Statistics on Pattern-avoiding Permutations

by

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Abstract

This thesis concerns the enumeration of pattern-avoiding permutations with respect to certain statistics.

Our first result is that the joint distribution of the pair of statistics 'number of fixed points' and 'number of excedances' is the same in 321-avoiding as in 132-avoiding permutations. This generalizes a recent result of Robertson, Saracino and Zeilberger, for which we also give another, more direct proof. The key ideas are to introduce a new class of statistics on Dyck paths, based on what we call a *tunnel*, and to use a new technique involving diagonals of non-rational generating functions.

Next we present a new statistic-preserving family of bijections from the set of Dyck paths to itself. They map statistics that appear in the study of pattern-avoiding permutations into classical statistics on Dyck paths, whose distribution is easy to obtain. In particular, this gives a simple bijective proof of the equidistribution of fixed points in the above two sets of restricted permutations.

Then we introduce a bijection between 321- and 132-avoiding permutations that preserves the number of fixed points and the number of excedances. A part of our bijection is based on the Robinson-Schensted-Knuth correspondence. We also show that our bijection preserves additional parameters.

Next, motivated by these results, we study the distribution of fixed points and excedances in permutations avoiding subsets of patterns of length 3. We solve all the cases of simultaneous avoidance of more than one pattern, giving generating functions which enumerate them. Some cases are generalized to patterns of arbitrary length. For avoidance of one single pattern we give partial results. We also describe the distribution of these statistics in involutions avoiding any subset of patterns of length 3. The main technique consists in using bijections between pattern-avoiding permutations and certain kinds of Dyck paths, in such a way that the statistics in permutations that we consider correspond to statistics on Dyck paths which are easier to enumerate.

Finally, we study another kind of restricted permutations, counted by the Motzkin numbers. By constructing a bijection into Motzkin paths, we enumerate them with respect to some parameters, including the length of the longest increasing and decreasing subsequences and the number of ascents.

Thesis Supervisor: Richard P. Stanley
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Introduction

The subject of *pattern-avoiding permutations*, also called *restricted permutations*, has blossomed in the past decade. A number of enumerative results have been proved, new bijections found, and connections to other fields established. A recent breakthrough [63] (see also [50, 2, 14, 3]) has been the proof of the so-called Stanley-Wilf conjecture, which gives an exponential upper bound on the number of permutations avoiding any given pattern.

However, the study of statistics on restricted permutations started developing very recently, and the interest in this topic is currently growing. On the one hand, the concept of pattern avoidance concerns permutations regarded as words $\pi = \pi_1\pi_2\cdots\pi_n$. On the other hand, concepts such as fixed points or excedances arise when we look at permutations as bijections $\pi : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$. It was not until recently that these two kinds of concepts were studied together.

An unexpected recent result of Robertson, Saracino and Zeilberger [73] gives a new and exciting extension to the classical result that the number of 321-avoiding permutations equals the number of 132-avoiding permutations. They show that one can refine this result by taking into account the number of fixed points in a permutation. Their proof is nontrivial and technically involved. A large part of the work in the present thesis is motivated by this result. A natural question that arises is whether the fact that the number of fixed points has the same distribution in both 321-avoiding and 132-avoiding permutations can be generalized to other statistics and to other patterns. In particular, this gives more interest to the problem of studying the distribution of statistics on pattern-avoiding permutations. Another natural question to consider is whether a bijection between 321-avoiding permutations and 132-avoiding permutations that preserves the number of fixed points can be described. It is somewhat surprising that, before the work on this thesis [27, 29, 32], none of the several known bijections between these two sets of permutations preserved the number of fixed points.

In Chapter 2 we prove a further refinement of the result just mentioned, namely that it still holds when we fix not only the number of fixed points but also the number of excedances. We introduce bijections between pattern-avoiding permutations and Dyck paths, that will play an important role throughout the thesis. They are presented in a graphical way which makes it easier to study their properties. After reducing the problem to the enumeration of Dyck paths with respect to certain statistics, the rest of the proof is based on manipulations of generating functions with additional variables.

A combinatorial proof of the original refinement that only takes into account fixed points is given in Chapter 3. This is the simplest known bijection between 321- and 132-avoiding permutations that preserves the number of fixed points. The main ingredient is a new unusual bijection from the set of Dyck paths to itself. We also present a generalization of it, which gives additional correspondences of statistics. This bijection has also applications to the enumeration of Dyck paths and restricted permutations with respect to several statistics.

We end the chapter by giving some new interpretations of the Catalan numbers.

In Chapter 4 we present a bijection between 321- and 132-avoiding permutations that preserves both the number of fixed points and the number of excedances, thus giving a combinatorial proof of the main result of Chapter 2. Our bijection is a composition of two bijections into Dyck paths, and the result follows from a new analysis of these bijections. The Robinson-Schensted-Knuth correspondence is a part of one of them, and from it stems the difficulty of the analysis. We also show that our bijection preserves additional statistics, which extends the previous results. Then our bijections are applied to refined restricted involutions.

In Chapter 5 we systematically enumerate permutations avoiding any subset of patterns of length 3 with respect to the number of fixed points and the number of excedances [28]. By means of the bijections between restricted permutations and Dyck paths described in previous chapters, additional restrictions on permutations correspond to certain conditions on the paths, and thus the problem reduces to enumerating such paths with respect to the statistics that fixed points and excedances are mapped to by these bijections.

First we consider permutations avoiding a single pattern. For the pattern 123 we give partial results, and we use them to prove a recent conjecture of Bóna and Guibert. For the other patterns we give the generating functions with variables marking the number of fixed points and the number of excedances, and in some cases we study other statistics as well. In one case the generating function is expressed as a continued fraction. For permutations avoiding simultaneously two or more patterns of length 3 we present all the corresponding generating functions for these two statistics, which are rational and have relatively simple expressions. In some cases we can generalize the results to avoidance of subsets of patterns where one of them has arbitrary length. Then we enumerate involutions avoiding any subset of patterns of length 3 with respect to the same two statistics. Lastly, the generating functions obtained in this chapter allow us to easily compute the expected number of fixed points in permutations avoiding patterns of length 3. We conclude with some remarks regarding possible extensions of our work.

Finally, in Chapter 6 we consider a different class of restricted permutations enumerated by the Motzkin numbers [30]. The definition of this class involves the notion of generalized patterns, which was introduced by Babson and Steingrímsson [4]. We give a bijection from this set of permutations to Motzkin paths, which maps permutation statistics such as the length of the longest increasing and decreasing subsequences, and the number of descents, to certain statistics on Motzkin paths that are easy to deal with. Similarly, by considering a new statistic on Motzkin paths we are able to enumerate another related class of restricted permutations with respect to the number of fixed points.

Chapter 1

Definitions and preliminaries

1.1 Permutations

1.1.1 Pattern avoidance

We will denote by $[n]$ the set $\{1, 2, \ldots, n\}$, and by $\mathcal{S}_n$ the symmetric group on $[n]$. A permutation $\pi \in \mathcal{S}_n$ will be written in one-line notation as $\pi = \pi_1\pi_2 \cdots \pi_n$. We will write the cardinality of a set A as $|A|$. In this section we define the classical notion of pattern avoidance, which will be used throughout most of this thesis. For a definition of *generalized patterns* see Section 6.1.1.

Let n, m be two positive integers with $m \leq n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ and $\sigma = \sigma_1\sigma_2 \cdots \sigma_m \in \mathcal{S}_m$ be two permutations. We say that π *contains* σ if there exist indices $i_1 < i_2 < \ldots < i_m$ such that $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ is in the same relative order as $\sigma_1\sigma_2 \cdots \sigma_m$ (that is, for all indices a and b, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$). In that case, $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ is called an *occurrence* of σ in π. In this context, σ is also called a *pattern*.

If π does not contain σ, we say that π *avoids* σ, or that it is σ*-avoiding*. For example, if $\sigma = 132$, then $\pi = 24531$ contains 132, because the subsequence $\pi_1\pi_3\pi_4 = 253$ has the same relative order as 132. However, $\pi = 42351$ is 132-avoiding. Denote by $\mathcal{S}_n(\sigma)$ the set of σ-avoiding permutations in $\mathcal{S}_n$. More generally, for any subset $A \subseteq \mathcal{S}_n$ and any pattern σ, define $A(\sigma) := A \cap \mathcal{S}_n(\sigma)$ to be the set of σ-avoiding permutations in A.

It is a natural generalization to consider permutations that avoid several patterns at the same time. If $\Sigma \subseteq \bigcup_{k\geq 1} \mathcal{S}_k$ is any finite set of patterns, denote by $\mathcal{S}_n(\Sigma)$ the set of permutations in $\mathcal{S}_n$ that avoid simultaneously all the patterns in Σ. These are also called Σ-avoiding permutations. For example, if $\Sigma = \{123, 231\}$, then $\mathcal{S}_4(\Sigma) = \{1432, 2143, 3214, 4132, 4213, 4312, 4321\}$.

1.1.2 Permutation statistics

Informally speaking, the notion of permutation can be viewed in two different ways. On one hand, a permutation can be regarded as a word $\pi = \pi_1\pi_2 \cdots \pi_n$, namely, as a sequence of numbers in some given order. From this description arises the concept of pattern avoidance discussed in the previous subsection. On the other hand, one can regard a permutation $\pi \in \mathcal{S}_n$ as a bijection $\pi : [n] \longrightarrow [n]$. Some concepts such as fixed points or excedances arise when we consider a permutation as a bijection. This double nature of permutations makes it interesting to study some of the following statistics together with the notion of pattern avoidance. There is a lot of mathematical literature devoted to permutation statistics (see for example [26, 35, 37, 39]).

We say that i is a *fixed point* of a permutation π if $\pi_i = i$. We say that i is an *excedance* of π if $\pi_i > i$. Denote by $\mathrm{fp}(\pi)$ and $\mathrm{exc}(\pi)$ the number of fixed points and the number of excedances of π respectively. The distribution of the statistics fp and exc in pattern-avoiding permutations will be one of the main topics of this thesis. An element of a permutation that is neither a fixed point nor an excedance, namely an i for which $\pi_i < i$, will be called a *deficiency*. Permutations without fixed points are also called *derangements*.

We say that $i \leq n-1$ is a *descent* of $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. Similarly, $i \leq n-1$ is an *ascent* of $\pi \in \mathcal{S}_n$ if $\pi_i < \pi_{i+1}$. Denote by $\mathrm{des}(\pi)$ and $\mathrm{asc}(\pi)$ the number of descents and the number of ascents of π respectively. A *right-to-left minimum* of π is an element π_i such that $\pi_i < \pi_j$ for all $j > i$.

Let $\mathrm{lis}(\pi)$ denote the length of the *longest increasing subsequence* of π, i.e., the largest m for which there exist indices $i_1 < i_2 < \cdots < i_m$ such that $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_m}$. Equivalently, in terms of forbidden patterns, $\mathrm{lis}(\pi)$ is the smallest m such that π avoids $12\cdots(m+1)$. The length of the *longest decreasing subsequence* is defined analogously, and it is denoted $\mathrm{lds}(\pi)$. Define the *rank* of π, denoted $\mathrm{rank}(\pi)$, to be the largest k such that $\pi(i) > k$ for all $i \leq k$. For example, if $\pi = 63528174$, then $\mathrm{fp}(\pi) = 1$, $\mathrm{exc}(\sigma) = 4$, $\mathrm{des}(\pi) = 4$, $\mathrm{lis}(\pi) = 3$, $\mathrm{lds}(\pi) = 4$ and $\mathrm{rank}(\pi) = 2$.

We say that a permutation $\pi \in \mathcal{S}_n$ is an *involution* if $\pi = \pi^{-1}$. Denote by $\mathcal{I}_n$ the set of involutions of length n.

1.1.3 Trivial operations

Often it will be convenient to represent a permutation $\pi \in \mathcal{S}_n$ by an $n \times n$ array with a cross in each one of the squares (i, π_i). We will denote by $\mathrm{arr}(\pi)$ the array corresponding to π. Figure 1-1 shows arr(63528174).

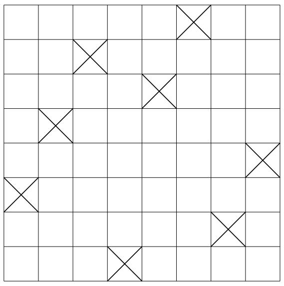


Figure 1-1: The array of $\pi = 63528174$.

The diagonal from the top-left corner to the bottom-right corner of the array will be referred to as *main diagonal*, and the diagonal perpendicular to it will be called *secondary diagonal*. Note that fixed points of π correspond to crosses on the main diagonal of the array, and excedances of π are represented by crosses to the right of this diagonal.

Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, define its reversal $\pi^R = \pi_n \ldots \pi_2\pi_1$ and its complementation $\bar{\pi} = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$. The array of $\bar{\pi}$ is obtained from the array of π by a flip along a vertical axis, so fixed points (resp. excedances) of π correspond to crosses on (resp. to the left of) the secondary diagonal of the array of $\bar{\pi}$. Similarly, define $\widehat{\pi}$ to be the permutation whose array is the one obtained from that of π by reflection along the secondary diagonal. Note that reflecting the array of π along the main diagonal we get the array of its inverse π^{-1}. For any set of permutations Σ, let $\bar{\Sigma}$ be the set obtained by reversing each of the elements of Σ. Define $\widehat{\Sigma}$ and Σ^{-1} analogously. The following trivial lemma will be used in Chapter 5.

Lemma 1.1 *Let $\Sigma \subset \bigcup_{k\geq 1} \mathcal{S}_k$ be a finite set of patterns, and let $\pi \in \mathcal{S}_n$. We have that*

(1) $\pi \in \mathcal{S}_n(\Sigma) \iff \bar{\pi} \in \mathcal{S}_n(\bar{\Sigma}) \iff \widehat{\pi} \in \mathcal{S}_n(\widehat{\Sigma}) \iff \pi^{-1} \in \mathcal{S}_n(\Sigma^{-1})$,

(2) $\mathrm{fp}(\widehat{\pi}) = \mathrm{fp}(\pi)$, $\mathrm{exc}(\widehat{\pi}) = \mathrm{exc}(\pi)$,

(3) $\mathrm{fp}(\pi^{-1}) = \mathrm{fp}(\pi)$, $\mathrm{exc}(\pi^{-1}) = n - \mathrm{fp}(\pi) - \mathrm{exc}(\pi)$.

In this thesis we will often be interested in the distribution of the statistics fp and exc among the permutations avoiding a certain pattern or set of patterns. Given any such set Σ, we define the generating function F_Σ as

$$F_\Sigma(x, q, z) := \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n. \tag{1.1}$$

If $\Sigma = \{\sigma\}$, we will write F_σ instead of $F_{\{\sigma\}}$. The following lemma restates the previous one in terms of generating functions.

Lemma 1.2 *Let Σ be any set of permutations. We have*

(1) $F_{\widehat{\Sigma}}(x, q, z) = F_\Sigma(x, q, z)$,

(2) $F_{\Sigma^{-1}}(x, q, z) = F_\Sigma(x/q, 1/q, qz)$.

Proof. To prove (1), consider the bijection between $\mathcal{S}_n(\Sigma)$ and $\mathcal{S}_n(\widehat{\Sigma})$ that maps π to $\widehat{\pi}$. The equation follows from parts (1) and (2) of Lemma 1.1.

Equation (2) follows similarly from parts (1) and (3) of the previous lemma, noticing that

$$\sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma^{-1})} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} z^n = \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi^{-1})} q^{\mathrm{exc}(\pi^{-1})} z^n$$
$$= \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} x^{\mathrm{fp}(\pi)} q^{n-\mathrm{fp}(\pi)-\mathrm{exc}(\pi)} z^n = \sum_{n\geq 0} \sum_{\pi \in \mathcal{S}_n(\Sigma)} \left(\frac{x}{q}\right)^{\mathrm{fp}(\pi)} \left(\frac{1}{q}\right)^{\mathrm{exc}(\pi)} (qz)^n.$$

□

If for two sets of patterns Σ_1 and Σ_2 we have that $F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(x, q, z)$ (i.e., the joint distribution of fixed points and excedances is the same in Σ_1-avoiding as in Σ_2-avoiding permutations), we will write $\Sigma_1 \approx \Sigma_2$. If we have that $F_{\Sigma_1}(x, q, z) = F_{\Sigma_2}(\frac{x}{q}, \frac{1}{q}, qz)$, we will write $\Sigma_1 \sim \Sigma_2$. In this notation, Lemma 1.2 says that $\widehat{\Sigma} \approx \Sigma$ and $\Sigma^{-1} \sim \Sigma$.

1.2 Dyck paths

A *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0, 0)$ and $(2n, 0)$ consisting of up-steps $(1, 1)$ and down-steps $(1, -1)$ which never goes below the x-axis. We shall denote by $\mathcal{D}_n$ the set of Dyck paths of length $2n$, and by $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$ the class of all Dyck paths. It is well-known that $|\mathcal{D}_n| = \mathbf{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number. If $D \in \mathcal{D}_n$, we will write $|D| = n$ to indicate the semilength of D. The generating function that enumerates Dyck paths according to their semilength is $\sum_{D\in\mathcal{D}} z^{|D|} = \sum_{n\geq 0} \mathbf{C}_n z^n = \frac{1-\sqrt{1-4z}}{2z}$, which we denote by $\mathbf{C}(z)$.

