Statistics on Pattern-avoiding permutations

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Overview

- Definitions
  - Pattern avoidance
  - Permutation statistics
  - Statistics on Dyck paths
- Equidistribution of fixed points and excedances
  - Proof by generating functions
  - Bijective proof for fixed points
  - Bijective proof of the general result
- Avoidance of subsets of patterns of length 3
  - Single restrictions
  - Simultaneous avoidance
  - Pattern-avoiding involutions
- Permutations avoiding generalized patterns

Definitions

$\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{S}_n, \quad \sigma \in \mathcal{S}_k$

π *avoids* σ if there do not exist $i_1 < \ldots < i_k$ s.t. $\pi_{i_1} \cdots \pi_{i_k}$ has type σ (i.e., $\pi_{i_a} < \pi_{i_b} \Longleftrightarrow \sigma_a < \sigma_b$)

Example:
$\underline{2}4\underline{53}1$ contains 132
42351 avoids 132

$\mathcal{S}_n(\sigma) := \{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}$

Permutation statistics

i is a *fixed point* of π if $\pi_i = i$
$\mathrm{fp}(\pi) :=$ number of fixed points of π

i is an *excedance* of π if $\pi_i > i$
$\mathrm{exc}(\pi) :=$ number of excedances of π

i is a *descent* of π if $\pi_i > \pi_{i+1}$
(otherwise i is an *ascent*)
$\mathrm{des}(\pi) :=$ number of descents of π

$\mathrm{lis}(\pi) :=$ length of longest increasing subseq. of π

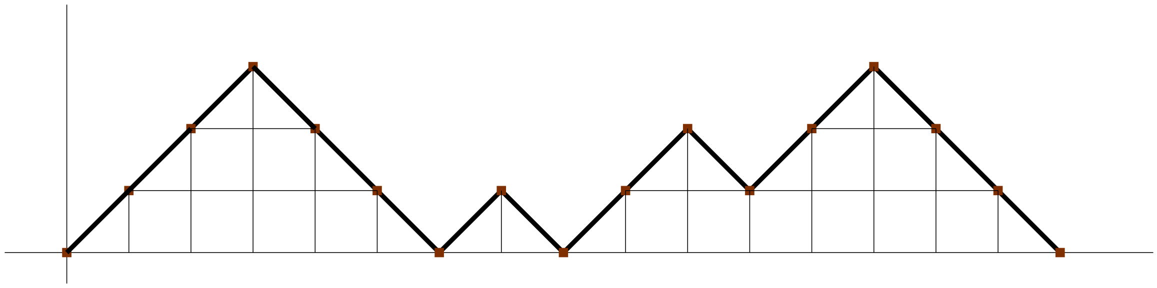
$\mathrm{lds}(\pi) :=$ length of longest decreasing subseq. of π

Ex: if $\pi = 4{\cdot}2{\cdot}17{\cdot}5{\cdot}36$, then
$\mathrm{fp}(\pi) = 2$, $\mathrm{exc}(\pi) = 2$, $\mathrm{des}(\pi) = 4$,
$\mathrm{lis}(\pi) = 3$, $\mathrm{lds}(\pi) = 3$

Dyck paths and Motzkin paths

Start and end at the x-axis; never go below it

Dyck path: steps $\mathbf{u} = (1,1)$ and $\mathbf{d} = (1,-1)$



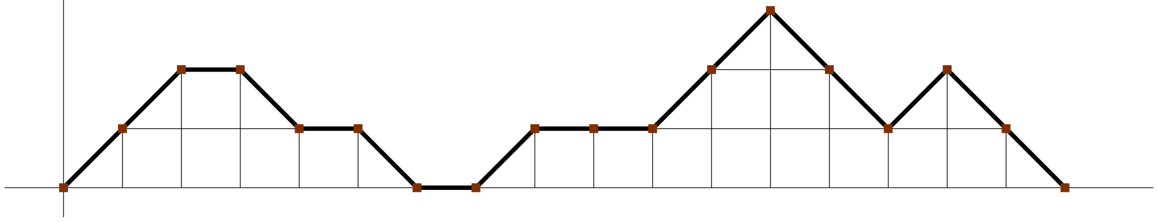
$\mathcal{D}_n$:= set of Dyck paths of length $2n$

$|\mathcal{D}_n| = \mathbf{C}_n$ (Catalan number)

$\mathcal{D} := \bigcup_{n \geq 0} \mathcal{D}_n$ class of all Dyck paths

$|D| := n$ if $D \in \mathcal{D}_n$

Motzkin path: steps $\mathbf{u} = (1,1)$, $\mathbf{d} = (1,-1)$ and $\mathbf{h} = (1,0)$



$\mathcal{M}_n$:= set of Motzkin paths of length n

$|\mathcal{M}_n| = \mathbf{M}_n$ (Motzkin number)

For a Dyck path D, define:

double rise: $\mathbf{uu}$ (two consecutive up-steps)

$\mathrm{dr}(D) :=$ number of double rises of D

peak: $\mathbf{ud}$ (up-step followed by down-step)

hill: peak at lowest height

$h(D) :=$ number of hills of D

tunnel: horizontal segment between two lattice points of D that stays always below D

