) : \beta_r(\pi) = k\}|$$

Proof uses a generalization of Φ_{ED}

Another application of the bijection

Deutsch, E. '03.

$$1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)}q^{\mathsf{exc}(\pi)}p^{\mathsf{des}(\pi)+1}z^n$$
$$=\frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{\spadesuit}}$$

where

$$\spadesuit = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2 - 2(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$$

Proof:

$$\mathcal{S}_n(132) \overset{\varphi_{\mathsf{Krat}}}{\longleftrightarrow} \mathcal{D}_n \overset{\Phi_{\mathsf{ED}}}{\longleftrightarrow} \mathcal{D}_n$$

fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	even up-steps
descents+1	$\leftrightarrow$	peaks	$\leftrightarrow$	"certain statistic"

easier to enumerate

A bijective proof (E., Pak '03) of

$$|\{\pi \in \mathcal{S}_n(321) : \mathrm{fp}(\pi) = k, \mathrm{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathrm{fp}(\pi) = k, \mathrm{exc}(\pi) = l\}|$$

Composition of bijections:

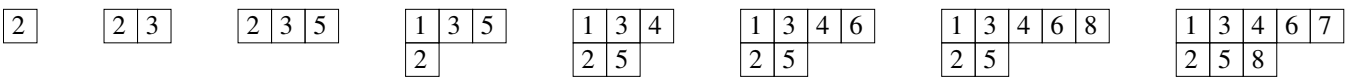
$\mathcal{S}_n(132)$	$\overset{\varphi_{\mathrm{Krat}}}{\longrightarrow}$	$\mathcal{D}_n$	$\overset{\Psi_{\mathrm{RSK}}}{\longleftarrow}$	$\mathcal{S}_n(321)$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	fixed points
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	excedances

$$\Psi_{\mathsf{RSK}} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$$

Step 1: RSK correspondence

$$\pi \mapsto (P, Q)$$

Example: $\pi = 23514687$

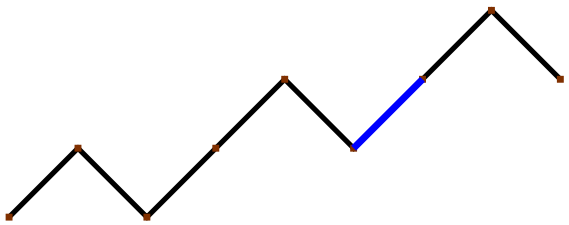


$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array}$$

Step 2: $(P, Q) \mapsto \Psi_{\mathrm{RSK}}(\pi)$

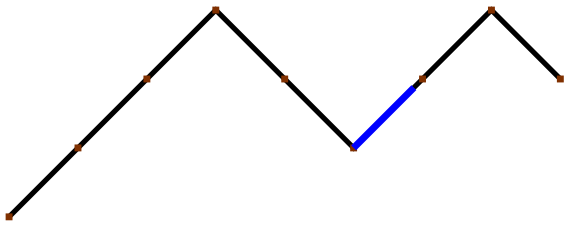
$P =$

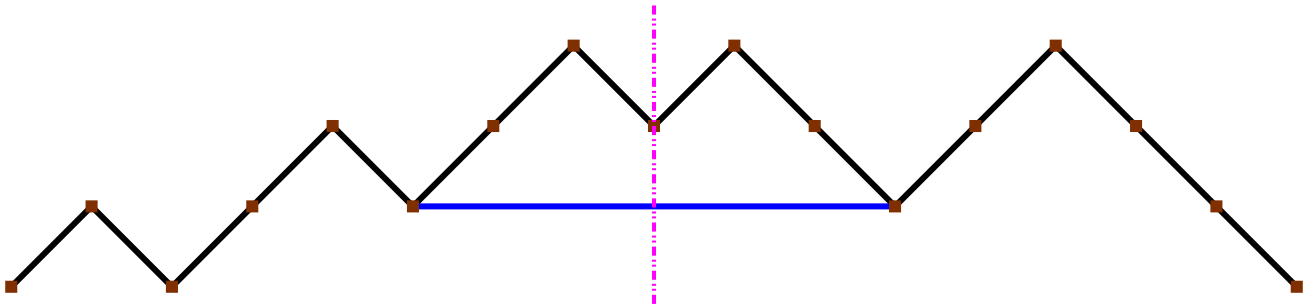
1	3	4	6	7
2	5	8		



$Q =$

1	2	3	6	7
4	5	8		





Ψ_{RSK} maps fixed points to centered tunnels and excedances to right tunnels

□

Statistics and simultaneous avoidance

$$F_{\sigma_1,\sigma_2} := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(\sigma_1,\sigma_2)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} z^n$$