Sometimes it will be convenient to encode each up-step by a letter **u** and each down-step by **d**, obtaining an encoding of the Dyck path as a *Dyck word*. We will use D to refer indistinctively to the Dyck path D or to the Dyck word associated to it. In particular, given $D_1 \in \mathcal{D}_{n_1}$, $\mathcal{D}_2 \in \mathcal{D}_{n_2}$, we will write D_1D_2 to denote the concatenation of D_1 and D_2 (note that, as seen in terms of lattice paths, D_2 has to be shifted $2n_1$ units to the right). If A is any sequence of up and down steps, $\text{length}(A)$ will denote the number of steps in the sequence. For example, if $A \in \mathcal{D}_n$, then $\text{length}(A) = 2n$.

1.2.1 Standard statistics

A *peak* of a Dyck path $D \in \mathcal{D}$ is an up-step followed by a down-step (i.e., an occurrence[1] of **ud** in the associated Dyck word). The coordinates of a peak are given by the point at the top of it. A *hill* is a peak at height 1, where the height is the y-coordinate of the peak. Denote by $h(D)$ the number of hills of D, and by $p_2(D)$ the number of peaks of D of height at least 2. A *valley* of D is a down-step followed by an up-step (i.e., an occurrence of **du** in the associated Dyck word). Denote by $\text{va}(D)$ the number of valleys of D. Clearly, both $p_2(D) + h(D)$ and $\text{va}(D) + 1$ equal the total number of peaks of D. A *double rise* of D is an up-step followed by another up-step (i.e., an occurrence **uu** in the Dyck word). Denote by $\text{dr}(D)$ the number of double rises of D.

An *odd rise* is an up-step in an odd position when the steps are numbered from left to right starting with 1 (or, equivalently, it is an up-step at odd level when the steps leaving the x-axis are considered to be at level 1). Denote by $\text{or}(D)$ the number of odd rises of D. *Even rises* and $\text{er}(D)$ are defined analogously. The x-coordinate of an odd or even rise is given by the rightmost end of the corresponding up-step.

A *return* of a Dyck path is a down-step landing on the x-axis. An *arch* is a part of the path joining two consecutive points on the x-axis. Clearly for any $D \in \mathcal{D}_n$ the number of returns equals the number of arches. Denote it by $\text{ret}(D)$. Define the x-coordinate of an arch as the x-coordinate of its leftmost point.

The *height* of D is the y-coordinate of the highest point of the path. Denote by $\mathcal{D}^{\leq k}$ the set of Dyck paths of height at most k. For any $D \in \mathcal{D}_n$, define $\nu(D)$ to be the height of the middle point of D, that is, the y-coordinate of the intersection of the vertical line $x = n$ with the path. For example, if $D \in \mathcal{D}_8$ is the path in Figure 1-2, then $h(D) = 1$, $p_2(D) = 4$, $\text{va}(D) = 4$, $\text{dr}(D) = 3$, $\text{or}(D) = 5$, $\text{er}(D) = 3$, $\text{ret}(D) = 2$, $\nu(D) = 2$, and its height is 3.

Define a *pyramid* to be a Dyck path that has only one peak, that is, a path of the form $\mathbf{u}^k\mathbf{d}^k$ with $k \geq 1$ (here the exponent indicates the number of times the letter is repeated). For a Dyck path $D \in \mathcal{D}_n$, denote by D^* the path obtained by reflection of D from the vertical line $x = n$. We say that D is *symmetric* if $D = D^*$. Denote by $\mathcal{D}s \subset \mathcal{D}$ the subclass of symmetric Dyck paths.

1.2.2 Tunnels

Here we introduce a new class of statistics on Dyck paths that will become very useful for the study of statistics on permutations avoiding patterns of length 3. They are based on

[1]In the context of Dyck words, the letters have to appear in consecutive positions to form an *occurrence* of a subword.

the notion of tunnel of a Dyck path.

For any $D \in \mathcal{D}$, define a *tunnel* of D to be a horizontal segment between two lattice points of D that intersects D only in these two points, and stays always below D. Tunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = A\mathbf{u}B\mathbf{d}C$, where $B \in \mathcal{D}$ (no restrictions on A and C). In the decomposition, the tunnel is the segment that goes from the beginning of the $\mathbf{u}$ to the end of the $\mathbf{d}$. If $D \in \mathcal{D}_n$, then D has exactly n tunnels, since such a decomposition can be given for each up-step $\mathbf{u}$ of D. The *length* of a tunnel is just its length as a segment, and the *height* is its y-coordinate. It will be useful to define the *depth* of a tunnel T as $\mathrm{depth}(T) := \frac{1}{2}\mathrm{length}(T) - \mathrm{height}(T) - 1$.

A tunnel of $D \in \mathcal{D}_n$ is called a *centered tunnel* if the x-coordinate of its midpoint (as a segment) is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition of the Dyck word $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to A and C having the same length, namely, $\mathrm{length}(A) = \mathrm{length}(C)$. Alternatively, this can be taken as a definition of centered tunnel. Denote by $\mathrm{ct}(D)$ the number of centered tunnels of D.

A tunnel of $D \in \mathcal{D}_n$ is called a *right tunnel* if the x-coordinate of its midpoint is strictly greater than n, that is, the midpoint of the tunnel is to the right of the vertical line through the middle of D. In terms of the decomposition $D = A\mathbf{u}B\mathbf{d}C$, this is equivalent to saying that $\mathrm{length}(A) > \mathrm{length}(C)$. Denote by $\mathrm{rt}(D)$ the number of right tunnels of D. In Figure 1-2, there is one centered tunnel drawn with a solid line, and four right tunnels drawn with dotted lines. Similarly, a tunnel is called a *left tunnel* if the x-coordinate of its midpoint is strictly less than n. Denote by $\mathrm{lt}(D)$ the number of left tunnels of D. Clearly, $\mathrm{lt}(D) + \mathrm{rt}(D) + \mathrm{ct}(D) = n$ for any $D \in \mathcal{D}_n$.

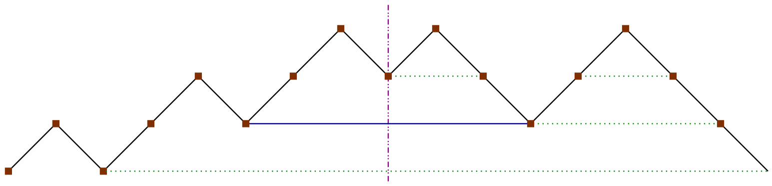


Figure 1-2: One centered and four right tunnels.

We will distinguish between right tunnels of $D \in \mathcal{D}_n$ that are entirely contained in the half plane $x \geq n$ and those that cross the vertical line $x = n$. These will be called *right-side tunnels* and *right-across tunnels*, respectively. In terms of Dyck words, a decomposition $D = A\mathbf{u}B\mathbf{d}C$ corresponds to a right-side tunnel if $\mathrm{length}(A) \geq n$, and to a right-across tunnel if $\mathrm{length}(C) < \mathrm{length}(A) < n$. In Figure 1-2 there are three right-side tunnels and one right-across tunnel. *Left-side tunnels* and *left-across tunnels* are defined analogously.

For any $D \in \mathcal{D}$, we define a *multitunnel* of D to be a horizontal segment between two lattice points of D such that D never goes below it. In other words, a multitunnel is just a concatenation of tunnels, so that each tunnel starts at the point where the previous one ends. Similarly to the case of tunnels, multitunnels are in obvious one-to-one correspondence with decompositions of the Dyck word $D = ABC$, where $B \in \mathcal{D}$ is not empty. In the decomposition, the multitunnel is the segment that connects the initial and final points of B.

A multitunnel of $D \in \mathcal{D}_n$ is called a *centered multitunnel* if the x-coordinate of its midpoint (as a segment) is n, that is, the tunnel is centered with respect to the vertical line through the middle of D. In terms of the decomposition $D = ABC$, this is equivalent to saying that A and C have the same length. Denote by $\mathrm{cmt}(D)$ the number of centered multitunnels of D.

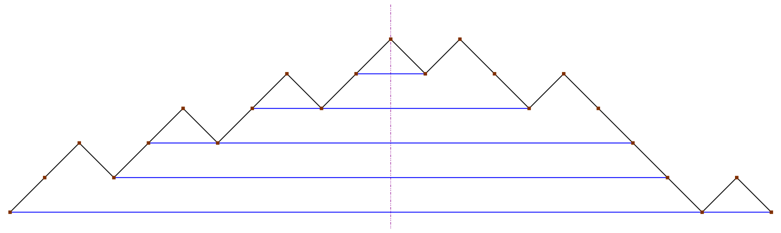


Figure 1-3: Five centered multitunnels, two of which are centered tunnels.

Additional interpretations of centered tunnels

Through the numerous known bijections between Dyck paths and other combinatorial objects counted by the Catalan numbers, the new statistics that we defined on Dyck paths give rise to corresponding statistics in other objects. Here we give a couple of examples that were suggested by Emeric Deutsch.

It is known [86, Exercise 6.19(n)] that the diagrams of n nonintersecting chords joining $2n$ points on the circumference of a circle are in bijection with $\mathcal{D}_n$. We can draw these points as the vertices of a regular $2n$-gon, and the chords as straight segments, so that one of the diagrams has n horizontal chords. The bijection to Dyck paths can be described as follows. Starting counterclockwise from the topmost vertex on the left, for each vertex draw an up-step in the path if the chord from that vertex is encountered for the first time, and a down-step otherwise. By means of this bijection, horizontal chords of the diagram correspond precisely to centered tunnels of the Dyck path (see Figure 1-4).

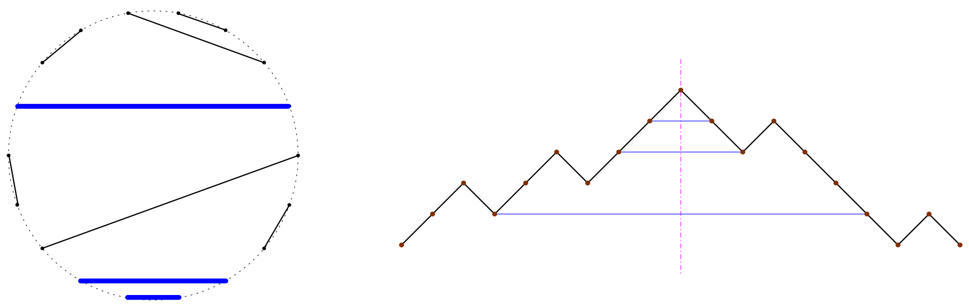


Figure 1-4: A bijection between nonintersecting chord diagrams and Dyck paths.

More generally, if we number the vertices of the $2n$-gon from 1 to $2n$ in the order in which they are read by the bijection, then, for $1 \leq i \leq n$, the chords parallel to the line between vertices i and $i+1$ correspond to tunnels of the Dyck path with midpoint at $x = i$ or at $x = n + i$.

Another class of objects in bijection with $\mathcal{D}_n$ is the set of plane trees with $n+1$ vertices. Consider the bijection described in [86, Exercise 6.19(e)]. Now, given a plane tree on $n+1$ vertices, label the vertices with integers from 0 to n in preorder (depth-first search) from left to right. Next, label the vertices again from 0 to n, but now in preorder from right to left. Then, the vertices other than the root for which the two labels coincide correspond to centered tunnels in the Dyck path. Besides, right tunnels correspond precisely to vertices for which the second label is less than the first one.

1.3 Combinatorial classes and generating functions

Here we direct the reader to [34] and [78] for a detailed account on combinatorial classes and the symbolic method. Let $\mathcal{A}$ be a class of unlabelled combinatorial objects and let $|\alpha|$ be the size of an object $\alpha \in \mathcal{A}$. If $\mathcal{A}_n$ denotes the objects in $\mathcal{A}$ of size n and $a_n = |\mathcal{A}_n|$, then the *ordinary generating function* of the class $\mathcal{A}$ is

$$A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|} = \sum_{n \geq 0} a_n z^n.$$

In our context, the size of a Dyck path is simply its semilength. From now on we will use the acronym GF as a shorthand for the term *generating function*.

There is a direct correspondence between set theoretic operations (or "constructions") on combinatorial classes and algebraic operations on GFs. Table 1.1 summarizes this correspondence for the operations that are used in this work. There "union" means union of disjoint copies, "product" is the usual cartesian product, and "sequence" forms an ordered sequence in the usual sense. Enumerations according to size and auxiliary parameters

Construction		*Operation on GFs*
Union	$\mathcal{A} = \mathcal{B} + \mathcal{C}$	$A(z) = B(z) + C(z)$
Product	$\mathcal{A} = \mathcal{B} \times \mathcal{C}$	$A(z) = B(z)C(z)$
Sequence	$\mathcal{A} = \mathrm{Seq}(\mathcal{B})$	$A(z) = \frac{1}{1-B(z)}$

Table 1.1: The basic combinatorial constructions and their translation into GFs.

$\chi_1, \chi_2, \ldots, \chi_r$ are described by multivariate GFs,

$$A(u_1, u_2, \ldots, u_r, z) = \sum_{\alpha \in \mathcal{A}} u_1^{\chi_1(\alpha)} u_2^{\chi_2(\alpha)} \cdots u_r^{\chi_r(\alpha)} z^{|\alpha|}.$$

Throughout this thesis the variable z is reserved for marking the length of a permutation and the semilength of a Dyck path, x is used for marking the number of fixed points of a permutation and the number of centered tunnels or tunnels of depth 0 of a Dyck path, and q is the variable that marks the number of excedances of a permutation and the number of right tunnels or tunnels of negative depth of a Dyck path, unless otherwise stated.

1.3.1 The Lagrange inversion formula

The Lagrange inversion formula (see for example [86, Theorem 5.4.2]) is a useful tool that provides a way to compute the coefficients of a generating function if it satisfies an equation of a certain form.

Theorem 1.3 ([86]) *Let $G(x) \in \mathbb{C}$ be a formal power series with $G(0) \neq 0$, and let $f(x)$ be defined by $f(x) = xG(f(x))$. Then, for any $k, n \in \mathbb{Z}$,*

$$n[x^n]f(x)^k = k[x^{n-k}]G(x)^n,$$

where $[z^n]A(z)$ denotes the coefficient of z^n in the expansion of $A(z)$.

1.3.2 Chebyshev polynomials

Chebyshev polynomials of the second kind are defined by $U_r(\cos\theta) = \frac{\sin(r+1)\theta}{\sin\theta}$ for $r \geq 0$. It can be checked that $U_r(t)$ is a polynomial of degree r in t with integer coefficients, and that the following recurrence holds:

$$\begin{cases} U_r(t) = 2tU_{r-1}(t) - U_{r-2}(t) & \text{for all } r \geq 2, \\ U_0(t) = 1,\ U_1(t) = 2t. \end{cases} \tag{1.2}$$

Chebyshev polynomials were invented for the needs of approximation theory, but are also widely used in various other branches of mathematics, including algebra, combinatorics, and number theory. The relation between restricted permutations and Chebyshev polynomials was discovered for the first time by Chow and West in [19], and later by Mansour and Vainshtein [59, 60], and Krattenthaler [52].

1.4 Patterns of length 3

For the case of patterns of length 3, it is known [51] that regardless of the pattern $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = \mathbf{C}_n$, the n-th Catalan number. While the equalities $|\mathcal{S}_n(132)| = |\mathcal{S}_n(231)| = |\mathcal{S}_n(312)| = |\mathcal{S}_n(213)|$ and $|\mathcal{S}_n(321)| = |\mathcal{S}_n(123)|$ are straightforward by reversal and complementation operations, the equality $|\mathcal{S}_n(321)| = |\mathcal{S}_n(132)|$ is more difficult to establish. Bijective proofs of this fact are given in [52, 70, 79, 91]. However, none of these bijections preserves either of the statistics fp or exc.

Patterns σ and σ' are said to be in the same *Wilf-equivalence* class if $|\mathcal{S}_n(\sigma)| = |\mathcal{S}_n(\sigma')|$ for all n. Partial results on the classification of forbidden patterns can be found in [5, 12, 13, 81, 82, 83].

1.4.1 Equidistribution of fixed points

It was not until recently that the concept of pattern avoidance, which regards a permutation as a word, was studied together with a statistic arising from viewing a permutation as a bijection. In the recent paper [73], Robertson, Saracino and Zeilberger consider restricted permutations with respect to the number of fixed points, obtaining the following refinement of the fact that $|\mathcal{S}_n(321)| = |\mathcal{S}_n(132)|$.

Theorem 1.4 ([73]) *The number of 321-avoiding permutations $\pi \in \mathcal{S}_n$ with $\mathrm{fp}(\pi) = i$ equals the number of 132-avoiding permutations $\pi \in \mathcal{S}_n$ with $\mathrm{fp}(\pi) = i$, for any $0 \leq i \leq n$.*

Their proof is nontrivial and technically involved. In the same paper, they study the distribution of fixed points for all six patterns of length 3.