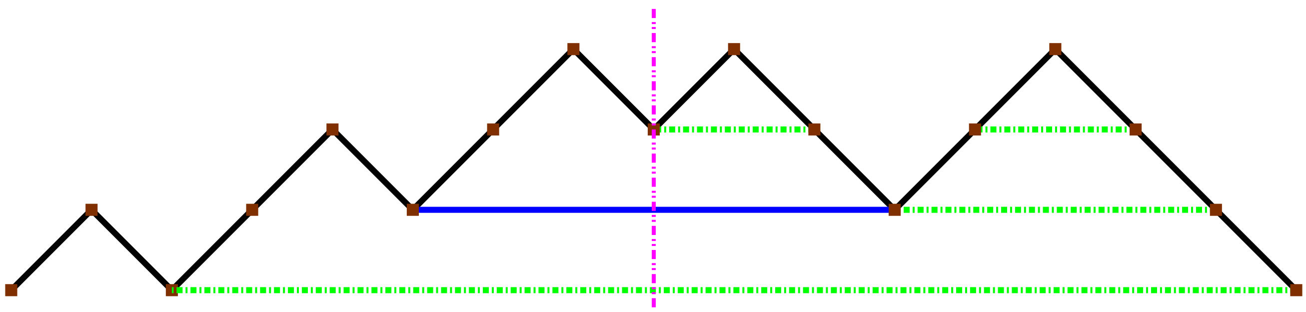
centered tunnel: x-coordinate of midpoint is at the middle of D

$\mathrm{ct}(D) :=$ number of centered tunnels of D

right tunnel: x-coordinate of midpoint is in the right half of D

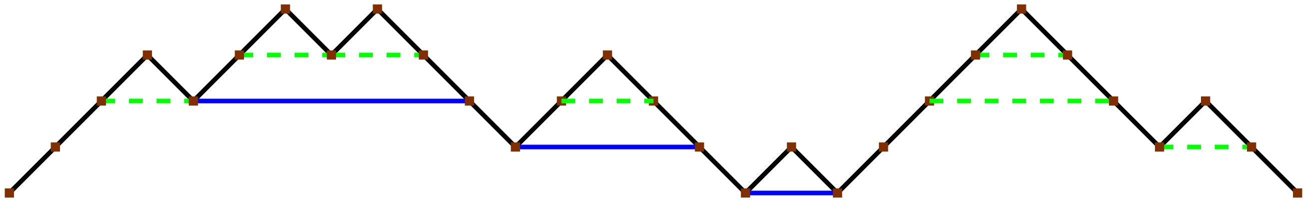
$\mathrm{rt}(D) :=$ number of right tunnels of D

$\mathrm{lt}(D) :=$ number of left tunnels of D



depth of a tunnel:

$$\mathrm{depth}(T) := \frac{1}{2}\mathrm{length}(T) - \mathrm{height}(T) - 1$$



tunnels of depth 0

tunnels of negative depth

Patterns of length 3

By trivial bijections,

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(321)|$$

$$|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$$

Proposition (Knuth '73).

$$|\mathcal{S}_n(123)| = |\mathcal{S}_n(132)| = \mathrm{C}_n = \frac{1}{n+1}\binom{2n}{n}$$

Equidistribution of statistics

Thm (Robertson, Saracino, Zeilberger '02). For any k, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k\}|$$

Their proof is not bijective

Questions:

- Is there a simple bijective proof?
- Can this theorem be generalized, considering other statistics in permutations?

Theorem. For any k, l, n,

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}| = |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$

Equivalently,

$$\sum_{\pi \in \mathcal{S}_n(321)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} = \sum_{\pi \in \mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)}$$

Example (n=4):

$\mathcal{S}_4(321)$

π	fp	exc
1234	4	0
1243	2	1
1324	2	1
1342	1	2
1423	1	1
2134	2	1
2143	0	2
2314	1	2
2341	0	3
2413	0	2
3124	1	1
3142	0	2
3412	0	2
4123	0	1

$\mathcal{S}_4(132)$

π	fp	exc
1234	4	0
2134	2	1
2314	1	2
2341	0	3
3124	1	1
3214	2	1
3241	1	2
3412	0	2
3421	0	2
4123	0	1
4213	1	1
4231	2	1
4312	0	2
4321	0	2

$$x^4 + 3x^2q + 2xq^2 + 2xq + q^3 + 4q^2 + q$$

Proof I:

- Non-bijective
- Transforms the problem into a Dyck path enumeration problem
- Uses nonstandard techniques in generating functions: diagonal of a nonrational g.f.

Proof II (with Emeric Deutsch):

- Bijective, but only for the statistic fp
- Key part: an unusual bijection for Dyck paths
- Generalizes to different statistics

Proof III (with Igor Pak):

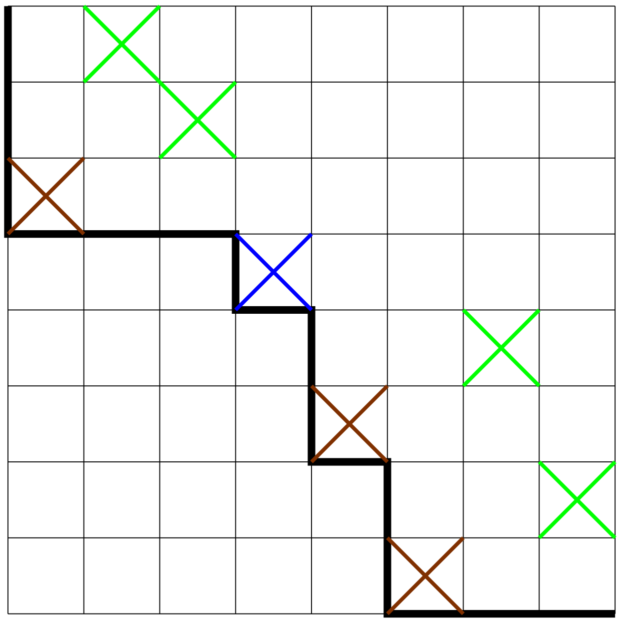
- Bijection preserving fp and exc
- Uses the RSK correspondence

I (sketch)