(define $F_{\sigma_1,\sigma_2,\sigma_3}$ similarly)

E. '03. Explicit expressions for F_{σ_1,σ_2} and $F_{\sigma_1,\sigma_2,\sigma_3}$, for any $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{S}_3$.

Examples:

$$F_{132,231} = \frac{1 - z - qz^2 + xqz^3}{(1 - xz)(1 - z - 2qz^2)}$$

$$F_{132,213} = \frac{1 - (1 + q)z - 2qz^2 + 4q(1 + q)z^3 - (xq^2 + xq + 5q^2)z^4 + 2xq^2z^5}{(1 - z)(1 - xz)(1 - qz)(1 - 4qz^2)}$$

$$F_{123,132,213} = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

Statistics and generalized patterns

E., Mansour '03.

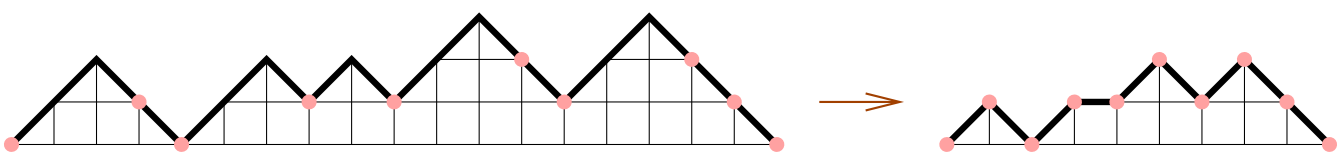
Bijection between $\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)$ and $\mathcal{M}_n$

$\widehat{\mathcal{D}}_n := \{D \in \mathcal{D}_n$ with no uuu (3 consec. up-steps)$\}$

$$\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23) \xrightarrow{\varphi_{\text{Krat}}} \widehat{\mathcal{D}}_n \xrightarrow{\theta_{\text{EM}}} \mathcal{M}_n$$

lds	$\leftrightarrow$	steps u and h
lis	$\leftrightarrow$	height+1
rises	$\leftrightarrow$	steps u

$$\theta_{\text{EM}} : \widehat{\mathcal{D}}_n \longrightarrow \mathcal{M}_n$$



$$\begin{aligned} uud &\to u \\ ud &\to h \\ d &\to d \end{aligned}$$

From the bijection,

E., Mansour '03.

$$\sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(132,1\text{-}23,\ 12\cdots(k+1))} v^{\mathrm{lds}(\pi)} y^{\#\{\text{rises of } \pi\}} z^n$$

$$= \frac{U_{k-1}\left(\frac{1-vz}{2z\sqrt{vy}}\right)}{z\sqrt{vy}\,U_k\left(\frac{1-vz}{2z\sqrt{vy}}\right)}$$

(U_i are Chebyshev polynomials of the second kind)

Asymptotic distribution of statistics

Deutsch, Hildebrand, Wilf '03 describe the limiting distribution of the parameter lis in $S_n(\sigma)$ for all $\sigma \in \mathcal{S}_3$

Let $d_n(\sigma) := |\{\pi \in \mathcal{S}_n(\sigma) : \mathrm{fp}(\pi) = 0\}|$

E. '03 (conjectured by Bóna, Guibert).
For all $n \geq 4$,

$$d_n(132) < d_n(123)$$

Open questions

- Find $|\mathcal{S}_n(\sigma)|$ for other patterns (e.g. $\sigma = 1324$)
- Stanley-Wilf conjecture
- Conjecture (Noonan, Zeilberger '96).
  For all σ, $|\mathcal{S}_n^r(\sigma)|$ is a P-recursive function of n
- Are there other statistics in restricted permutations having the same distribution for different patterns?
- Find distribution of statistics in permutations avoiding longer patterns (e.g., of length 4)