Two questions arise naturally with this result in sight. The first one is whether there exists a simple bijection between $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$ that preserves the number of fixed points. This would give a better understanding of why fixed points are equidistributed in both sets of pattern-avoiding permutations. There does not seem to be an intuitive reason why Theorem 1.4 holds, especially since from the definitions fixed points do not seem to be related to the notion of pattern avoidance. The second question is whether this theorem can be generalized to other statistics or to other patterns. These two issues are discussed in Chapters 2, 3 and 4.

Chapter 2

Equidistribution of fixed points and excedances

In this chapter we give a generalization of Theorem 1.4. We show that the joint distribution of the number of fixed points and the number of excedances is the same in $\mathcal{S}_n(321)$ as in $\mathcal{S}_n(132)$. This result is stated in the following theorem.

Theorem 2.1 *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$ *and* $\mathrm{exc}(\pi) = j$ *equals the number of* 132*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$ *and* $\mathrm{exc}(\pi) = j$*, for any* $0 \leq i, j \leq n$*.*

Note that in terms of the generating functions F_σ defined in equation (1.1), this result can be expressed equivalently as $F_{321}(x, q, z) = F_{132}(x, q, z)$.

The proof that we give in this chapter is analytical. A bijective and more elegant proof of this result is presented in Chapter 4. However, we find it necessary to discuss the analytical proof here as well because it is interesting in its own right. One of the key points in the reasoning is the use of a nonstandard technique in generating functions, which involves taking the diagonal of a non-rational power series. To the best of our knowledge, this approach gives the first instance of an application of the computation of diagonals of non-rational generating functions to solve a combinatorial problem.

The proof presented in this chapter is done in two parts. First, in Section 2.1 we use a bijection between pattern-avoiding permutations and Dyck paths to obtain the GF $F_{321}(x, q, z)$ for the number of fixed points and excedances in 321-avoiding permutations. Then, in Section 2.2, we use another bijection to reduce the analogous problem from 132-avoiding permutations to a problem of Dyck paths. In this section we also give a proof Theorem 1.4, which is a weaker version of Theorem 2.1 that only considers the statistic fp. In Section 2.3 we show that $F_{132}(x, q, z) = F_{321}(x, q, z)$ using the technique of diagonals of GFs mentioned above. To do this, we introduce an extra variable marking a new parameter in the GF F_{132}. Then, using combinatorial properties, we deduce an identity that determines this four-variable GF. Finally, we conjecture an expression for it and check that our expression satisfies the identity, hence it is the correct GF.

2.1 The bijection $\psi_\llcorner$

In this section we define a bijection $\psi_\llcorner$ between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$, which appeared originally in [41, pg. 89] in a slightly different form, and later was used by Richard Stanley in connection to pattern avoidance. This will allow us to find an expression for the GF

$$F_{321}(x,q,z)=\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}z^n,$$

which enumerates 321-avoiding permutations with respect to fixed points and excedances.

We will give three equivalent definitions of the bijection $\psi_{\llcorner}$. Let $\pi=\pi_1\pi_2\cdots\pi_n\in\mathcal{S}_n(321)$. For $i\in[n]$, define $a_i=\max\{j:\{1,2,\ldots,j\}\subseteq\{\pi_1,\pi_2,\ldots,\pi_i\}\}$ (j can be 0, in which case $\{1,2,\ldots,j\}=\emptyset$). Now build the Dyck path $\psi_{\llcorner}(\pi)$ by adjoining, for each i from 1 to n, one up-step followed by $\max\{a_i-\pi_i+1,0\}$ down-steps. For example, for $\pi=23147586$ we get $a_1=a_2=0$, $a_3=3$, $a_4=a_5=4$, $a_6=a_7=5$, $a_8=8$, and the corresponding Dyck path is given in Figure 2-1.

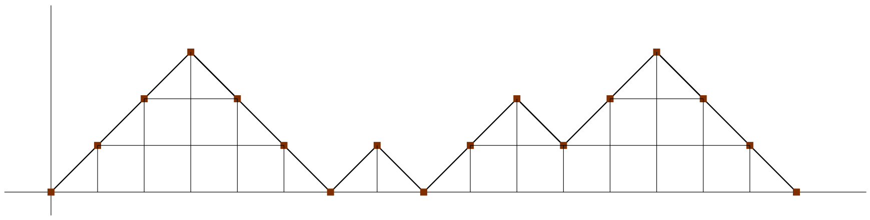


Figure 2-1: The Dyck path $\psi_{\llcorner}(23147586)$.

Here is an alternative way to define this bijection. Let $\pi_{i_1},\pi_{i_2},\ldots,\pi_{i_k}$ be the right-to-left minima of π, from left to right. For example, the right-to-left minima of 23147586 are $1,4,5,6$. Then, $\psi_{\llcorner}(\pi)$ is precisely the path that starts with i_1 up-steps, then has, for each j from 2 to k, $\pi_{i_j}-\pi_{i_{j-1}}$ down-steps followed by i_j-i_{j-1} up-steps, and finally ends with $n+1-\pi_{i_k}$ down-steps.

The third way to define $\psi_{\llcorner}$ is the easiest one to visualize, and the one that gives us a better intuition for how the bijection works. Consider the array of crosses $\mathrm{arr}(\pi)$ as defined in Section 1.1.3. By definition, excedances correspond to crosses strictly to the right of the main diagonal of the array. It is known (see e.g. [69]) that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. Therefore, the elements that are not excedances are precisely the right-to-left minima of π. Consider the path with *east* and *south* steps along the edges of the squares of $\mathrm{arr}(\pi)$ that goes from the upper-left corner to the lower-right corner of the array, leaving all the crosses to the right and remaining always as close to the main diagonal as possible. Let U be such path. Then $\psi_{\llcorner}(\pi)$ can be obtained from U just by reading an up-step for every south step of U, and a down-step for every east step of U. Figure 2-2 shows a picture of this bijection, again for $\pi=23147586$.

Proposition 2.2 *The bijection* $\psi_{\llcorner}:\mathcal{S}_n(321)\longrightarrow\mathcal{D}_n$ *satisfies*

(1) $\mathrm{fp}(\pi)=h(\psi_{\llcorner}(\pi))$,

(2) $\mathrm{exc}(\pi)=\mathrm{dr}(\psi_{\llcorner}(\pi))$,

for all $\pi\in\mathcal{S}_n(321)$.

Proof. To see this, just observe that fixed points of π correspond to crosses on the main diagonal of the array, which produce hills in the path. On the other hand, for each cross

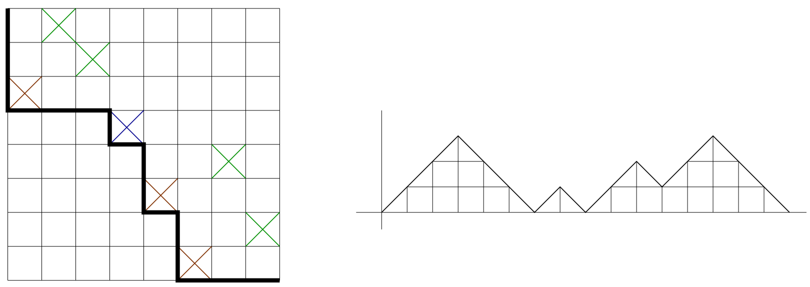


Figure 2-2: The bijection ψ_{L}.

corresponding to an excedance, the south step of U on the same row as the cross gives an up-step in $\psi_{\mathsf{L}}(\pi)$ which is followed by another up-step, thus forming a double rise. □

Therefore, counting 321-avoiding permutations according to the number of fixed points and excedances is equivalent to counting Dyck paths according to the number of hills and double rises. More precisely,

$$F_{321}(x,q,z) = \sum_{D\in\mathcal{D}} x^{h(D)} q^{\mathrm{dr}(D)} z^{|D|}.$$

We can give an equation for F_{321} using the symbolic method summarized in Section 1.3. A recursive definition for the class $\mathcal{D}$ is given by the fact that every non-empty Dyck path D can be decomposed in a unique way as $D = \mathbf{u}A\mathbf{d}B$, where $A, B \in \mathcal{D}$. Clearly if A is empty, $h(D) = h(B) + 1$ and $\mathrm{dr}(D) = \mathrm{dr}(B)$, and otherwise $h(D) = h(B)$ and $\mathrm{dr}(D) = \mathrm{dr}(A) + \mathrm{dr}(B) + 1$. Hence, we obtain the following equation for F_{321}:

$$F_{321}(x,q,z) = 1 + z(x + q(F_{321}(1,q,z) - 1))F_{321}(x,q,z). \tag{2.1}$$

Substituting first $x = 1$, we obtain that $F_{321}(1,q,z) = \frac{1+(q-1)z-\sqrt{1-2(1+q)z+(1-q)^2z^2}}{2qt}$. Now, solving (2.1) for $F_{321}(x,q,z)$ gives

$$F_{321}(x,q,z) = \frac{2}{1+(1+q-2x)z+\sqrt{1-2(1+q)z+(1-q)^2z^2}}. \tag{2.2}$$

To conclude this section, we want to remark that applying this bijection one can also obtain the GF that enumerates 321-avoiding permutations with respect to fixed points, excedances and descents. It follows easily from the description of ψ_{L} that the number $\mathrm{des}(\pi)$ of descents of a 321-avoiding permutation π equals the number of occurrences of **uud** in the Dyck word of $\psi_{\mathsf{L}}(\pi)$. Using the same decomposition as before, we obtain the following result.

Theorem 2.3

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(321)} x^{\mathrm{fp}(\pi)} q^{\mathrm{exc}(\pi)} p^{\mathrm{des}(\pi)} z^n$$
$$= \frac{2}{1+(1+q-2x)z+\sqrt{1-2(1+q)z+((1+q)^2-4qp)z^2}}.$$

2.2 The bijection φ

In this section we define a bijection φ between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$. This bijection will be used extensively throughout this work, because of its convenient properties.

Given any permutation $\pi \in \mathcal{S}_n$, consider its array $\mathrm{arr}(\pi)$ as defined in Section 1.1.3. The *diagram* of π can be obtained from it as follows. For each cross, shade the cell containing it and the squares that are due south and due east of it. The diagram is defined as the region that is left unshaded. It is shown in [69] that this gives a bijection between $\mathcal{S}_n(132)$ and Young diagrams that fit in the shape $(n-1, n-2, \dots, 1)$. Consider now the path determined by the border of the diagram of π, that is, the path with *north* and *east* steps that goes from the lower-left corner to the upper-right corner of the array, leaving all the crosses to the right, and staying always as close to the diagonal connecting these two corners as possible. Define $\varphi(\pi)$ to be the Dyck path obtained from this path by reading an up-step for each north step and a down-step for each east step (that is, we rotate it 45°). Since the path in the array does not go below the diagonal, $\varphi(\pi)$ does not go below the x-axis. Figure 2-3 shows an example when $\pi = 67435281$.

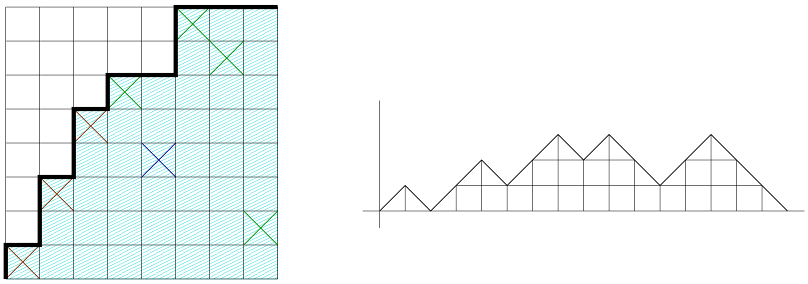


Figure 2-3: The bijection φ.

The bijection φ is essentially the same bijection between $\mathcal{S}_n(132)$ and $\mathcal{D}_n$ given by Krattenthaler [52] (see also [36]), up to reflection of the path from a vertical line.

Next we define the inverse map $\varphi^{-1} : \mathcal{D}_n \longrightarrow \mathcal{S}_n(132)$. Given a Dyck path $D \in \mathcal{D}_n$, the first step needed to reverse the above procedure is to transform D into a path U from the lower-left corner to the upper-right corner of an $n \times n$ array, not going below the diagonal connecting these two corners. Then, the squares to the left of this path form a Young diagram contained in the shape $(n-1, n-2, \dots, 1)$, and we can shade all the remaining squares. From this diagram, the permutation $\pi \in \mathcal{S}_n(132)$ can be recovered as follows: row by row, put a cross in the leftmost shaded square such that there is exactly one cross in each column. Start from the top and continue downward until all crosses are placed.

The bijection φ is useful here because it transforms fixed points and excedances of the permutation into centered tunnels and right tunnels of the Dyck path respectively. These two properties, along with a few more that will be used in upcoming chapters, are shown in the next proposition. Denote by $\mathrm{nlis}(\pi)$ the number of increasing subsequences of π of length $\mathrm{lis}(\pi)$.

Proposition 2.4 *The bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ satisfies*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\varphi(\pi))$,

(3) $\mathrm{des}(\pi) = \mathrm{va}(\varphi(\pi))$,

(4) $\mathrm{lis}(\pi) =$ *height of* $\varphi(\pi)$,

(5) $\mathrm{nlis}(\pi) = \#\{$*peaks of* $\varphi(\pi)$ *at maximum height*$\}$,

(6) $\mathrm{lds}(\pi) = \#\{$*peaks of* $\varphi(\pi)\}$,

(7) $\mathrm{rank}(\pi) = \frac{1}{2}\big(n - \nu(\varphi(\pi))\big)$,

for all $\pi \in \mathcal{S}_n(132)$.

Proof. For the proof of the first six equalities, instead of using $D = \varphi(\pi)$, it will be convenient to consider the associated path U from the lower-left corner to the upper-right corner of $\mathrm{arr}(\pi)$ with north and east steps. We will talk about tunnels of U to refer to the corresponding tunnels of D under this trivial transformation.

We now show how to associate a unique tunnel of D to each cross of the array $\mathrm{arr}(\pi)$. Observe that given a cross in position (i, j), U has a north step in row i and an east step in column j. In D, these two steps correspond to steps $\mathbf{u}$ and $\mathbf{d}$ respectively, so they determine a decomposition $D = A\mathbf{u}B\mathbf{d}C$ (see Figure 2-4), and therefore a tunnel of D (it is not hard to see that $\mathbf{u}$ and $\mathbf{d}$ are at the same level). According to whether the cross was to the left of, to the right of, or on the main diagonal or $\mathrm{arr}(\pi)$, the associated tunnel will be respectively a left, right, or centered tunnel of D. Thus, fixed points give centered tunnels and excedances give right tunnels.

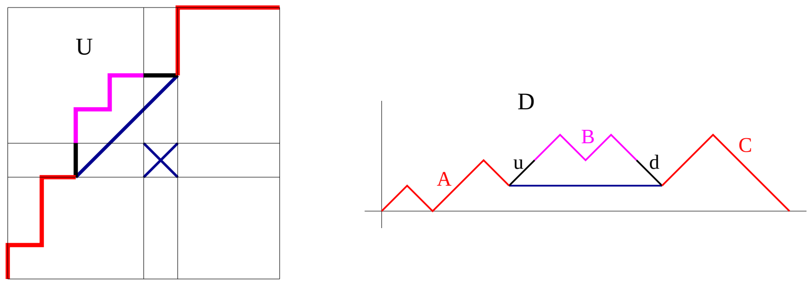


Figure 2-4: A cross and the corresponding tunnel.

To show (3), observe that from the description of φ^{-1}, a sequence of consecutive north steps of U gives rise to an increasing run of crosses in the rows of $\mathrm{arr}(\pi)$ where those steps lie. Descents of the permutation occur precisely in the rows of the array where there is a north step of U that is preceded by an east step. And these are just the valleys of $\varphi(\pi)$.

Property (4) is shown in [52], but here we give a more graphical proof. Given an increasing subsequence of π, consider the crosses of $\mathrm{arr}(\pi)$ that form such subsequence. The tunnels of $\varphi(\pi)$ corresponding to these crosses are all at different heights, and their projections on the x-axis are nested intervals (i.e., pairwise contained in each other). Reciprocally, any tower of tunnels of $\varphi(\pi)$ whose projections on the x-axis are nested corresponds to an increasing subsequence of π. The maximum number of tunnels in such a tower is the height of the path, so (4) follows. Furthermore, the number of such towers having as many tunnels as possible equals the number of peaks of $\varphi(\pi)$ at maximum height (the highest tunnel of the tower determines the peak), which proves (5).

Part (6) follows from the description of φ^{-1} and the observation that the crosses of the arr(π) located in the positions of the peaks (inner corners) of U form a decreasing subsequence of π of maximum length.

To prove the last equality of the proposition, notice that $\operatorname{rank}(\pi)$ is the largest m such that an $m\times m$ square fits in the upper-left corner of the diagram of π. Therefore, the height of $\varphi(\pi)$ at the middle is exactly $\nu(\varphi(\pi)) = n - 2\operatorname{rank}(\pi)$. $\square$

2.2.1 Enumeration of centered tunnels

As a consequence of the first two parts of Proposition 2.4, our problem of counting permutations according to fixed points and excedances is equivalent to counting Dyck paths according to centered and right tunnels. In particular, the GF

$$F_{132}(x,q,z) := \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}z^n$$

that we want to find can be expressed as

$$F_{132}(x,q,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}z^{|D|}.$$

If we try to imitate the same method that we used in Section 2.1 for 321-avoiding permutations, the next step now would be to enumerate Dyck paths with respect to centered tunnels and right tunnels. Unfortunately, the decomposition of Dyck paths that we used to count hills and double rises no longer works here. The reason for this is that if we write $D = \mathbf{u}A\mathbf{d}B$ with $A, B \in \mathcal{D}$, then $\mathrm{ct}(A)$ and $\mathrm{ct}(B)$ do not give information about $\mathrm{ct}(D)$, and similarly $\mathrm{rt}(D)$ cannot be obtained from $\mathrm{rt}(A)$ and $\mathrm{rt}(B)$.