Step 1

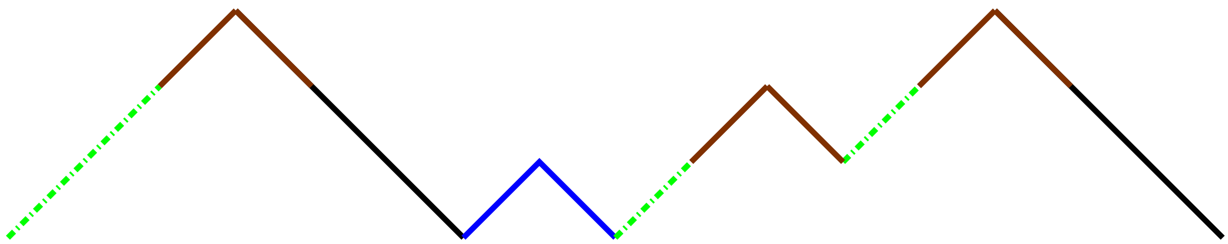
$\psi_{\llcorner} : \mathcal{S}_n(321) \xrightarrow{\sim} \mathcal{D}_n$ (Stanley '03)

Example: $\pi = 23147586 \in \mathcal{S}_8(321)$



Fixed points <––> hills

Excedances <––> double rises



$$F_{321}(x,q,z) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(321)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} z^n$$
$$= \sum_{D\in\mathcal{D}} x^{h(D)} q^{\mathsf{dr}(D)} z^{|D|}$$

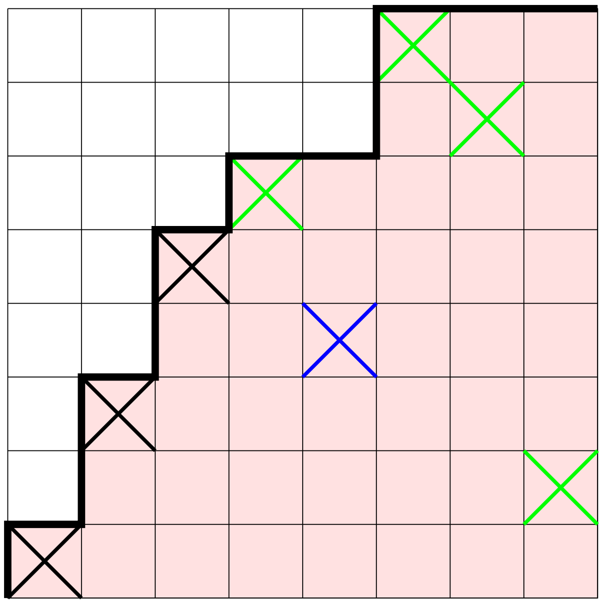
Hills and double rises in Dyck paths are easy to enumerate. We get

$$F_{321}(x,q,z) = \frac{2}{1+z\,(1+q-2x)+\sqrt{1-2z(1+q)+z^2(1-q)^2}}$$

Step 2

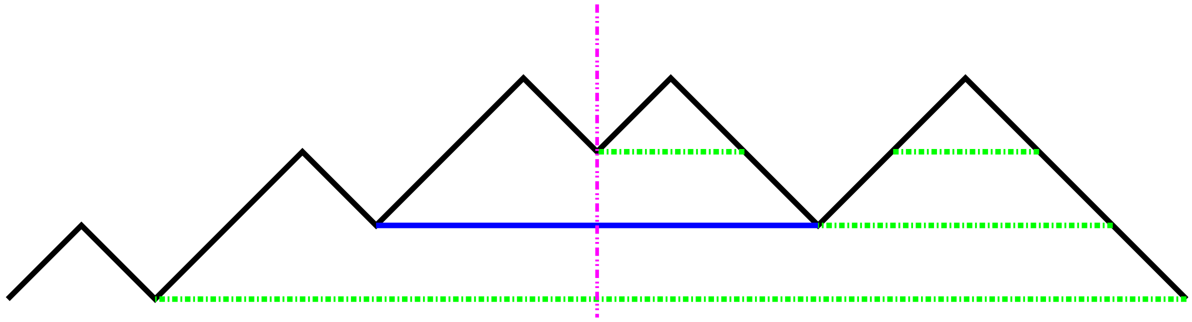
$\varphi : \mathcal{S}_n(132) \xrightarrow{\sim} \mathcal{D}_n$ (Krattenthaler '01)

Example: $\pi = 67435281 \in \mathcal{S}_8(132)$



Fixed points <–––> centered tunnels

Excedances <–––> right tunnels

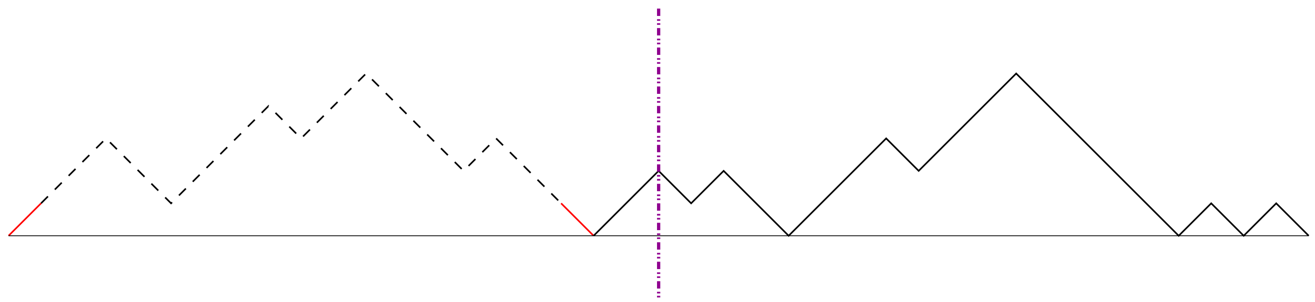


$$
\begin{aligned}
F_{132}(x,q,t) &:= \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} t^n \\
&= \sum_{D\in\mathcal{D}} x^{\mathsf{ct}(D)} q^{\mathsf{rt}(D)} t^{|D|}
\end{aligned}
$$

We want to show that

$$F_{321}(x,q,t) = F_{132}(x,q,t)$$

Problem: We cannot compute F_{132} using the standard decomposition of Dyck paths, because we need to keep track of the center.



Solution: Add a new variable that allows the reference line to be shifted from the center

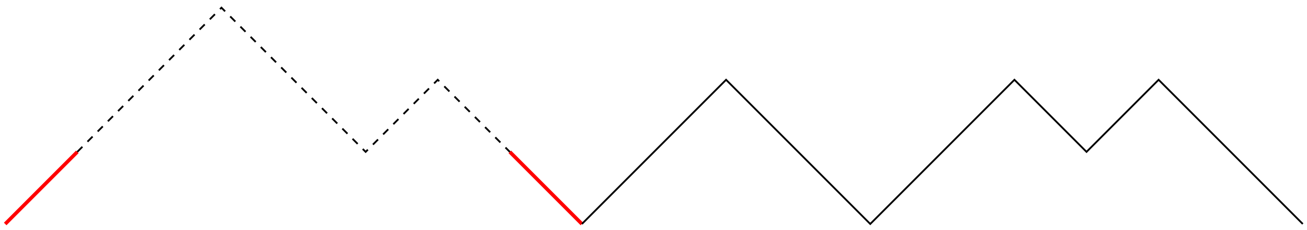
Step 3

For $r \in \mathbb{Z}$ and $D \in \mathcal{D}_n$, let
$\mathsf{ct}_r(D) :=$ # tunnels of D with center at $x = n - r$
$\mathsf{lt}_r(D) :=$ # tunnels of D with center in $x < n - r$

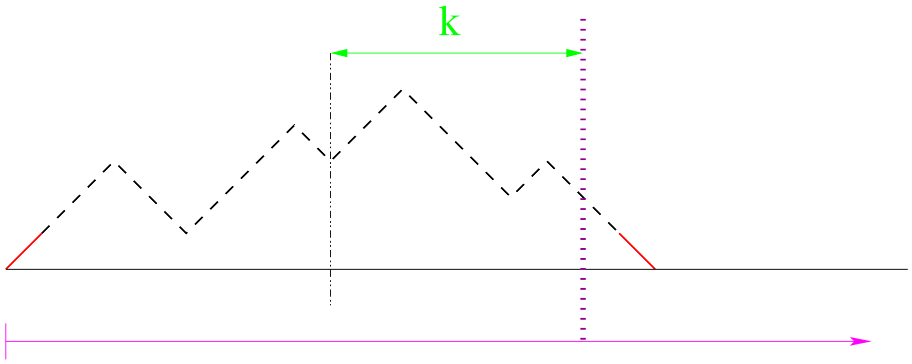
$$T(x, q, z, v) := \sum_{n,r \geq 0} \sum_{D \in \mathcal{D}_n} x^{\mathsf{ct}_r(D)} q^{\mathsf{lt}_r(D)} v^r z^n$$

Want to find an equation for T

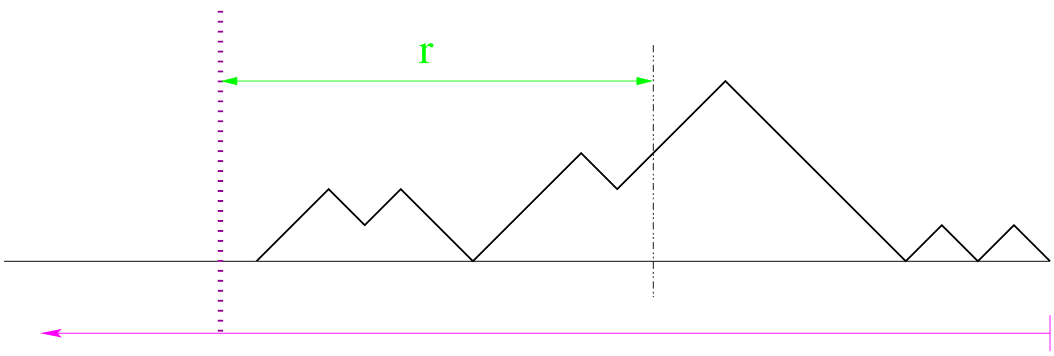
Use decomposition of Dyck paths as $D = \mathbf{u}A\mathbf{d}B$



$$H_1(x,q,z,v) := \sum_{\substack{n\geq 1\\ k\geq -n}} \sum_{A\in\mathcal{D}_{n-1}} x^{\mathsf{ct}_{-k}(\mathbf{u}A\mathbf{d})} q^{\mathsf{lt}_{-k}(\mathbf{u}A\mathbf{d})} v^k z^n$$

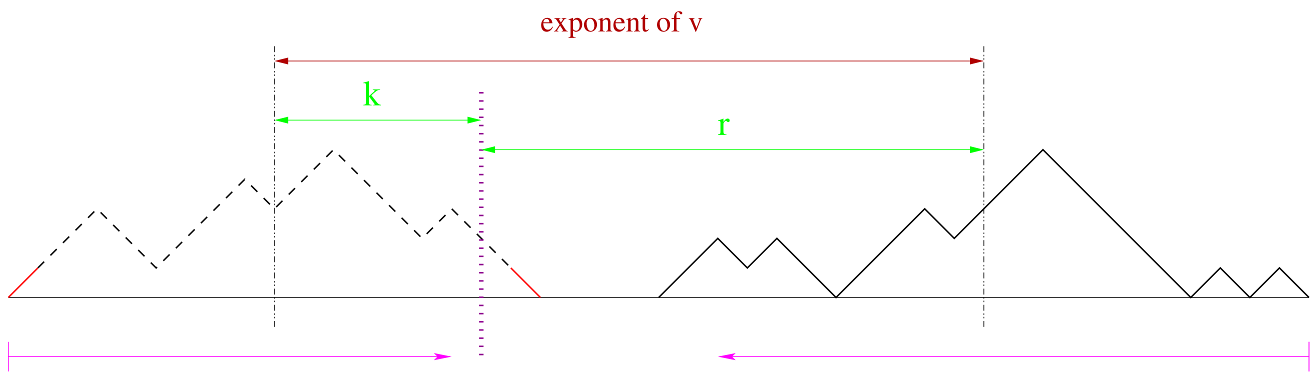


$$H_2(x,q,z,v) := \sum_{\substack{n\geq 0\\ r\geq -n}} \sum_{B\in\mathcal{D}_{n}} x^{\mathsf{ct}_{r}(B)} q^{\mathsf{lt}_{r}(B)} v^r z^n$$