For the case of counting only centered tunnels, however, we can use another decomposition. Let us forget for a moment about the number of right tunnels. Now we show how to obtain an expression for $F_{132}(x,1,z) = \sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}z^n = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}z^{|D|}$. Denote by $\mathrm{CT}(D)$ the set of centered tunnels of D, so that $|\mathrm{CT}(D)| = \mathrm{ct}(D)$. The trick is to consider Dyck paths where some centered tunnels are marked. That is, we will count pairs (D,S) where $D\in\mathcal{D}$ and $S\subseteq \mathrm{CT}(D)$ (S is the set of marked tunnels). Each such pair is given weight $(x-1)^{|S|}z^{|D|}$, so that for a fixed D, the sum of weights of all pairs (D,S) will be

$$\sum_{S\subseteq \mathrm{CT}(D)} (x-1)^{|S|}z^{|D|} = ((x-1)+1)^{|\mathrm{CT}(D)|}z^{|D|} = x^{\mathrm{ct}(D)}z^{|D|},$$

which is precisely the weight that D has in $F_{132}(x,1,z)$.

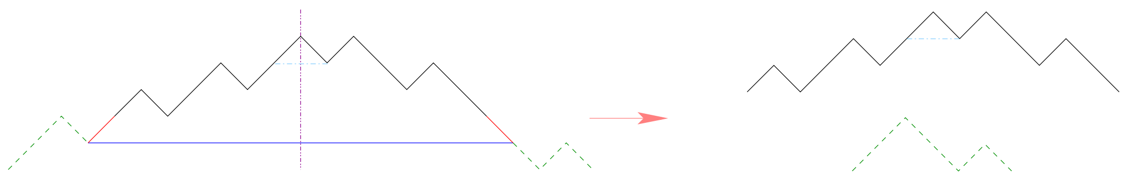


Figure 2-5: Decomposing Dyck paths with marked centered tunnels.

Dyck paths with no marked tunnels (i.e., pairs $(D,\emptyset)$) are enumerated by $\mathbf{C}(z) = \frac{1-\sqrt{1-4z}}{2z}$, the GF for Catalan numbers. On the other hand, for an arbitrary Dyck path

D with some centered tunnel marked (i.e., a pair (D, S) with $S \neq \emptyset$), we can consider the decomposition induced by the longest marked tunnel, say $D = A\mathbf{u}B\mathbf{d}C$. Then, AC (seen as the concatenation of Dyck words) gives an arbitrary Dyck path with no marked centered tunnels, and B is an arbitrary Dyck path where some centered tunnels may be marked (see Figure 2-5). This decomposition translates into the following equation for GFs:

$$F_{132}(x,1,z) = \mathbf{C}(z) + (x-1)z\mathbf{C}(z)F_{132}(x,1,z). \tag{2.3}$$

Solving it, we obtain

$$F_{132}(x,1,z) = \frac{2}{1+2(1-x)z+\sqrt{1-4z}},$$

which is precisely the expression that we had for $F_{321}(x,1,z)$ in (2.2). This gives a new and perhaps simpler proof of the main result in [73], namely that $|\{\pi \in \mathcal{S}_n(321) : \mathrm{fp}(\pi) = i\}| = |\{\pi \in \mathcal{S}_n(132) : \mathrm{fp}(\pi) = i\}|$ for all $i \leq n$ (see Theorem 1.4).

2.2.2 Enumeration of shifted tunnels

The same kind of decomposition can be used to count tunnels whose midpoints are at a given distance from the middle of the path. For $D \in \mathcal{D}$ and $r \in \mathbb{Z}$, let $\mathrm{ct}_r(D)$ be the number of tunnels of D whose midpoint lies on the vertical line $x = n - r$, and let

$$W_r(x,z) = \sum_{D \in \mathcal{D}} x^{\mathrm{ct}_r(D)} z^{|D|}$$

be the corresponding generating function. Note that

$$W_0(x,z) = F_{132}(x,1,z) = \frac{\mathbf{C}(z)}{1-(x-1)z\mathbf{C}(z)}$$

by (2.3). Let

$$\mathbf{C}_{<i}(z) = \sum_{j=0}^{i-1} \mathbf{C}_j z^j$$

be the series for the Catalan numbers truncated at degree i.

Analogously to the previous subsection, we will consider Dyck paths where some of the tunnels with midpoint on the line $x = n - r$ are marked. Each such marked tunnel will be given weight $x - 1$, so that for a fixed D, the contribution to $W_r(x,z)$ of the sum over all marked subsets of tunnels will be precisely $x^{\mathrm{ct}_r(D)} z^{|D|}$.

Dyck paths with no marked tunnels are enumerated by $\mathbf{C}(z)$. On the other hand, for an arbitrary Dyck path D with some marked tunnel with midpoint at $x = n - r$, we can consider the decomposition induced by the longest such marked tunnel, say $D = A\mathbf{u}B\mathbf{d}C$, where necessarily $\mathrm{length}(A) + 2r = \mathrm{length}(C)$. Then, AC (seen as the concatenation of Dyck words) gives an arbitrary Dyck path of length at least $2r$ with no marked tunnels, which produces the term $\mathbf{C}(z) - \mathbf{C}_{<r}(z)$ in the GF. The central part B is an arbitrary Dyck path where some centered tunnels may be marked (indeed, all the other marked tunnels of D with midpoint at $x = n - r$ become marked centered tunnels of B). This decomposition

gives the following formula for W_r.

$$\begin{aligned} W_r(x,z) &= \mathbf{C}(z) + (x-1)z\left[\mathbf{C}(z) - \mathbf{C}_{<r}(z)\right]W_0(x,z) \\ &= \frac{\mathbf{C}(z)\,(1-(x-1)z\mathbf{C}_{<r}(z))}{1-(x-1)z\mathbf{C}(z)} = \frac{2\,(1-(x-1)z\mathbf{C}_{<r}(z))}{1+2(1-x)z+\sqrt{1-4z}}. \end{aligned}$$

2.3 Enumeration of right tunnels: the method of the diagonal of a power series

To count right tunnels we will need a different approach. The technique that we use is based on the concept of *diagonal* of a power series.

Definition 1 *Let*

$$A(x,y) = \sum a_{i,j} x^i y^j$$

be a formal power series in the variables x and y. The diagonal *of A, denoted* $\mathrm{diag}_{x,y}^{z}\, A$*, is the power series in a single variable z defined by*

$$\mathrm{diag}_{x,y}^{z}\, A = \sum a_{n,n} z^n.$$

For convenience, rather than dealing with right tunnels, in this section we will consider left tunnels instead. Clearly, if we reflect $D \in \mathcal{D}_n$ from a vertical line $x = n$ we have that $\mathrm{ct}(D^*) = \mathrm{ct}(D)$, $\mathrm{rt}(D^*) = \mathrm{lt}(D)$ and $\mathrm{lt}(D^*) = \mathrm{rt}(D)$, so the problems of enumerating right or left tunnels respectively are equivalent. In particular,

$$F_{132}(x,q,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)} q^{\mathrm{lt}(D)} z^{|D|}.$$

A tunnel of $D \in \mathcal{D}_n$ is categorized as a centered or left tunnel depending on its position with respect to the line $x = n$. The first step of this new approach is to generalize the concepts of centered and left tunnels, allowing the vertical line that we use as a reference to be shifted from the center of the Dyck path. In the previous section we already defined $\mathrm{ct}_r(D)$ to be the number of tunnels of D whose midpoint lies on the vertical line $x = n - r$ (we call this the *reference line*), for any $D \in \mathcal{D}$ and $r \in \mathbb{Z}$. Similarly, let $\mathrm{lt}_r(D)$ be the number of tunnels of D whose midpoint lies on the half-plane $x < n - r$. Notice that by definition, ct_0 and lt_0 are respectively the statistics ct and lt defined previously.

We also add a new variable v to F_{132} which marks the distance from the reference line to the actual middle of the path. Define

$$T(x,q,z,v) := \sum_{n,r\geq 0} \sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_r(D)} q^{\mathrm{lt}_r(D)} v^r z^n. \tag{2.4}$$

Our next goal is to find an equation that determines $T(x,q,z,v)$. The idea is to use again the decomposition of a Dyck path as $D = \mathbf{u}A\mathbf{d}B$, where $A, B \in \mathcal{D}$. The difference is that now the GFs involve sums not only over Dyck paths but also over the possible positions of the reference line.

Let

$$H_1(x,q,z,v) := \sum_{\substack{n\geq 1 \\ k\geq -n}} \sum_{A\in\mathcal{D}_{n-1}} x^{\mathrm{ct}_{-k}(\mathbf{u}A\mathbf{d})} q^{\mathrm{lt}_{-k}(\mathbf{u}A\mathbf{d})} v^k z^n \tag{2.5}$$

be the GF for the first part $\mathbf{u}A\mathbf{d}$ of the decomposition, where the reference line can be anywhere to the right of the left end of the path (Figure 2-6). Similarly, let

$$H_2(x,q,z,v) := \sum_{\substack{n\geq 0 \\ r\geq -n}} \sum_{B\in\mathcal{D}_n} x^{\mathrm{ct}_r(B)} q^{\mathrm{lt}_r(B)} v^r z^n \tag{2.6}$$

be the GF for the second part B of the decomposition, where now the reference line can be anywhere to the left of the right end of the path.

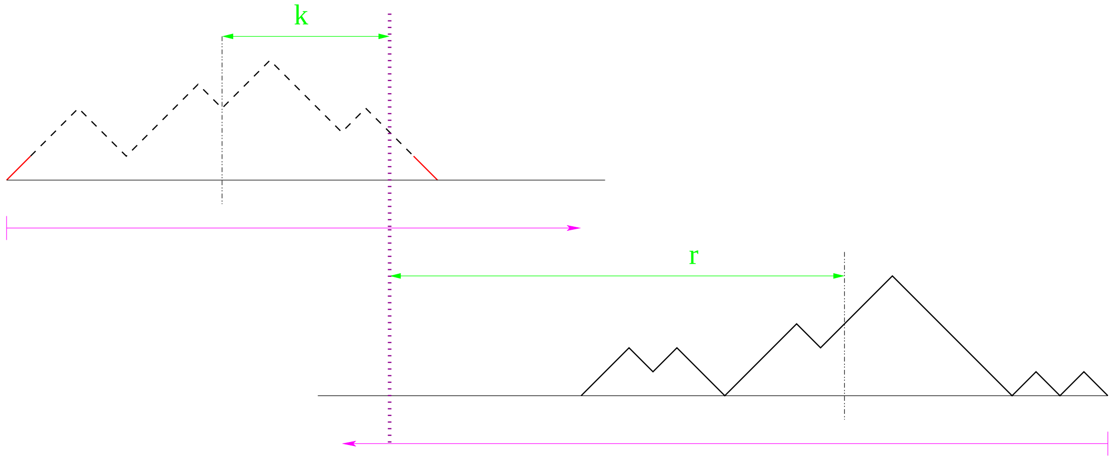


Figure 2-6: H_1 and H_2.

We would like to express the generating function for paths $\mathbf{u}A\mathbf{d}B$ in terms of H_1 and H_2. The product of these two GFs counts pairs $(\mathbf{u}A\mathbf{d}, B)$, but if we want the reference line to coincide in $\mathbf{u}A\mathbf{d}$ and in B, then the two parts are not necessarily placed next to each other (Figure 2-6). The exponent of v in H_1 indicates how far to the right the reference line is from the middle of the path $\mathbf{u}A\mathbf{d}$. The exponent of v in H_2 indicates how far to the left the reference line is from the middle of the path B. In the product H_1H_2, the exponent of v is the distance from the middle of the path $\mathbf{u}A\mathbf{d}$ to the middle of the path B if we draw them so that the reference lines coincide. Now comes one of the key points of the argument. The terms that correspond to an actual path $D = \mathbf{u}A\mathbf{d}B$ are those in which the two parts are placed next to each other in the picture (B begins where $\mathbf{u}A\mathbf{d}$ ends), and this happens precisely when the exponent of v is half the sum of lengths of $\mathbf{u}A\mathbf{d}$ and B. But the semilength of each path is the exponent of z in the corresponding GF, so the sum of semilengths is the exponent of z in the product H_1H_2. Hence, the terms that correspond to actual paths $D = \mathbf{u}A\mathbf{d}B$ are exactly those in which the exponent of v equals the exponent of z (Figure 2-7). The generating function consisting of only such terms is nothing else than a diagonal.

We also need another variable y to mark the distance between the reference line and the middle of the new path $D = \mathbf{u}A\mathbf{d}B$. Considering that D starts at $(0,0)$, the x-coordinate

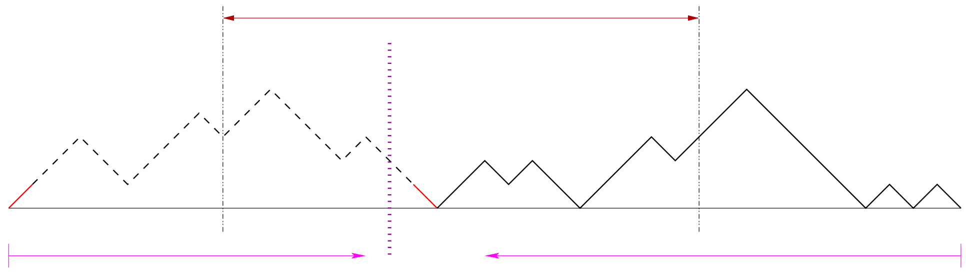


Figure 2-7: Terms with equal exponent in z and v.

of the middle of this path is given by the exponent of z in the product H_1H_2, which is the sum of the exponents of z in H_1 and H_2. The x-coordinate of the reference line is given by the exponent of z in H_1 plus the exponent of v in H_1. Hence, the difference between these two x-coordinates is given by the exponent of z in H_2 minus the exponent of v in H_1.

Let

$$P(x,q,z,v,y) := H_1(x,q,z,v/y)H_2(x,q,zy,v),$$

and let its series expansion in v and z be

$$P(x,q,z,v,y) = \sum_{\substack{n\geq 0\\ j\geq -n}} P_{j,n}(x,q,y)v^j z^n.$$

Consider the diagonal (in v and z) of P,

$$\operatorname{diag}_{v,z}^t\ P := \sum_{n\geq 0} P_{n,n}(x,q,y)t^n.$$

Now, the above argument implies that this diagonal equals precisely

$$H_3(x,q,t,y) := \sum_{\substack{n\geq 1\\ -n\leq r\leq n}} \sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_r(D)}q^{\mathrm{lt}_r(D)}y^r t^n, \tag{2.7}$$

that is, the sum over arbitrary non-empty (since $\mathbf{u}A\mathbf{d}$ was non-empty) Dyck paths D, where the reference line can be anywhere between the left end and the right end of the path.

We have found an equation that relates H_1, H_2 and H_3, thus proving the following lemma.