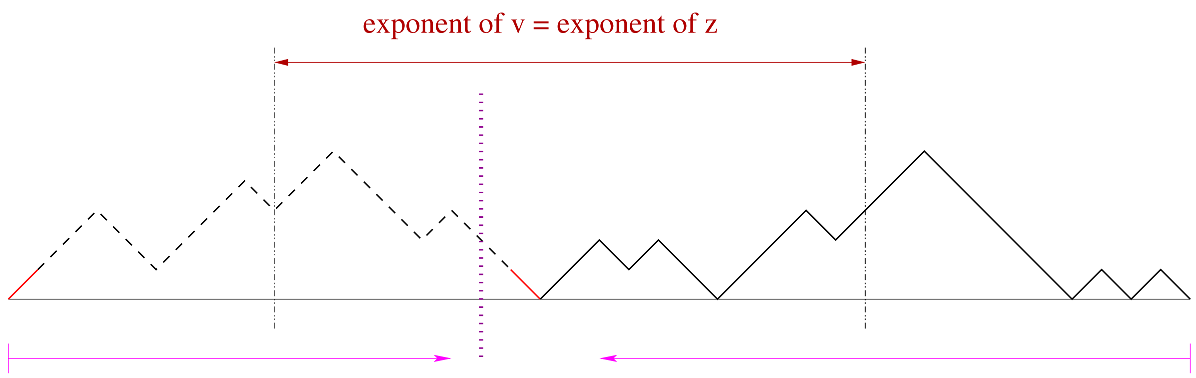
H_1 and H_2 can be expressed in terms of T

Multiplying these two functions, we get



But we want only the terms in which the two parts of the path are next to each other.
This happens precisely when

$$\text{exponent of } v = \text{exponent of } z$$



More formally, we have

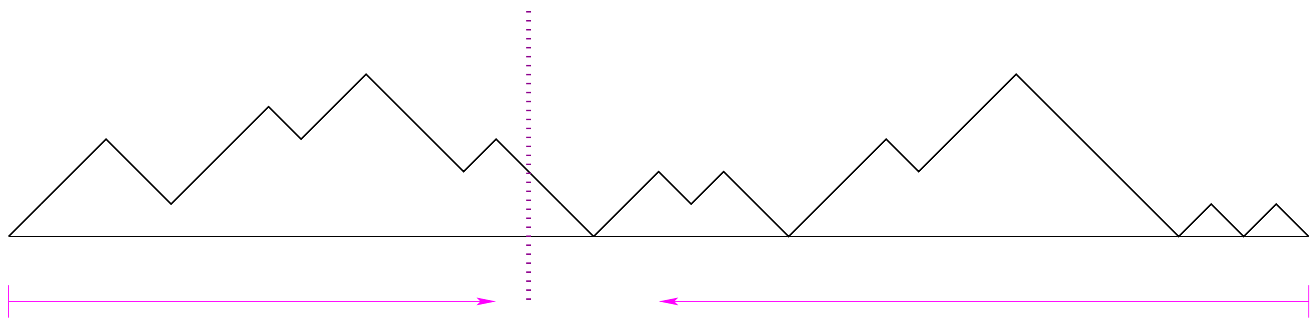
$$P(x,q,z,v,y) := H_1(x,q,z,\frac{v}{y})\ H_2(x,q,zy,v)$$
$$= \sum_{\substack{n\geq 0\\ j\geq -n}} P_{j,n}(x,q,y)v^j z^n$$

and we want

$$\text{diag } P := \sum_{n\geq 0} P_{n,n}(x,q,y)\, t^n$$

(diagonal of a nonrational generating function)

$$\text{diag } P = \sum_{\substack{n\geq 1\\ -n\leq r\leq n}} \sum_{D\in\mathcal{D}_n} x^{\mathsf{ct}_r(D)} q^{\mathsf{lt}_r(D)} y^r t^n$$



This gives an equation for T

II

A simple bijective proof of

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k\}|$$

Composition of bijections:

$$\mathcal{S}_n(132) \xleftrightarrow{\ \varphi\ } \mathcal{D}_n \xleftrightarrow{\ \Phi\ } \mathcal{D}_n \xleftrightarrow{\ \psi_{\llcorner}\ } S_n(321)$$

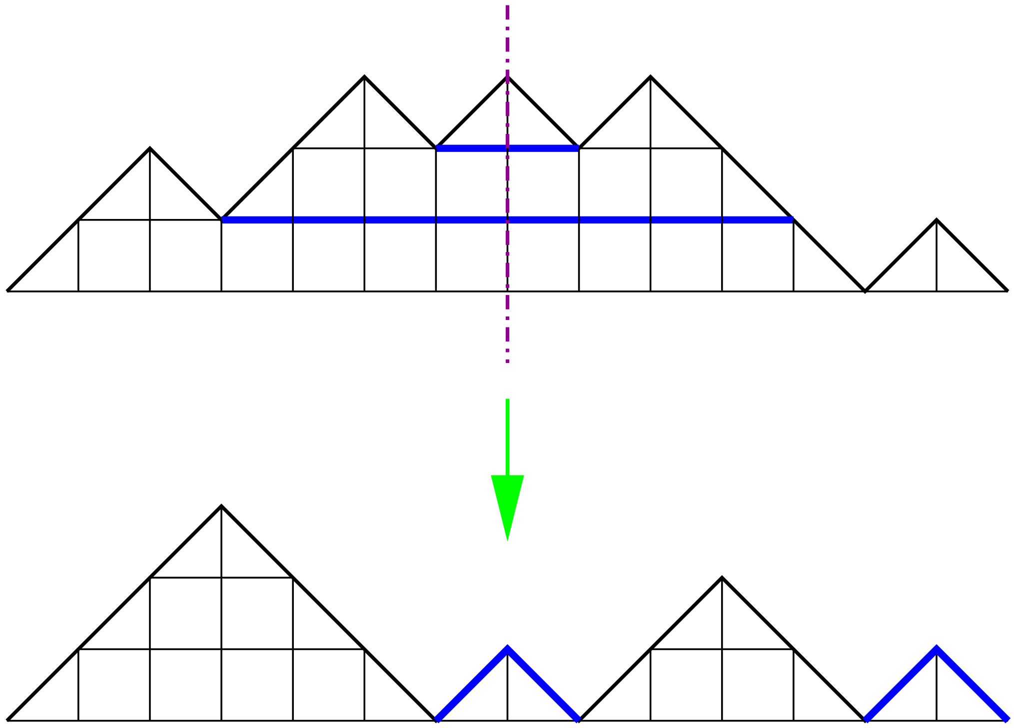
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills	$\leftrightarrow$	fixed points

$\Phi : \mathcal{D}_n \xrightarrow{\sim} \mathcal{D}_n$ (Deutsch, E.)

1. Each $\mathbf{u}$ in D has a matching $\mathbf{d}$ (together they determine a tunnel)

2. Read the steps of D in zigzag: $1, 2n, 2, 2n-1, \ldots$

3. For each step, if its corresponding matching step has not yet been read, draw an up-step in $\Phi(D)$. Otherwise, draw a down-step



Φ maps centered tunnels to hills

$\square$

More generally:

Theorem (Deutsch, E.). Let

$$\alpha_r(\pi) := |\{i : \pi_i = i + r\}|$$

$$\beta_r(\pi) := |\{i : i > r, \pi_i = i\}|$$

Then, for any k, r, n,

$$|\{\pi \in \mathcal{S}_n(132) : \alpha_r(\pi) = k\}| \\ = |\{\pi \in \mathcal{S}_n(321) : \beta_r(\pi) = k\}|$$

Proof uses a generalization of Φ

Another application of Φ

Theorem (Deutsch, E.).

$$1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} p^{\mathsf{des}(\pi)+1} z^n$$
$$= \frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{\spadesuit}}$$

where

$$\spadesuit = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2 \\ -2(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$$

Proof:

$\mathcal{S}_n(132)$	$\overset{\varphi}{\longleftrightarrow}$	$\mathcal{D}_n$	$\overset{\Phi}{\longleftrightarrow}$	$\mathcal{D}_n$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	hills
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	even up-steps
descents+1	$\leftrightarrow$	peaks	$\leftrightarrow$	$\underbrace{\text{“certain statistic”}}_{\text{easier to enumerate}}$

III

Generalization to excedances

A bijective proof (E., Pak) of

$$|\{\pi \in \mathcal{S}_n(321) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$
$$= |\{\pi \in \mathcal{S}_n(132) : \mathsf{fp}(\pi) = k,\ \mathsf{exc}(\pi) = l\}|$$

Composition of bijections:

$$\begin{array}{ccccc}
\mathcal{S}_n(132) & \xrightarrow{\varphi} & \mathcal{D}_n & \xleftarrow{\Psi} & \mathcal{S}_n(321) \\
\text{fixed points} & \leftrightarrow & \text{centered tunnels} & \leftrightarrow & \text{fixed points} \\
\text{excedances} & \leftrightarrow & \text{right tunnels} & \leftrightarrow & \text{excedances}
\end{array}$$

$$\Psi : \mathcal{S}_n(321) \xrightarrow{\sim} \mathcal{D}_n$$

Step 1: RSK correspondence

$$\pi \mapsto (P, Q)$$

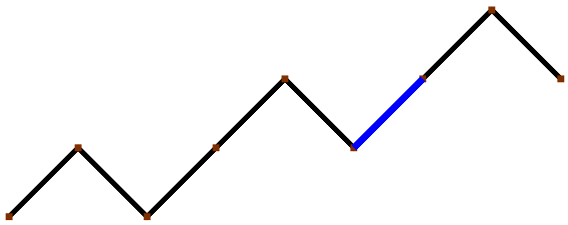
Example: $\pi = 23514687$

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 3 & 5 \\ \hline\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 8 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array}$$

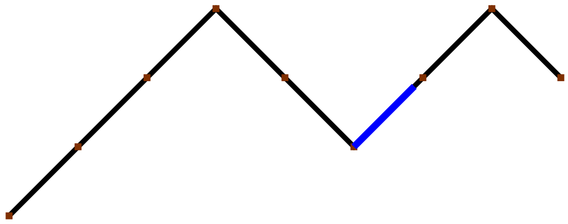
$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array}$$