Lemma 2.5 *Let H_1, H_2 and H_3 be defined respectively by (2.5), (2.6), and (2.7). Then,*

$$\operatorname{diag}_{v,z}^t\ H_1(x,q,z,v/y)H_2(x,q,zy,v) = H_3(x,q,t,y). \tag{2.8}$$

The next step is to express these three GFs in terms of T, so that (2.8) will in fact give an equation for T. First, note that given $D\in\mathcal{D}_n$, we have that $\mathrm{ct}_{-r}(D) = \mathrm{ct}_r(D^*)$ and $\mathrm{lt}_{-r}(D) = n - \mathrm{lt}_r(D^*) - \mathrm{ct}_r(D^*)$, since the total number of tunnels of D^* is n. Thus,

$$\begin{aligned}\sum_{n,r\geq 0}\sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_{-r}(D)}q^{\mathrm{lt}_{-r}(D)}v^r z^n &= \sum_{n,r\geq 0}\sum_{D\in\mathcal{D}_n}\left(\frac{x}{q}\right)^{\mathrm{ct}_r(D^*)}\left(\frac{1}{q}\right)^{\mathrm{lt}_r(D^*)} v^r(qz)^n\\ &= T(x/q,1/q,qz,v). \end{aligned}\tag{2.9}$$

Also, note that if $|D| = n$ and $r \geq n$, then $\mathrm{ct}_r(D) = \mathrm{lt}_r(D) = \mathrm{ct}_{-r}(D) = 0$ and $\mathrm{lt}_{-r}(D) = n$. In particular,

$$\sum_{\substack{n\geq 0\\ r>n}} \sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_r(D)} q^{\mathrm{lt}_r(D)} v^r z^n = \sum_{\substack{n\geq 0\\ r>n}} \mathbf{C}_n v^r z^n = \sum_{n\geq 0} \mathbf{C}_n \frac{v^{n+1}}{1-v} z^n = \frac{v}{1-v}\mathbf{C}(zv). \tag{2.10}$$

For H_1 we can write

$$\begin{aligned} H_1(x,q,z,v) &= \sum_{\substack{n\geq 0\\ k\geq -n-1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(\mathbf{u}A\mathbf{d})} q^{\mathrm{lt}_{-k}(\mathbf{u}A\mathbf{d})} v^k z^{n+1} \\ &= z\left[\sum_{\substack{n\geq 0\\ k>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(\mathbf{u}A\mathbf{d})} q^{\mathrm{lt}_{-k}(\mathbf{u}A\mathbf{d})} v^k z^n + \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_0(\mathbf{u}A\mathbf{d})} q^{\mathrm{lt}_0(\mathbf{u}A\mathbf{d})} z^n \right.\\ &\qquad \left. + \sum_{\substack{n\geq 0\\ 0<r\leq n+1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_r(\mathbf{u}A\mathbf{d})} q^{\mathrm{lt}_r(\mathbf{u}A\mathbf{d})} v^{-r} z^n \right]. \end{aligned} \tag{2.11}$$

For $k > 0$, $\mathrm{ct}_{-k}(\mathbf{u}A\mathbf{d}) = \mathrm{ct}_{-k}(A)$ and $\mathrm{lt}_{-k}(\mathbf{u}A\mathbf{d}) = \mathrm{lt}_{-k}(A) + 1$, so the first sum on the right hand side of (2.11) equals

$$\begin{aligned} & q\sum_{\substack{n\geq 0\\ k>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(A)} q^{\mathrm{lt}_{-k}(A)} v^k z^n \\ = \;& q\left[\sum_{\substack{n\geq 0\\ k\geq 0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_{-k}(A)} q^{\mathrm{lt}_{-k}(A)} v^k z^n - \sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_0(A)} q^{\mathrm{lt}_0(A)} z^n\right] \\ = \;& q\left[T(x/q, 1/q, qz, v) - T(x,q,z,0)\right], \end{aligned}$$

by (2.9). For the second sum in (2.11), note that $\mathrm{ct}_0(\mathbf{u}A\mathbf{d}) = \mathrm{ct}_0(A) + 1$ and $\mathrm{lt}_0(\mathbf{u}A\mathbf{d}) = \mathrm{lt}_0(A)$, so the sum equals

$$x\sum_{n\geq 0} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_0(A)} q^{\mathrm{lt}_0(A)} z^n = x\,T(x,q,z,0).$$

Using that for $r > 0$ $\mathrm{ct}_r(\mathbf{u}A\mathbf{d}) = \mathrm{ct}_r(A)$ and $\mathrm{lt}_r(\mathbf{u}A\mathbf{d}) = \mathrm{lt}_r(A)$, the third sum in (2.11) can be written as

$$\begin{aligned} &\sum_{\substack{n\geq 0\\ r>0}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_r(A)} q^{\mathrm{lt}_r(A)} v^{-r} z^n - \sum_{\substack{n\geq 0\\ r>n+1}} \sum_{A\in\mathcal{D}_n} x^{\mathrm{ct}_r(A)} q^{\mathrm{lt}_r(A)} v^{-r} z^n \\ &\qquad = T(x,q,z,v^{-1}) - T(x,q,z,0) - \frac{1}{v(v-1)}\mathbf{C}(zv^{-1}), \end{aligned}$$

by (2.10). Thus,

$$H_1(x,q,z,v) = \\ = z\left[qT(\frac{x}{q},\frac{1}{q},qz,v) + (x-q-1)T(x,q,z,0) + T(x,q,z,\frac{1}{v}) + \frac{1}{v(1-v)}\mathbf{C}(\frac{z}{v})\right]. \quad (2.12)$$

For H_2, a very similar reasoning implies that

$$H_2(x,q,z,v) = T(x,q,z,v) - T(x,q,z,0) + T(\frac{x}{q},\frac{1}{q},qz,\frac{1}{v}) + \frac{1}{1-v}\mathbf{C}(\frac{qz}{v}). \quad (2.13)$$

Finally, for H_3 we get that

$$H_3(x,q,t,y) = \\ = T(x,q,t,y) + T(\frac{x}{q},\frac{1}{q},qt,\frac{1}{y}) - T(x,q,t,0) - \frac{y}{1-y}\mathbf{C}(ty) + \frac{1}{1-y}\mathbf{C}(\frac{qt}{y}) - 1. \quad (2.14)$$

Substituting these expressions for H_1, H_2 and H_3 in (2.8) we obtain an equation for T. Note that the common factor z in $H_1(x,q,z,v)$ guarantees that this equation will express the coefficients of the series expansion in t of $H_3(x,q,t,y)$ in terms of coefficients of T of smaller order in the series expansion in z of $H_1(x,q,z,v)H_2(x,q,z,v)$, so it uniquely determines T as a GF. The final step of the proof is to guess an expression for T and check that it satisfies this equation.

Proposition 2.6 *We have*

$$T(x,q,z,v) = \frac{\dfrac{1-v+(q-1)zv\mathbf{C}(zv)}{1-v+(q-1)zvF_{321}(1,q,z)} - (x-1)zv\mathbf{C}(zv)}{[1-qz(F_{321}(1,q,z)-1)-xz](1-v)}. \quad (2.15)$$

Before proving this proposition, we observe that it implies Theorem 2.1. Indeed, we have by definition

$$T(x,q,z,0) = \sum_{n\geq 0}\sum_{D\in\mathcal{D}_n} x^{\mathrm{ct}_0(D)}q^{\mathrm{lt}_0(D)}z^n = F_{132}(x,q,z).$$

But if Proposition 2.6 holds, then

$$T(x,q,z,0) = \frac{1}{1-qt(F_{321}(1,q,z)-1)-xt} = F_{321}(x,q,z),$$

where the last equality follows from (2.1). So, all that remains is to prove Proposition 2.6.

Proof. The computations that follow have been done using *Maple*. Let $\widetilde{H}_1$, $\widetilde{H}_2$ and $\widetilde{H}_3$ be the expressions obtained respectively from (2.12), (2.13) and (2.14) when T is substituted with the expression given in (2.15). All we have to check is that

$$\mathrm{diag}^t_{v,z}\ \widetilde{H}_1(x,q,z,v/y)\widetilde{H}_2(x,q,zy,v) = \widetilde{H}_3(x,q,t,y).$$

Let $\widetilde{P}(x,q,z,v,y) := \widetilde{H}_1(x,q,z,v/y)\widetilde{H}_2(x,q,zy,v)$. We want to compute $\mathrm{diag}^t_{v,z}\ \widetilde{P}$. In [86, Chapter 6], a general method is described for obtaining diagonals of rational functions. This theory does not apply to our function $\widetilde{P}$, because it is not rational. However, we will show that in this particular case we can modify the technique to obtain $\operatorname{diag}^t_{v,z} \widetilde{P}$.

The series expansion of $\widetilde{P}$ in v and z,

$$\widetilde{P}(x,q,z,v,y) = \sum_{\substack{n\geq 0\\ j\geq -n}} \widetilde{P}_{j,n}(x,q,y)v^j z^n = \sum_{n,i\geq 0} \widetilde{P}_{i-n,n}(x,q,y)v^i \left(\frac{z}{v}\right)^n,$$

converges for $|v| < \beta$, $|\frac{z}{v}| < \alpha$, if $\alpha, \beta > 0$ are taken sufficiently small. Similarly,

$$\operatorname{diag}^t_{v,z} \widetilde{P} = \sum_{n\geq 0} \widetilde{P}_{n,n}(x,q,y)t^n$$

converges for $|t|$ sufficiently small. Fix such a t that also satisfies $|t| < \alpha\beta^2$. The series

$$\widetilde{P}(x,q,z,\frac{t}{z},y) = \sum_{\substack{n\geq 0\\ j\geq -n}} \widetilde{P}_{j,n}(x,q,y)t^j z^{n-j}$$

will converge for $|\frac{t}{z}| < \beta$ and $|\frac{z^2}{t}| < \alpha$. Regarded as a function of z, it will converge for $|z|$ in the annulus $\frac{|t|}{\beta} < |z| < \sqrt{\alpha|t|}$, which is non-empty because $|t| < \alpha\beta^2$. In particular, it converges on some circle $|z| = \rho$ in the annulus. By [46, Theorem 1],

$$\operatorname{diag}^t_{v,z} \widetilde{P} = \frac{1}{2\pi i}\int_{|z|=\rho} \widetilde{P}(x,q,z,\frac{t}{z},y)\frac{dz}{z}.$$

It can be checked that the singularities of $\widetilde{P}(x,q,z,\frac{t}{z},y)/z$ (as a function of z) that lie inside the circle $|z| = \rho$ are all simple poles. These poles are

$$z_1 = 0,\ z_2 = t,\ z_3 = \frac{t}{y},\ z_{4,5} = \frac{(1+q)y \pm (1-q)\sqrt{y(y-4qt)}}{2y(y+t(1-q)^2)}t,$$

$$z_{6,7} = \frac{1+q \pm (1-q)\sqrt{1-4ty}}{2(q+ty(1-q)^2)}t.$$

There are also branch points for $z = \pm\frac{1}{2}\sqrt{\frac{t}{y}}$ and $z = \pm\frac{1}{2}\sqrt{\frac{t}{qy}}$, but they lie outside the circle for an appropriate choice of ρ in the annulus $\frac{|t|}{\beta} < \rho < \sqrt{\alpha|t|}$. The remaining singularities do not depend on t and lie outside the circle.

So, by the residue theorem, the integral can be obtained by summing up the residues at the poles inside $|z| = \rho$. Computing them in *Maple*, we see that all the residues are 0 except for those in z_2 and z_3. Thus,

$$\operatorname{diag}^t_{v,z} \widetilde{P} = \operatorname{Res}_{z=t} \widetilde{P}(x,q,z,\frac{t}{z},y)\frac{1}{z} + \operatorname{Res}_{z=\frac{t}{y}} \widetilde{P}(x,q,z,\frac{t}{z},y)\frac{1}{z}.$$

Computing these residues we get precisely $\widetilde{H}_3(x,q,t,y)$. $\square$

2.4 Some other bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$

Considering the array of crosses associated to a permutation, as we did to define $\psi_{\llcorner}$, some other known bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ can easily be viewed in a systematic way, as paths with east and south steps from the upper-left corner to the lower-right corner of the $n \times n$ array. For each of these bijections, our canonical example will be $\pi = 23147586$. One such bijection was established by Billey, Jockusch and Stanley in [7, p. 361]. Denote it by $\phi_{\llcorner}$. Consider the path with east and south steps that leaves the crosses corresponding to excedances to the right, and stays always as far from the main diagonal as possible (Figure 2-8). Then $\phi_{\llcorner}(\pi)$ can be obtained from it just by reading an up-step for every east step of this path and a down-step for every south step.

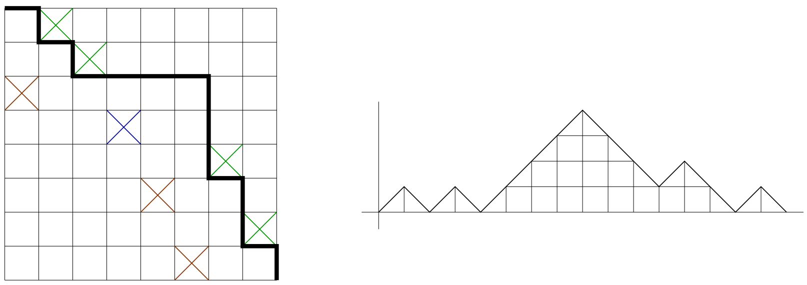


Figure 2-8: The bijection $\phi_{\llcorner}$.

In [52], Krattenthaler describes a bijection from $\mathcal{S}_n(123)$ to $\mathcal{D}_n$. If we omit the last step, consisting of reflecting the path over a vertical line, and compose the bijection with the reversal operation, that maps a permutation $\pi_1\pi_2\cdots\pi_n$ into $\pi_n\cdots\pi_2\pi_1$, we get a bijection from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$. Denote it by $\psi_{\neg}$. In the array representation, $\psi_{\neg}(\pi)$ corresponds (by the same trivial transformation as before) to the path with east and south steps that leaves all the crosses to the left and remains always as close to the main diagonal as possible (Figure 2-9).

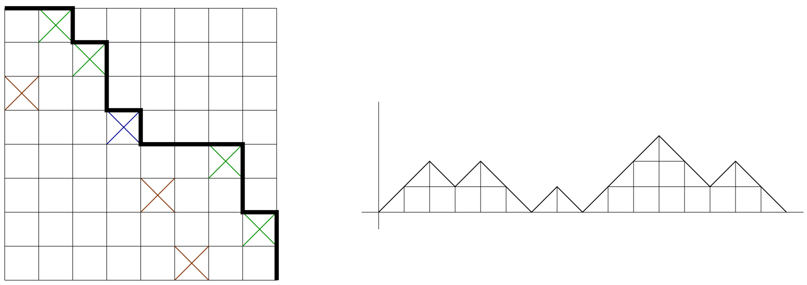


Figure 2-9: The bijection $\psi_{\neg}$.

Our first bijection is related to this last one by $\psi_{\neg}(\pi) = \psi_{\llcorner}(\pi^{-1})$. In a similar way, we could still define a fourth bijection $\phi_{\neg} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ by $\phi_{\neg}(\pi) := \phi_{\llcorner}(\pi^{-1})$ (Figure 2-10).

Combining these bijections and their inverses, one can get some automorphisms on Dyck paths and on 321-avoiding permutations with interesting properties. Recall from Section 1.2.1 that $\mathrm{va}(D)$ and $p_2(D)$ denote respectively the number of valleys and the number of peaks of height at least 2 of D. It can be checked that $\psi_{\llcorner} \circ \phi_{\llcorner}^{-1}$ is an involution on $\mathcal{D}_n$ with the property that $\mathrm{va}(\psi_{\llcorner} \circ \phi_{\llcorner}^{-1}(D)) = \mathrm{dr}(D)$ and $\mathrm{dr}(\psi_{\llcorner} \circ \phi_{\llcorner}^{-1}(D)) = \mathrm{va}(D)$.

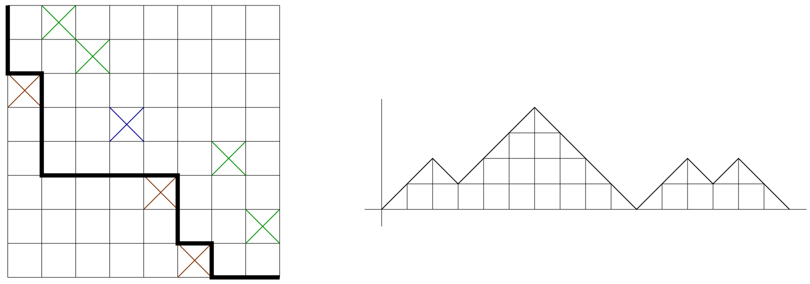


Figure 2-10: The bijection $\phi_{\neg}$.

Indeed, this follows from the fact that excedances are mapped to valleys by $\phi_{\llcorner}$ and to double rises by $\psi_{\llcorner}$. This bijection gives a new proof of the symmetry of the bivariate distribution of the pair $(\mathrm{va}, \mathrm{dr})$ of statistics on Dyck paths. A different involution with this property was introduced in [21].

Another involution on $\mathcal{D}_n$ is given by $\psi_{\llcorner} \circ \psi_{\neg}^{-1}$. This one shows the symmetry of the distribution of the pair (dr, p_2), because $\mathrm{dr}(\psi_{\llcorner} \circ \psi_{\neg}^{-1}(D)) = p_2(D)$ and $p_2(\psi_{\llcorner} \circ \psi_{\neg}^{-1}(D)) = \mathrm{dr}(D)$. In addition, it preserves the number of hills, i.e., $h(\psi_{\llcorner} \circ \psi_{\neg}^{-1}(D)) = h(D)$. To see this, just note that both $\psi_{\neg}$ and $\psi_{\llcorner}$ map fixed points to hills, whereas excedances are mapped to peaks of height at least 2 by $\psi_{\neg}$ and to double rises by $\psi_{\llcorner}$.

It is well known that the number of permutations in $\mathcal{S}_n$ with k excedances equals the number of permutations in $\mathcal{S}_n$ with $k+1$ weak excedances (recall that i is a *weak excedance* of π if $\pi_i \geq i$). We can show combinatorially that the analogous results for 321-avoiding and for 132-avoiding permutations hold as well.

Proposition 2.7 *Fix $n, k \geq 0$. The following quantities are equal to the* Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$.

(1) *The number of 321-avoiding permutations $\pi \in \mathcal{S}_n$ with k excedances.*

(2) *The number of 321-avoiding permutations $\pi \in \mathcal{S}_n$ with $k+1$ weak excedances.*

(3) *The number of 132-avoiding permutations $\pi \in \mathcal{S}_n$ with k excedances.*

(4) *The number of 132-avoiding permutations $\pi \in \mathcal{S}_n$ with $k+1$ weak excedances.*

(5) *The number of 132-avoiding permutations $\pi \in \mathcal{S}_n$ with k descents.*

Proof. By Proposition 2.2, excedances of $\pi \in \mathcal{S}_n(321)$ correspond to double rises of $\psi_{\llcorner}(\pi)$. It is known that the number of Dyck paths in $\mathcal{D}_n$ with k double rises is given by the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$.

To prove the equality (1)=(2), consider the involution on $\mathcal{S}_n(321)$ that maps π to $(\phi_{\llcorner}^{-1}(\psi_{\llcorner}(\pi)))^{-1}$. Excedances of $\pi \in \mathcal{S}_n(321)$ give double rises in $\psi_{\llcorner}(\pi)$. On the other hand, the bijection $\phi_{\llcorner}^{-1}$ maps valleys to excedances. Combining this together, we have that $\mathrm{exc}(\pi) = \mathrm{dr}(\psi_{\llcorner}(\pi)) = n - \mathrm{va}(\psi_{\llcorner}(\pi)) - 1 = n - \mathrm{exc}(\phi_{\llcorner}^{-1}(\psi_{\llcorner}(\pi))) - 1$, where the second equality follows from the fact that each up-step of a Dyck path is either the beginning of a double rise or the beginning of a peak, so the number of peaks plus double rises equals the semilength of the path. Finally, we use that the number of excedances of a permutation in $\mathcal{S}_n$ plus the number of weak excedances of its inverse is n.

The equalities (1)=(3) and (2)=(4) are immediate consequences of Theorem 2.1. Finally, to see that (1)=(5), notice that if $\pi \in \mathcal{S}_n(321)$, then $\mathrm{exc}(\pi) = \mathrm{va}(\phi_{\mathsf{L}}(\pi))$. On the other hand, if $\pi \in \mathcal{S}_n(132)$, then $\mathrm{des}(\pi) = \mathrm{va}(\varphi(\pi))$ by Proposition 2.4. Thus, $\varphi^{-1} \circ \phi_{\mathsf{L}} : \mathcal{S}_n(321) \longrightarrow \mathcal{S}_n(132)$ maps excedances to descents. $\square$

Chapter 3

A simple and unusual bijection for Dyck paths

In this chapter we introduce a new bijection Φ from the set of Dyck paths to itself. This bijection has the property that it maps nontrivial statistics that appear in the study of pattern-avoiding permutations into classical statistics on Dyck paths, which have been widely studied in the literature and whose distribution is easy to obtain.

Recall that in Section 2.2 we tried to enumerate Dyck paths with respect to the number of centered and right tunnels, but we were unable to do it directly. Intuitively, the problem is that unlike hills, peaks, or double rises, which are characteristics of a Dyck path that are defined locally, the notion of tunnel may involve an arbitrarily large number of steps of the path. This is essentially why the standard decompositions of Dyck paths do not work to enumerate centered and right tunnels. The bijection Φ has the advantage that it transforms tunnel-like statistics into locally defined statistics that behave well under the usual decompositions of Dyck paths. As a consequence, several enumeration problems regarding permutation statistics on restricted permutations can be solved more easily considering their counterpart in terms of Dyck paths.

Another important application of Φ is that it allows us to give a simple bijective proof of Theorem 1.4, which is a weaker version of Theorem 2.1 considering only the number of fixed points. Some results in this chapter are joint work with Emeric Deutsch [29].

In Section 3.1 we present the bijection Φ, and in Section 3.2 we study its properties. In Section 3.3 we give a generalization of Φ, namely a family of bijections depending on an integer parameter r, from which the main bijection Φ is the particular case $r = 0$. These bijections give correspondences involving new statistics on Dyck paths, which generalize ct and rt. We give multivariate generating functions for them. Section 3.4 discusses several applications of these bijections to enumeration of statistics on 321- and 132-avoiding permutations. In particular, we generalize Theorem 1.4, and we find a multivariate generating function for fixed points, excedances and descents in 132-avoiding permutations. Finally, in Section 3.5 we discuss new interpretations of Catalan numbers that follow from our work.

3.1 The bijection Φ

In this section we describe a bijection Φ from $\mathcal{D}_n$ to itself. Let $D \in \mathcal{D}_n$. Each up-step of D has a corresponding down-step together with which it determines a tunnel. Match each such pair of steps. Let $\tau \in \mathcal{S}_{2n}$ be the permutation defined by

$$\tau_i = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd;} \\ 2n+1-\dfrac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

In two-line notation,

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ 1 & 2n & 2 & 2n-1 & 3 & 2n-2 & \cdots & n-1 & n+2 & n & n+1 \end{pmatrix}.$$

Then $\Phi(D)$ is created as follows. For i from 1 to $2n$, consider the τ_i-th step of D (i.e., D is read in zigzag). If its corresponding matching step has not yet been read, define the i-th step of $\Phi(D)$ to be an up-step, otherwise let it be a down-step. In the first case, we say that the τ_i-th step of D *opens* a tunnel, in the second we say that it *closes* a tunnel.

The bijection Φ applied to the Dyck paths of semilength at most 3 is shown in Figure 3-1. Figure 3-2 shows Φ applied to the example of the Dyck path $D =$ **uuduudududdddud**.

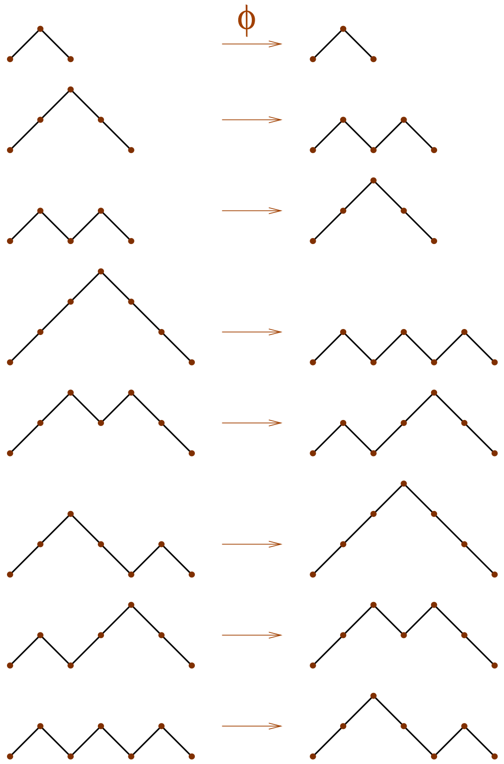


Figure 3-1: The bijection Φ for paths of length at most 3.

It is clear from the definition that $\Phi(D)$ is a Dyck path. Indeed, it never goes below the x-axis because at any point the number of down-steps drawn so far can never exceed the number of up-steps, since each down-step is drawn when the second step of a matching pair in D is read, and in that case the first step of the pair has already produced an up-step in $\Phi(D)$. Also, $\Phi(D)$ ends in $(2n, 0)$ because each of the matched pairs of D produces an up-step and a down-step in $\Phi(D)$.

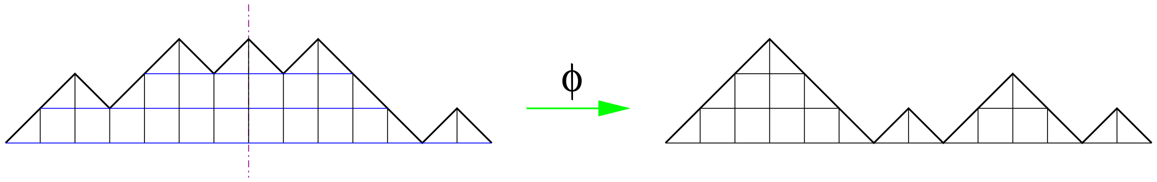


Figure 3-2: An example of Φ.

To show that Φ is indeed a bijection, we will describe the inverse map Φ^{-1}. Given $D' \in \mathcal{D}_n$, the following procedure recovers the $D \in \mathcal{D}_n$ such that $\Phi(D) = D'$. Consider the permutation τ defined above, and let $W = w_1 w_2 \cdots w_{2n}$ be the word obtained from D' as follows. For i from 1 to $2n$, if the i-th step of D' is an up-step, let $w_{\tau_i} = o$, otherwise let $w_{\tau_i} = c$. W contains the same information as D', with the advantage that the o's are located in the positions of D in which a tunnel is opened when D is read in zigzag, and the c's are located in the positions where a tunnel is closed. Equivalently, the o's are located in the positions of the left walls of the left and centered tunnels of D, and in the positions of the right walls of the right tunnels. For an example see Figure 3-3.

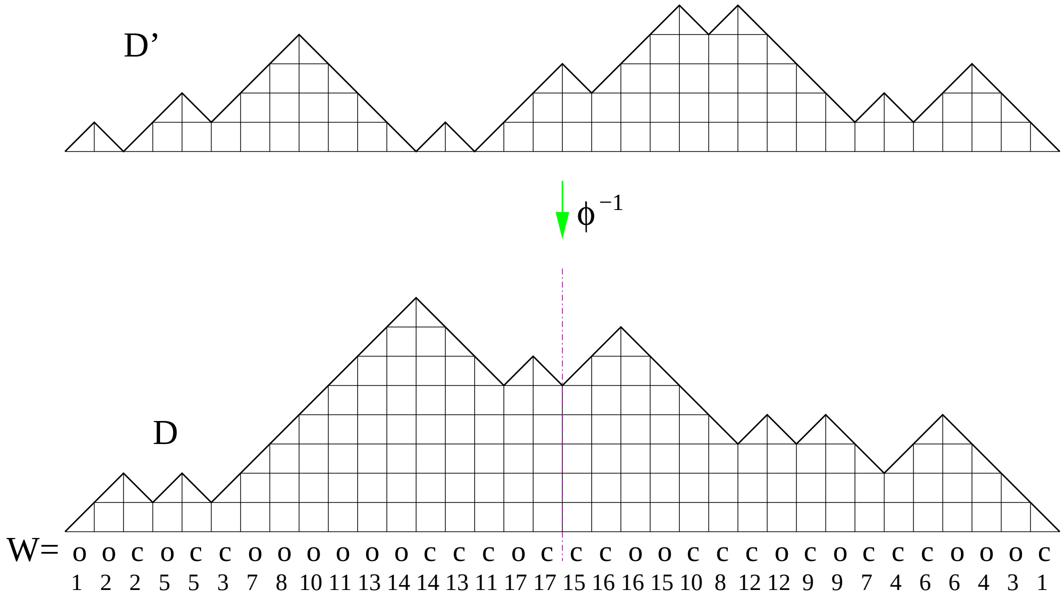


Figure 3-3: The inverse of Φ.

Now we define a matching between the o's and the c's in W, so that each matched pair will give a tunnel in D. We will label the o's with $1, 2, \ldots, n$ and similarly the c's, to indicate that an o and a c with the same label are matched. By left (resp. right) half of W we mean the symbols w_i with $i \leq n$ (resp. $i > n$). For i from 1 to $2n$, if $w_{\tau_i} = o$, place in it the smallest label that has not been used yet. If $w_{\tau_i} = c$, match it with the unmatched o in the same half of W as w_{τ_i} with largest label, if such an o exists. If it does not, match w_{τ_i} with the unmatched o in the opposite half of W with smallest label. Note that since D' was a Dyck path, at any time the number of c's read so far does not exceed the number of o's, so each c has some o to be paired up with.

Once the symbols in W have been labelled, D can be recovered by reading the labels from left to right, drawing an up-step for each label that is read for the first time, and a down-step for each label that appears the second time. In Figure 3-3 the labelling is shown under W.

3.2 Properties of Φ

Lemma 3.1 *Let $D = ABC$ be a decomposition of a Dyck path D, where B is a Dyck path, and A and C have the same length. Then $\Phi(ABC) = \Phi(AC)\Phi(B)$. In particular, $\Phi(\mathbf{u}B\mathbf{d}) = \mathbf{ud}\Phi(B)$.*

Proof. It follows immediately from the definition of Φ, since the path D is read in zigzag while $\Phi(D)$ is built from left to right. □

Theorem 3.2 *Let D be any Dyck path, and let $D' = \Phi(D)$. We have the following correspondences:*

(1) $\mathrm{ct}(D) = h(D')$,

(2) $\mathrm{rt}(D) = \mathrm{er}(D')$,

(3) $\mathrm{lt}(D) + \mathrm{ct}(D) = \mathrm{or}(D')$,

(4) $\mathrm{cmt}(D) = \mathrm{ret}(D')$.

Proof. First we show (1). Consider a centered tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$. Applying Lemma 3.1 twice, we get $D' = \Phi(A\mathbf{u}B\mathbf{d}C) = \Phi(AC)\Phi(\mathbf{u}B\mathbf{d}) = \Phi(AC)\mathbf{ud}\Phi(B)$, so we have a hill $\mathbf{ud}$ in D'. Reciprocally, any hill in D', say $D' = X\mathbf{ud}Y$, where $X, Y \in \mathcal{D}$, comes from a centered tunnel $D = Z_1\mathbf{u}\Phi^{-1}(Y)\mathbf{d}Z_2$, where Z_1 and Z_2 are respectively the first and second halves of $\Phi^{-1}(X)$.

The proof of (4) is very similar. Recall that $\mathrm{ret}(D')$ equals the number of arches of D'. Given a centered multitunnel corresponding to the decomposition $D = ABC$, we have $\Phi(D) = \Phi(AC)\Phi(B)$, so D' has an arch starting at the first step of $\Phi(B)$, which is nonempty.

To show (2), consider a right tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$, where $\mathrm{length}(A) > \mathrm{length}(C)$. Of the two steps $\mathbf{u}$ and $\mathbf{d}$ delimiting the tunnel, $\mathbf{d}$ will be encountered before $\mathbf{u}$ when D is read in zigzag, since $\mathrm{length}(A) > \mathrm{length}(C)$. So $\mathbf{d}$ will open a tunnel, producing an up-step in D'. Besides, this up-step will be at an even position, since $\mathbf{d}$ was in the right half of D. Reciprocally, an even rise of D' corresponds to a step in the right half of D that opens a tunnel when D is read in zigzag, so it is necessarily a right tunnel.

Relation (3) follows from (2) and the fact that the total number of tunnels of D is $\mathrm{lt}(D) + \mathrm{ct}(D) + \mathrm{rt}(D) = n$, and the total number of up-steps of D' is $\mathrm{or}(D') + \mathrm{er}(D') = n$. □

One interesting application of this bijection is that it can be used to enumerate Dyck paths according to the number of centered, left, and right tunnels, and number of centered multitunnels. We are looking for a multivariate generating function for these four statistics, namely

$$\widetilde{R}(x, u, v, w, z) = \sum_{D \in \mathcal{D}} x^{\mathrm{ct}(D)} u^{\mathrm{lt}(D)} v^{\mathrm{rt}(D)} w^{\mathrm{cmt}(D)} z^{|D|}.$$

By Theorem 3.2, this GF can be expressed as

$$\widetilde{R}(x,u,v,w,z) = R(\frac{x}{u},u,v,w,z), \tag{3.1}$$

where

$$R(t,u,v,w,z) = \sum_{D\in\mathcal{D}} t^{h(D)}u^{\mathrm{or}(D)}v^{\mathrm{er}(D)}w^{\mathrm{ret}(D)}z^{|D|}.$$

We can derive an equation for R using again that every nonempty Dyck path D can be decomposed in a unique way as $D = \mathbf{u}A\mathbf{d}B$, where $A, B \in \mathcal{D}$. The number of hills of $\mathbf{u}A\mathbf{d}B$ is $h(B)+1$ if A is empty, and $h(B)$ otherwise. The odd rises of A become even rises of $\mathbf{u}A\mathbf{d}B$, and the even rises of A become odd rises of $\mathbf{u}A\mathbf{d}B$. Thus, we have $\mathrm{er}(\mathbf{u}A\mathbf{d}B) = \mathrm{or}(A) + \mathrm{er}(B)$, and $\mathrm{or}(\mathbf{u}A\mathbf{d}B) = \mathrm{er}(A) + \mathrm{or}(B) + 1$, where the extra odd rise comes from the first step $\mathbf{u}$. We also have $\mathrm{ret}(\mathbf{u}A\mathbf{d}B) = \mathrm{ret}(B) + 1$. Hence, we obtain the following equation for R:

$$R(t,u,v,w,z) = 1 + uwz(R(1,v,u,1,z) - 1 + t)R(t,u,v,w,z). \tag{3.2}$$

Denote $R_1 := R(1,u,v,1,z)$, $\widehat{R}_1 := R(1,v,u,1,z)$. Substituting $t = w = 1$ in (3.2), we obtain

$$R_1 = 1 + uz\widehat{R}_1R_1, \tag{3.3}$$

and interchanging u and v,

$$\widehat{R}_1 = 1 + vzR_1\widehat{R}_1. \tag{3.4}$$

Solving (3.3) and (3.4) for $\widehat{R}_1$, gives

$$\widehat{R}_1 = \frac{1 + (u-v)z - \sqrt{1 - 2(v+u)z + (v-u)^2z^2}}{2uz}.$$

Thus, from (3.2),

$$\begin{aligned} R(t,u,v,w,z) &= \frac{1}{1 - uwz(\widehat{R}_1 - 1 + t)} \\ &= \frac{2}{2 - w + (v+u-2tu)wz + w\sqrt{1-2(v+u)z+(v-u)^2z^2}}. \end{aligned} \tag{3.5}$$

Now, switching to $\widetilde{R}$, we obtain the following theorem.