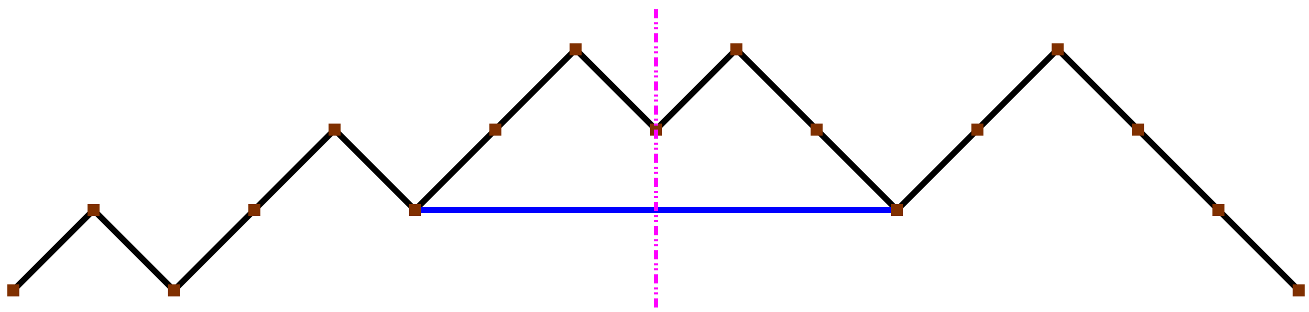
Step 2: $(P, Q) \mapsto \Psi(\pi)$

$$P = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 6 & 7 \\ \hline 2 & 5 & 8 \\ \cline{1-3}\end{array} \longrightarrow$$



$$Q = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 7 \\ \hline 4 & 5 & 8 \\ \cline{1-3}\end{array} \longrightarrow$$

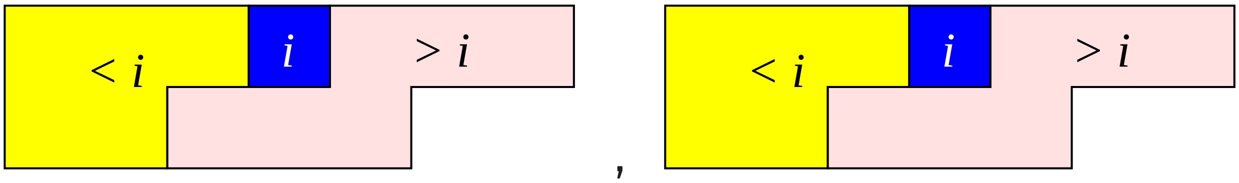




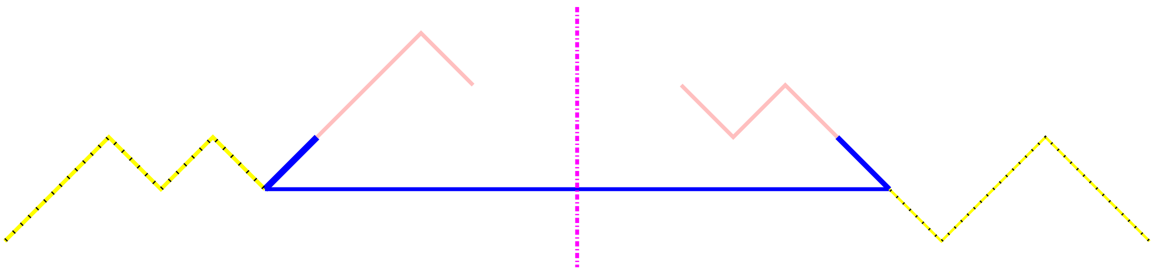
Ψ maps fixed points to centered tunnels:

$$\left.\begin{array}{c}\pi \text{ 321-avoiding} \\ \pi_i = i\end{array}\right\} \Rightarrow \pi = \underbrace{\cdots\cdots}_{<i} i \underbrace{\cdots\cdots\cdots}_{>i}$$

Then, both P and Q have shape



and i produces a centered tunnel



It can also be checked that Ψ maps excedances to right tunnels

$\square$

Avoidance of subsets of patterns of length 3

Defn:

$$\mathcal{S}_n(\sigma_1, \ldots, \sigma_m) = \bigcap_{i=1}^{m} \mathcal{S}_n(\sigma_i)$$

For a set of patterns $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$, let

$$F_{\Sigma}(x, q, z) := \sum_{n \geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n$$

Notation:

$$\Sigma_1 \approx \Sigma_2 \quad \text{if} \quad F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(x, q, z)$$

$$\Sigma_1 \sim \Sigma_2 \quad \text{if} \quad F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(\frac{x}{q}, \frac{1}{q}, qz)$$

Single restrictions

a) 123
b) $132 \approx 213 \approx 321$
c) 231 $\sim$ **c')** 312

a) We only have long summation formulas for

$$|\{\pi \in \mathcal{S}_n(123) : \mathrm{fp}(\pi) = k\}|$$

b) Done

c,c') Theorem.

$$F_{312}(x,q,z) = \cfrac{1}{1 - zK_0 + \cfrac{z}{1 - zK_1 + \cfrac{z}{1 - zK_2 + \cfrac{z}{1 - zK_3 + \cfrac{z}{\ddots}}}}},$$

where

$$K_n = (x-1)\,\mathbf{C}_n\, q^n z^n + (q-1) \sum_{j=0}^{n-1} \mathbf{C}_j\, q^j z^j$$

Proof idea: bijection to Dyck paths mapping fixed points to tunnels of depth 0 and excedances to tunnels of negative depth

Simultaneous avoidance

Double restrictions

$$\begin{array}{c}
\textbf{a)}\ \{123,132\} \approx \{123,213\}\\
\textbf{b)}\ \{231,321\} \quad \sim \quad \textbf{b')}\ \{312,321\}\\
\textbf{c)}\ \{132,213\}\\
\textbf{d)}\ \{231,312\}\\
\textbf{e)}\ \{132,231\} \approx \{213,231\} \sim \textbf{e')}\ \{132,312\} \approx \{213,312\}\\
\textbf{f)}\ \{132,321\} \approx \{213,321\}\\
\textbf{g)}\ \{123,231\} \quad \sim \quad \textbf{g')}\ \{123,312\}\\
\textbf{h)}\ \{123,321\}
\end{array}$$

Theorem. Explicit expressions for $F_\Sigma(x,q,z)$ for every $\Sigma \subset \mathcal{S}_3$, $|\Sigma| \geq 2$

Examples:

$$F_{132,231} = \frac{1 - z - qz^2 + xqz^3}{(1-xz)(1-z-2qz^2)}$$

$$F_{132,213} =$$

$$\frac{1-(1+q)z-2qz^2+4q(1+q)z^3-(xq^2+xq+5q^2)z^4+2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}$$

$$F_{123,132,213} = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

Idea of the proof: bijections between pattern-avoiding permutations and Dyck paths with certain restrictions, so that fp and exc correspond to statistics that are easier to enumerate.

We can generalize some cases to $\Sigma = \{\sigma_1, \sigma_2\}$ where $\sigma_1 \in \mathcal{S}_3$ and $\sigma_2 \in \mathcal{S}_k$.

Theorem. Expressions for

$$F_{132,12\cdots(k+1)}(x,1,z)$$

$$F_{312,(k+1)k\cdots 1}(x,q,z)$$

$$F_{132,(k+1)k\cdots 1}(x,q,z)$$

for any k

Pattern-avoiding involutions

Let $\mathcal{I}_n(\Sigma) := \{\pi \in \mathcal{S}_n(\Sigma) : \pi = \pi^{-1}\}$

Theorem. Expressions for

$$G_{\Sigma}(x,q,z) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{I}_n(\Sigma)} x^{\mathsf{fp}(\pi)} q^{\mathsf{exc}(\pi)} z^n$$

for every $\Sigma \subset \mathcal{S}_3$

Note: Proof III restricts to a bijection

$\mathcal{I}_n(132)$	$\xrightarrow{\varphi}$	$\mathcal{E}_n$	$\xleftarrow{\Psi}$	$\mathcal{I}_n(321)$
fixed points	$\leftrightarrow$	centered tunnels	$\leftrightarrow$	fixed points
excedances	$\leftrightarrow$	right tunnels	$\leftrightarrow$	excedances

where $\mathcal{E}_n := \{D \in \mathcal{D}_n : D \text{ symmetric}\}$

Expected number of fixed points

X_n^σ random variable for # of fixed points in $\mathcal{S}_n(\sigma)$

Proposition.

$$E[X_n^{123}] = \begin{cases} \frac{n+1}{2n-1} & \text{if } n \text{ is even} \\ \frac{2\,\mathbf{C}_{n-1} - \mathbf{C}^2_{(n-1)/2}}{\mathbf{C}_n} & \text{if } n \text{ is odd} \end{cases}$$

$$E[X_n^{132}] = E[X_n^{213}] = E[X_n^{321}] = 1$$

$$\sum_{n\geq 1} E[X_n^{231}]\mathbf{C}_n z^n = \sum_{n\geq 1} E[X_n^{312}]\mathbf{C}_n z^n = \frac{1-\sqrt{-1+4z+2\sqrt{1-4z}}}{2z}$$

Generalized patterns

(Babson, Steingrímsson '00)

Dashes between some letters of σ

If no dash, elements have to be adjacent in π

Example:

$\underline{35}42\underline{6}1\underline{7}$ is an occurrence of 12-3-4, but

$\underline{3}5\underline{4}2\underline{6}1\underline{7}$ is not

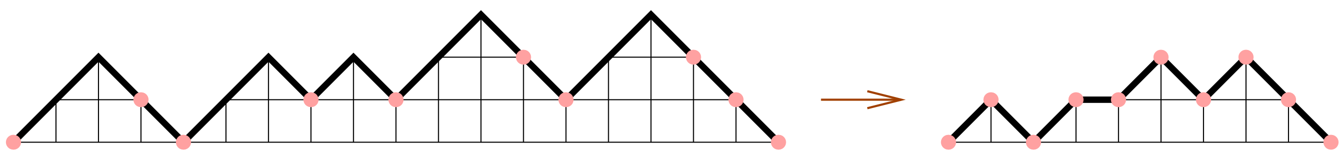
Theorem (E., Mansour).
Bijection between $\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23)$ and $\mathcal{M}_n$

$\widehat{\mathcal{D}}_n := \{D \in \mathcal{D}_n$ with no $\mathbf{uuu}$ (3 consec. up-steps)$\}$

$$\mathcal{S}_n(1\text{-}3\text{-}2, 1\text{-}23) \xrightarrow{\varphi} \widehat{\mathcal{D}}_n \xrightarrow{\theta} \mathcal{M}_n$$

lds	$\leftrightarrow$	steps $\mathbf{u}$ and $\mathbf{h}$
lis	$\leftrightarrow$	height+1
ascents	$\leftrightarrow$	steps $\mathbf{u}$

$$\theta : \widehat{\mathcal{D}}_n \longrightarrow \mathcal{M}_n$$



$$\begin{array}{rcl} \mathbf{uud} & \rightarrow & \mathbf{u} \\ \mathbf{ud} & \rightarrow & \mathbf{h} \\ \mathbf{d} & \rightarrow & \mathbf{d} \end{array}$$

Consequence of the bijection:

Theorem (E., Mansour).

$$\sum_{n\geq 0}\ \sum_{\substack{\pi\in\mathcal{S}_n(132,1\text{-}23,\\ 12\cdots(k+1))}} v^{\mathsf{lds}(\pi)} y^{\#\{\text{ascents of }\pi\}} z^n$$

$$= \frac{U_{k-1}\left(\frac{1-vz}{2z\sqrt{vy}}\right)}{z\sqrt{vy}\,U_k\left(\frac{1-vz}{2z\sqrt{vy}}\right)}$$

where $U_m(\cos t) = \frac{\sin(m+1)t}{\sin t}$
(Chebyshev polynomials of the second kind)