Theorem 3.3 *The multivariate generating function for Dyck paths according to centered, left, and right tunnels, centered multitunnels, and semilength is*

$$\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}u^{\mathrm{lt}(D)}v^{\mathrm{rt}(D)}w^{\mathrm{cmt}(D)}z^{|D|}$$
$$= \frac{2}{2 - w + (v+u-2x)wz + w\sqrt{1-2(v+u)z+(v-u)^2z^2}}.$$

As pointed out by Alex Burstein, Lagrange inversion applied to equation (3.2) gives a nice expression for the coefficients of $\widetilde{R}$.

Corollary 3.4 *Fix integers $c, l, r, m \geq 0$ and let $n = c + l + r$. The number of Dyck paths $D \in \mathcal{D}_n$ with $\mathrm{ct}(D) = c$, $\mathrm{lt}(D) = l$, $\mathrm{rt}(D) = r$ and $\mathrm{cmt}(D) = m$ is given by*

$$\frac{m-c}{n-m}\binom{m}{c}\binom{n-m}{l}\binom{n-m}{r}$$

if $c < m < c + l < n$, and it is 1 if $c = m = n$.

Proof. The case $c = m = n$ is trivial, since the only path in $\mathcal{D}_n$ with n centered tunnels is $D = \mathbf{u}^n\mathbf{d}^n$. For the rest of the proof we assume that $0 \leq c < m < c + l < n$.

We start by applying Lagrange inversion formula (Theorem 1.3) to equation (3.2) for variable w, being $f(w) = R(t, u, v, w, z) - 1$, $G(w) = uz(\widehat{R}_1 - 1 + t)(w + 1)$, $n = m$ and $k = 1$ in the theorem. We get that

$$[w^m](R(t, u, v, w, z) - 1) = \frac{1}{m}[w^{m-1}](uz(\widehat{R}_1 - 1 + t)(w + 1))^m = u^m z^m(\widehat{R}_1 - 1 + t)^m.$$

Taking the coefficient of t^c,

$$[t^c w^m](R(t, u, v, w, z) - 1) = \binom{m}{c} u^m z^m (\widehat{R}_1 - 1)^{m-c}. \tag{3.6}$$

Isolating R_1 in equation (3.3) and substituting it in (3.4) we get

$$\widehat{R}_1 = 1 + \frac{vz\widehat{R}_1}{1 - uz\widehat{R}_1},$$

which is equivalent to

$$\widehat{R}_1 - 1 = z\widehat{R}_1(u(\widehat{R}_1 - 1) + v).$$

We apply Lagrange inversion formula again, now for variable z, with $f(z) = \widehat{R}_1 - 1$, $G(z) = (z + 1)(uz + v)$ and $n = s$. This gives us (for $s \neq 0$),

$$[z^s](\widehat{R}_1 - 1)^k = \frac{k}{s}[z^{s-k}](z + 1)^s(uz + v)^s,$$

so

$$[u^{s-r}v^r z^s](\widehat{R}_1 - 1)^k = \frac{k}{s}\binom{s}{r}[z^{s-k}](z + 1)^s z^{s-r} = \frac{k}{s}\binom{s}{r}\binom{s}{r-k}. \tag{3.7}$$

Now, taking the appropriate coefficients of u, v and z in equation (3.6) gives

$$\begin{aligned}
[t^c u^{n-r} v^r w^m z^n](R(t, u, v, w, z) - 1) &= \binom{m}{c}[u^{n-m-r}v^r z^{n-m}](\widehat{R}_1 - 1)^{m-c} \\
&= \binom{m}{c}\frac{m-c}{n-m}\binom{n-m}{r}\binom{n-m}{r-m+c},
\end{aligned}$$

where the last equality follows from (3.7) with $s = n - m$ and $k = m - c$. Thus, using that $n - r = c + l$ and $n - m - (r - m + c) = l$, we get that for $n \geq 1$,

$$[t^c u^{c+l} v^r w^m z^n]R(t, u, v, w, z) = \frac{m-c}{n-m}\binom{m}{c}\binom{n-m}{r}\binom{n-m}{l}.$$

But by relation (3.1), this coefficient is precisely $[x^c u^l v^r w^m z^n]\widetilde{R}(x,u,v,w,z)$, which is the number of paths $D \in \mathcal{D}_n$ with $\mathrm{ct}(D) = c$, $\mathrm{lt}(D) = l$, $\mathrm{rt}(D) = r$ and $\mathrm{cmt}(D) = m$. $\square$

3.3 Generalizations

Here we present a generalization Φ_r of the bijection Φ, which depends on a nonnegative integer parameter r. Given $D \in \mathcal{D}_n$, copy the first $2r$ steps of D into the first $2r$ steps of $\Phi_r(D)$. Now, read the remaining steps of D in zigzag in the following order: $2r+1$, $2n$, $2r+2$, $2n-1$, $2r+3$, $2n-2$, and so on. For each of these steps, if its corresponding matching step in D has not yet been encountered, draw an up-step in $\Phi_r(D)$, otherwise draw a down-step. Note that for $r = 0$ we get the same bijection Φ as before.

Note that Φ_r can be defined exactly as Φ with the difference that instead of τ, the permutation that gives the order in which the steps of D are read is $\tau^{(r)} \in \mathcal{S}_{2n}$, defined as

$$\tau_i^{(r)} = \begin{cases} i & \text{if } i \leq 2r; \\ \dfrac{i+1}{2} + r & \text{if } i > 2r \text{ and } i \text{ is odd;} \\ 2n+1-\dfrac{i}{2}-r & \text{if } i > 2r \text{ and } i \text{ is even.} \end{cases}$$

Figure 3-4 shows an example of the bijection Φ_r for $r = 2$ applied to the path $D =$ **uduuduuduududddudd**.

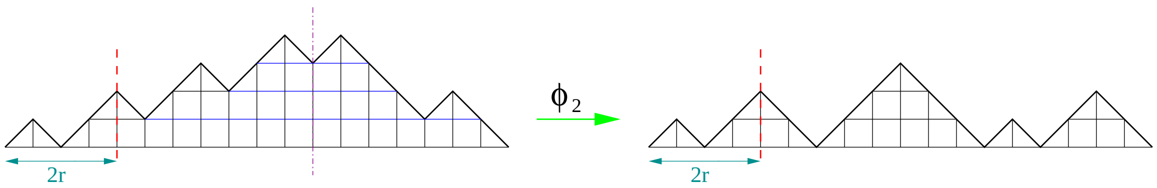


Figure 3-4: An example of Φ_2.

It is clear from the definition that $\Phi_r(D)$ is a Dyck path. A reasoning similar to the one used for Φ shows that Φ_r is indeed a bijection.

The properties of Φ given in Theorem 3.2 generalize to analogous properties of Φ_r. We will prove them using the next lemma, which follows immediately from the definition of Φ_r.

Lemma 3.5 *Let $r \geq 0$, and let $D = ABC$ be a decomposition of a Dyck path D, where B is a Dyck path, and* $\mathrm{length}(A) = \mathrm{length}(C) + 2r$. *Then $\Phi_r(ABC) = \Phi_r(AC)\Phi(B)$.*

Theorem 3.6 *Let $r \geq 0$, let D be any Dyck path, and let $D' = \Phi_r(D)$. We have the following correspondences:*

(1) $\#\{$*tunnels of D with midpoint at $x = n+r$*$\} = \#\{$*hills of D' in $x > 2r$*$\}$,

(2) $\#\{$*tunnels of D with midpoint in $x > n+r$*$\} = \#\{$*even rises of D' in $x > 2r$*$\}$,

(3) $\#\{$*tunnels of D with midpoint in $x \leq n+r$*$\} = \#\{$*odd rises of D' in $x > 2r$*$\}$ $+ \#\{$*up-steps of D' in $x \leq 2r$*$\}$,

(4) $\#\{$*multitunnels of D with midpoint at $x = n+r$*$\} = \#\{$*arches of D' in $x \geq 2r$*$\}$.

Notice that the statistic on the left hand side of (1) is the same statistic $\mathrm{ct}_{-r}(D)$ that we considered in Section 2.3. Similarly, the statistics appearing in (2) and (3) are related to the statistic $\mathrm{lt}_{-r}(D)$ used earlier.

Proof. Fist we show (1). A tunnel given by the decomposition $D = A\mathbf{u}B\mathbf{d}C$ has its midpoint at $x = n + r$ exactly when $\mathrm{length}(A) = \mathrm{length}(C) + 2r$. Applying Lemmas 3.5 and 3.1, $D' = \Phi_r(A\mathbf{u}B\mathbf{d}C) = \Phi_r(AC)\Phi(\mathbf{u}B\mathbf{d}) = \Phi_r(AC)\Phi(\mathbf{ud})\Phi(B) = \Phi_r(AC)\mathbf{ud}\Phi(B)$, and $\mathbf{ud}$ is a hill of D' in $x > 2r$, since $\mathrm{length}(\Phi_r(AC)) \geq 2r$. Reciprocally, any hill of D' in $x > 2r$, say $D' = X\mathbf{ud}Y$, where $X, Y \in \mathcal{D}$ and $\mathrm{length}(X) \geq 2r$, comes from a tunnel with midpoint at $x = n + r$, namely $D = Z_1\mathbf{u}\Phi^{-1}(Y)\mathbf{d}Z_2$, where $Z_1Z_2 = \Phi_r^{-1}(X)$ and $\mathrm{length}(Z_1) = \mathrm{length}(Z_2) + 2r$.

The proof of (4) is very similar. A multitunnel given by $D = ABC$ has its midpoint at $x = n + r$ exactly when $\mathrm{length}(A) = \mathrm{length}(C) + 2r$. In this case, $\Phi_r(D) = \Phi_r(AC)\Phi(B)$ by Lemma 3.5, so D' has an arch starting at the first step of $\Phi(B)$. Notice that this arch is in $x \geq 2r$ because $\mathrm{length}(\Phi_r(AC)) \geq 2r$.

To show (2), consider a tunnel in D with midpoint in $x > n + r$. This is equivalent to saying that it is given by a decomposition $D = A\mathbf{u}B\mathbf{d}C$ with $\mathrm{length}(A) > \mathrm{length}(C) + 2r$. In particular, the tunnel is contained in the halfspace $x \geq 2r$, so the two steps $\mathbf{u}$ and $\mathbf{d}$ delimiting the tunnel are in the part of D that is read in zigzag in the process to obtain $\Phi_r(D)$, and $\mathbf{d}$ will be encountered before $\mathbf{u}$, since $\mathrm{length}(A) - 2r > \mathrm{length}(C)$. So $\mathbf{d}$ will open a tunnel, producing an up-step of D' in $x > 2r$. Besides, this up-step will be at an even position, since $\mathbf{d}$ is in $x > n + r$, that is, in the right half of the part of D that is read in zigzag. Reciprocally, an even rise of D' in $x > 2r$ corresponds to a step of D in $x > n+r$ that opens a tunnel when D is read according to $\tau^{(r)}$, so it is necessarily a tunnel with midpoint to the right of $x = n + r$.

Relation (3) follows from (2) and the fact that the total number of tunnels of D is $\#\{\text{tunnels of } D \text{ with midpoint in } x > n + r\} + \#\{\text{tunnels of } D \text{ with midpoint in } x \leq n + r\} = n$, and the total number of up-steps of D' is $\#\{\text{even rises of } D' \text{ in } x > 2r\} + \#\{\text{odd rises of } D' \text{ in } x > 2r\} + \#\{\text{up-steps of } D' \text{ in } x \leq 2r\} = n$. ☐

Similarly to how we used the properties of Φ to prove Theorem 3.3, we can use the properties of Φ_r to prove a more general theorem. Our goal is to enumerate Dyck paths according to the number of tunnels with midpoint on, to the right of, and to the left of an arbitrary vertical line $x = n+r$, and multitunnels with midpoint on that line. In generating function terms, we are looking for an expression for

$$E(t, u, v, w, y, z) := \sum_{\substack{n\geq 0 \\ 0\leq r\leq n}} \sum_{D\in\mathcal{D}_n} t^{\#\{\text{tun. of } D \text{ w/ midp. at } x=n+r\}} u^{\#\{\text{tun. of } D \text{ w/ midp. in } x\leq n+r\}}$$
$$v^{\#\{\text{tun. of } D \text{ w/ midp. in } x>n+r\}} w^{\#\{\text{multitun. of } D \text{ w/ midp. at } x=n+r\}} y^r z^n.$$

Note that, as in Chapter 2, the variable y marks the position of the vertical line $x = n+r$ with respect to which the tunnels are classified. The following theorem gives an expression for E.

Theorem 3.7 *Let E, R and $\mathbf{C}$ be defined as above. Then,*

$$\begin{aligned} E(t,u,v,w,y,z) &= \frac{\mathbf{C}(uyz)R(t,u,v,w,z)}{1-yu^2z^2\mathbf{C}^2(uyz)R(1,u,v,1,z)R(1,v,u,1,z)} \\ &= \frac{2\delta_2(2+(v-u)z+\delta_1)}{[2+(u+v-2tu)wz+w\delta_1]\,[(\delta_1+(v-u)z)\delta_2-4uyz]}, \end{aligned}$$

where $\delta_1 := \sqrt{1-2(u+v)z+(u-v)^2z^2}-1$, $\delta_2 := \sqrt{1-4uyz}-1$.

Proof. By Theorem 3.6, the generating function E can be expressed as

$E(t,u,v,w,y,z) =$

$$\sum_{\substack{n\geq 0\\ 0\leq r\leq n}} \sum_{D\in\mathcal{D}_n} t^{\#\{\text{hills of } D \text{ in } x>2r\}} u^{\#\{\text{odd rises of } D \text{ in } x>2r\}+\#\{\text{up-steps of } D \text{ in } x\leq 2r\}}$$
$$v^{\#\{\text{even rises of } D \text{ in } x>2r\}} w^{\#\{\text{arches of } D \text{ in } x\geq 2r\}} y^r z^n. \tag{3.8}$$

For each path D in this summation, the y-coordinate of its intersection with the vertical line $x=2r$ has to be even. Fix $h\geq 0$. We will now focus only on the paths $D\in\mathcal{D}$ for which this intersection has y-coordinate equal to $2h$. Let $D=AB$, where A and B are the parts of the path respectively to the left and to the right of $x=2r$. Then, $\#\{\text{hills of } D \text{ in } x>2r\} = \#\{\text{hills of } B\}$, $\#\{\text{odd rises of } D \text{ in } x>2r\} = \#\{\text{odd rises of } B\}$, $\#\{\text{up-steps of } D \text{ in } x\leq 2r\} = \#\{\text{up-steps of } A\}$, and $\#\{\text{arches of } D \text{ in } x\geq 2r\} = \#\{\text{arches of } B\}$.

B can be any path starting at height $2h$ and landing on the x-axis, never going below it. If $h>0$, consider the first down-step of B that lands at height $2h-1$. Then B can be decomposed as $B=B_1\mathbf{d}B'$, where B_1 is any Dyck path, and B' is any path starting at height $2h-1$ and landing on the x-axis, never going below it. Applying this decomposition recursively, B can be written uniquely as $B=B_1\mathbf{d}B_2\mathbf{d}\cdots B_{2h}\mathbf{d}B_{2h+1}$, where the B_i's for $1\leq i\leq 2h+1$ are arbitrary Dyck paths. The number of hills and number of arches of B are given by those of B_{2h+1}. The odd rises of B are the odd rises of the B_i's with odd subindex plus the even rises of those with even subindex. In a similar way one can describe the even rises of B. The semilength of B is the sum of semilengths of the B_i's plus h, which comes from the $2h$ additional down-steps. Thus, the generating function for all paths B of this form, where t, u, v, and z mark respectively number of hills, number of odd rises, number of even rises, and semilength, is

$$z^hR^h(1,u,v,1,z)R^h(1,v,u,1,z)R(t,u,v,w,z). \tag{3.9}$$

Similarly, A can be decomposed uniquely as $A=A_1\mathbf{u}A_2\mathbf{u}\cdots A_{2h}\mathbf{u}A_{2h+1}$. The number of up-steps of A is the sum of the number of up-steps of each A_i, plus a $2h$ term that comes from the additional up-steps. The generating function for paths A of this form, where u marks the number of up-steps, and y and z mark both the semilength, is

$$z^hy^hu^{2h}\mathbf{C}^{2h+1}(uyz). \tag{3.10}$$

The product of (3.9) and (3.10) gives the generating function for paths $D=AB$ where the height of the intersection point of D with the vertical line between A and B is $2h$, where the variables mark the same statistics as in (3.8). Note that the exponent of y is half the distance between the origin of D and this vertical line. Summing over h, we obtain

$$E(t,u,v,w,y,z) = \sum_{h\geq 0} z^{2h}y^h u^{2h}\mathbf{C}^{2h+1}(uyz)R^h(1,u,v,1,z)R^h(1,v,u,1,z)R(t,u,v,w,z)$$
$$= \frac{\mathbf{C}(uyz)R(t,u,v,w,z)}{1-yu^2z^2\mathbf{C}^2(uyz)R(1,u,v,1,z)R(1,v,u,1,z)}.$$

The second expression in the statement of the theorem follows from the formula (3.5) that we had for R. □

From this theorem one can easily deduce an expression for the GF T defined in equation (2.4). This gives an alternative proof of Proposition 2.15.

3.4 Connection to pattern-avoiding permutations

The bijection Φ has applications to enumeration of statistics on pattern-avoiding permutations. The first one is that it can be used together with the bijections defined in Chapter 2 to give a bijective proof of Theorem 1.4. Here we show a more general result. We use the bijection Φ_r to give a combinatorial proof of the following generalization of Theorem 1.4. Note that the particular case $r = 0$ gives a new bijective proof of such theorem.

Theorem 3.8 *Fix $r, n \geq 0$. For any $\pi \in \mathcal{S}_n$, define $\alpha_r(\pi) = \#\{i : \pi_i = i + r\}$, $\beta_r(\pi) = \#\{i : i > r, \pi_i = i\}$. Then, the number of* 321*-avoiding permutations $\pi \in \mathcal{S}_n$ with $\beta_r(\pi) = k$ equals the number of* 132*-avoiding permutations $\pi \in \mathcal{S}_n$ with $\alpha_r(\pi) = k$, for any $0 \leq k \leq n$.*

Proof. Recall that the bijection $\psi_{\llcorner} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ defined in Section 2.1 satisfies that $\mathrm{fp}(\pi) = h(\psi_{\llcorner}(\pi))$. More precisely, it can be easily checked that i is a fixed point of π if and only if $\psi_{\llcorner}(\pi)$ has a hill with x-coordinate $2i - 1$. This implies that $\beta_r(\pi) = \#\{\text{hills of } \psi_{\llcorner}(\pi) \text{ in } x > 2r\}$.

The second bijection that we use is $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$, defined in Section 2.2. In Proposition 2.4 we showed that $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$. Recall that in the proof of this proposition, we associated a unique tunnel of D to each cross of the array $\mathrm{arr}(\pi)$. An element i with $\pi_i = i + r$ is represented by a cross $(i, i + r)$ in the array. From the description of the association between crosses and tunnels, it follows that such a cross $(i, i + r)$ corresponds to a tunnel of $\varphi(\pi)$ with midpoint r units to the right of the center. That is, an element i with $\pi_i = i + r$ gives a tunnel with midpoint at $x = n + r$. Therefore, we have that $\alpha_r(\pi) = \#\{\text{tunnels of } \varphi(\pi) \text{ with midpoint at } x = n + r\}$.

Now all we need to do is use Φ_r and property (1) given in Theorem 3.6. From this it follows that the bijection $\psi_{\llcorner}^{-1} \circ \Phi_r \circ \varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{S}_n(321)$ has the property that $\beta_r(\psi_{\llcorner}^{-1} \circ \Phi_r \circ \varphi(\pi)) = \#\{\text{hills of } \Phi_r \circ \varphi(\pi) \text{ in } x > 2r\} = \#\{\text{tunnels of } \varphi(\pi) \text{ with midpoint at } x = n + r\} = \alpha_r(\pi)$. □

While in Section 2.1 we describe a simple way to enumerate 321-avoiding permutations with respect to the statistics fp and exc, the analogous enumeration for 132-avoiding permutations is done in a more intricate way in Section 2.2. Here we use the properties of Φ to give a more direct derivation of the multivariate generating function for 132-avoiding permutations according to the number of fixed points and the number of excedances.

Corollary 3.9 (of Theorem 3.3)

$$\sum_{n\geq 0}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}v^{\mathrm{exc}(\pi)}z^n = \frac{2}{1+(1+v-2x)z+\sqrt{1-2(1+v)z+(1-v)^2z^2}}. \tag{3.11}$$

Proof. Proposition 2.4 shows that φ maps fixed points to centered tunnels, and excedances to right tunnels, i.e., $\mathrm{fp}(\pi)=\mathrm{ct}(\varphi(\pi))$ and $\mathrm{exc}(\pi)=\mathrm{rt}(\varphi(\pi))$. Therefore, the left hand side of (3.11) equals $\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}v^{\mathrm{rt}(D)}z^{|D|}$. The result now is obtained applying Theorem 3.3 for $u=w=1$. $\square$

Comparing this expression (3.11) with equation (2.2), we obtain another proof of Theorem 2.1.

As a further application, we can use the bijection Φ to give the following refinement of Corollary 3.9, which gives an expression for the multivariate generating function for number of fixed points, number of excedances, and number of descents in 132-avoiding permutations. The analogous result for 321-avoiding permutations is given in Theorem 2.3.

Theorem 3.10 *Let*

$$L(x,q,p,z) := 1+\sum_{n\geq 1}\sum_{\pi\in\mathcal{S}_n(132)} x^{\mathrm{fp}(\pi)}q^{\mathrm{exc}(\pi)}p^{\mathrm{des}(\pi)+1}z^n.$$

Then

$$L(x,q,p,z) = \frac{2(1+xz(p-1))}{1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{f_1(q,z)}}, \tag{3.12}$$

where $f_1(q,z)=1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2-2q(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$.

Proof. We use again that φ maps fixed points to centered tunnels, and excedances to right tunnels. It is shown in Proposition 2.4 that it also maps descents of the permutation to valleys of the corresponding Dyck path. Clearly, the number of valleys of any nonempty Dyck path equals the number of peaks minus one (in the empty path both numbers are 0). Thus, L can be expressed as

$$L(x,q,p,z) = \sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}p^{\#\{\text{peaks of } D\}}z^{|D|}.$$

By Theorem 3.2, Φ maps centered tunnels into hills and right tunnels into even rises. Let us see what peaks are mapped to by Φ. Given a peak $\mathbf{ud}$ in $D\in\mathcal{D}$, D can be written as $D=A\mathbf{ud}C$, where A and C are the parts of the path before and after the peak respectively. This decomposition corresponds to a tunnel of D that goes from the beginning of the $\mathbf{u}$ to the end of the $\mathbf{d}$. Assume first that the peak occurs in the left half (i.e., $\mathrm{length}(A)<\mathrm{length}(C)$). When D is read in zigzag, the $\mathbf{u}$ opens a tunnel that is closed by the $\mathbf{d}$ two steps later. This produces in $\Phi(D)$ an up-step followed by a down-step two positions ahead, that is, an occurrence of $\mathbf{u}\star\mathbf{d}$ in the Dyck word of $\Phi(D)$, where $\star$ stands for any symbol (either a $\mathbf{u}$ or a $\mathbf{d}$).

If the peak occurs in the right half of D (i.e., $\mathrm{length}(A)>\mathrm{length}(C)$), the reasoning is analogous, with the difference that the $\mathbf{d}$ opens a tunnel that is closed by the $\mathbf{u}$ two steps ahead. So, such a peak produces also an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$. Reciprocally, we claim that an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ can come only from a peak of D either in the left or in the right half. Indeed, using the notation from the procedure above describing the inverse of Φ, an occurrence of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ corresponds to either an occurrence of oc in the left half of W or an occurrence of co in the right half of W. In both cases, the algorithm given above will match these two letters c and o with each other, so they correspond to an occurrence of $\mathbf{ud}$ in D.

If the peak occurs in the middle (i.e., length(A) = length(C)), then by Lemma 3.1, $\Phi(A\mathbf{ud}C) = \Phi(AC)\mathbf{ud}$, so it is mapped to an occurrence of $\mathbf{ud}$ at the end of $\Phi(D)$. Clearly we have such an occurrence only when D has a peak in the middle.

Thus, we have shown that peaks in D are mapped by Φ to occurrences of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)$ and occurrences of $\mathbf{ud}$ at the end of $\Phi(D)$, or, equivalently, to occurrences of $\mathbf{u}\star\mathbf{d}$ in $\Phi(D)\mathbf{d}$ (here $\Phi(D)\mathbf{d}$ is a Dyck path followed by a down-step). Denote by $\lambda(D)$ the number of occurrences of $\mathbf{u}\star\mathbf{d}$ in $D\mathbf{d}$. This implies that L can be written as

$$L(x,q,p,z) = \sum_{D\in\mathcal{D}} x^{h(D)} q^{\mathrm{er}(D)} p^{\lambda(D)} z^{|D|}.$$

We are left with a Dyck path enumeration problem, which is solved in the following lemma. Let J be defined in Lemma 3.11. It is easy to see that we have $L(x,q,p,z) = 1 + J(x,1,p,1,q,p,z)$. Making use of (3.13) and (3.14), it follows at once that

$$L(x,q,p,z) = \frac{1 - xz + xpz}{1 - xz - zK_1},$$

where K_1 is given by

$$zK_1^2 - [1 - z - qz + q(1-p)^2 z^2]K_1 + p^2 qz = 0.$$

From these equations we obtain (3.12). $\square$

Lemma 3.11 *Denote by* $\mathrm{ih}(D)$ *(*$\mathrm{fh}(D)$*) the number of initial (final) hills in D (obviously, their only possible values are 0 and 1). Denote by* $\mu(D)$ *the number of occurrences of* $\mathbf{u}\star\mathbf{d}$ *in* D. *Then the generating function*

$$J(x,t,s,u,v,y,z) := \sum x^{h(D)} t^{\mathrm{ih}(D)} s^{\mathrm{fh}(D)} u^{\mathrm{or}(D)} v^{\mathrm{er}(D)} y^{\mu(D)} z^{|D|},$$

where the summation is over all nonempty Dyck paths, is given by

$$J(x,t,s,u,v,y,z) = \frac{uz[xts + (1 - xu(1-t)(1-s)z)K]}{1 - xuz - uzK}, \tag{3.13}$$

where K is given by

$$uzK^2 - [1 - (u+v)z + uv(1-y)^2 z^2]K + y^2 vz = 0. \tag{3.14}$$

Proof. Every nonempty Dyck path has one of the following four forms: $\mathbf{ud}$, $\mathbf{ud}B$, $\mathbf{u}A\mathbf{d}$, or $\mathbf{u}A\mathbf{d}B$, where A and B are nonempty Dyck paths. The generating functions of these four pairwise disjoint sets of Dyck paths are

(i) $xtsuz$,

(ii) $xtuzJ(x,1,s,u,v,y,z)$,

(iii) $uzJ(1, y, y, v, u, y, z)$,

(iv) $uzJ(1, y, y, v, u, y, z)J(x, 1, s, u, v, y, z)$,

respectively. Only the third factor in (iii) and (iv) needs an explanation: the hills of A are not hills in **u**A**d**; an initial (final) hill in A gives a **uud** (**udd**) in **u**A**d**; an odd (even) rise in A becomes an even (odd) rise in **u**A**d**.

Consequently, the generating function J satisfies the functional equation

$$\begin{aligned} J(x,t,s,u,v,y,z) &= xtsuz + xtuzJ(x,1,s,u,v,y,z) \\ &\quad + uzJ(1,y,y,v,u,y,z) + uzJ(1,y,y,v,u,y,z)J(x,1,s,u,v,y,z). \end{aligned} \tag{3.15}$$

From equation (3.15) it is clear that, whether interested or not in the statistics "number of initial (final) hills", we had to introduce them for the sake of the statistic marked by the variable y. Also, without any additional effort we could use two separate variables to mark the number of **uud**'s and the number of **udd**'s, and obtain a slightly more general generating function, although we do not need it here.

Denoting $H = J(x, 1, s, u, v, y, z)$, $K = J(1, y, y, v, u, y, z)$, equation (3.15) becomes

$$J = xtsuz + xtuzH + uzK + uzHK. \tag{3.16}$$

Setting here $t = 1$, we obtain

$$H = xsuz + xuzH + uzK + uzHK. \tag{3.17}$$

Solving (3.17) for H and introducing it into (3.16), we obtain (3.13).

It remains to show that K satisfies the quadratic equation (3.14). Setting $x = 1$, $t = y$, $s = y$ in (3.16), and interchanging u and v, we get

$$K = y^2vz + yvzM + vz\widehat{K} + vzM\widehat{K}, \tag{3.18}$$

where $M = J(1, 1, y, v, u, y, z)$ and $\widehat{K}$ is K with u and v interchanged, namely $\widehat{K} = J(1, y, y, u, v, y, z)$.

Now in (3.16) we set $x = 1$, $t = 1$, $s = y$, and we interchange u and v, to get

$$M = yvz + vzM + vz\widehat{K} + vzM\widehat{K}. \tag{3.19}$$

Eliminating M from (3.18) and (3.19), we obtain

$$vz(2yvz - y^2vz + 1 - vz)\widehat{K} + (vz - 1)K + vzK\widehat{K} + y^2vz = 0. \tag{3.20}$$

Finally, eliminating $\widehat{K}$ from (3.20) and the equation obtained from (3.20) by interchanging u and v, we obtain equation (3.14). Note that, as expected, J is symmetric in the variables t and s and linear in each of these two variables. □

From Theorem 3.10 one can see that the first terms of $L(x, q, p, z)$ are

$$1 + xpz + (qp^2 + x^2p)z^2 + (q^2p^2 + qp^2 + xqp^3 + xqp^2 + x^3p)z^3 + \cdots,$$

corresponding to Dyck paths of semilength at most 3 (or equivalently, to 132-avoiding permutations of length at most 3).

3.5 Some new interpretations of the Catalan numbers

Our first new interpretation follows immediately from the results in this chapter. Note that any nonempty Dyck path $D \in \mathcal{D}_n$ has a multitunnel that goes from $(0,0)$ to $(2n,0)$. We call this the *basic multitunnel*.

Proposition 3.12 *Let $n \geq 0$. The number of Dyck paths of length $2n+2$ with no centered multitunnels other than the basic one is $\mathbf{C}_n$.*

Proof. We could give a non-bijective argument using generating functions, but now the bijection Φ yields a simple combinatorial proof. We know from part (4) of Theorem 3.2 that the set of paths $D \in \mathcal{D}_{n+1}$ with $\mathrm{cmt}(D) = 1$ is in bijection with the set of Dyck paths of length $2n+2$ with only one return. But these are precisely elevated Dyck paths of the form $\mathbf{u}A\mathbf{d}$, where $A \in \mathcal{D}_n$. The number of them is $\mathbf{C}_n$. $\square$

Proposition 3.13 *The following quantities are equal to $\mathbf{C}_n$:*

(1) *The total number of fixed points in elements of $\mathcal{S}_n(321)$.*

(2) *The total number of fixed points in elements of $\mathcal{S}_n(132)$.*

(3) *The total number of centered tunnels in Dyck paths of length $2n$.*

Proof. By the first part of Theorem 3.2, (3) equals the total number of hills in Dyck paths of length $2n$. To prove that this number is $\mathbf{C}_n$, we define the following bijection between paths in $\mathcal{D}_n$ with a marked hill and the set $\mathcal{D}_n$ itself. Given a path with a distinguished hill $A\mathbf{ud}B \in \mathcal{D}_n$, where $A, B \in \mathcal{D}$, map it to the path $\mathbf{u}A\mathbf{d}B \in \mathcal{D}_n$. This is obviously a bijection, since each $D \in \mathcal{D}_n$ can be expressed uniquely as $D = \mathbf{u}A\mathbf{d}B$, with $A, B \in \mathcal{D}$.

The equality (2)=(3) is a consequence of the first part of Proposition 2.4. Finally, by Proposition 2.2, fixed points in 321-avoiding permutations are in one-to-one correspondence with hills of Dyck paths, which proves (1). Another argument to compute (1) directly follows from the reasoning preceding equation (5.21). $\square$

For the next interpretation of Catalan numbers consider the directed graph G drawn in Figure 3-5. Its nodes are the infinite set $\{q_{i,j} : i, j \geq 0, i+j \text{ even}\}$. The set of edges is $\{(q_{i,j} \to q_{i+1,j+1}), (q_{i+1,j+1} \to q_{i,j}) : i, j \geq 0, i+j \text{ even}\} \cup \{(q_{0,2j-2} \to q_{0,2j}), (q_{0,2j} \to q_{0,2j}) : j \geq 2\} \cup \{(q_{2i-2,0} \to q_{2i,0}), (q_{2i,0} \to q_{2i,0}) : i \geq 2\} \cup \{(q_{0,0} \to q_{0,0})\}$. Let $\mathbf{F}_n$ denote the n-th *Fine* number (see Section 4.4).

Proposition 3.14 *Let G be the directed graph described above, and let G' be the graph obtained from it by removing the edge $(q_{0,0} \to q_{0,0})$. Fix $n \geq 0$.*

(1) *The number of paths in G from $q_{0,0}$ to $q_{0,0}$ with n steps is $\mathbf{C}_n$.*

(2) *The number of paths in G' from $q_{0,0}$ to $q_{0,0}$ with n steps is $\mathbf{F}_n$.*

(3) *The number of paths in G from $q_{0,0}$ to $q_{0,0}$ with $n+1$ steps not having $q_{0,0}$ as an interior point is $\mathbf{C}_n$.*

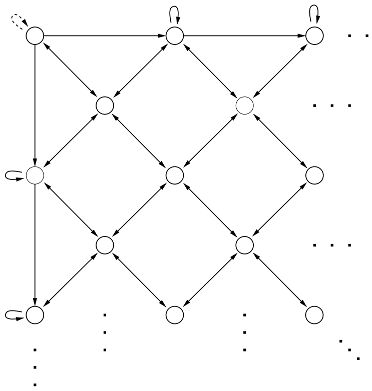


Figure 3-5: The graph G.

Proof. We will construct a bijection between $\mathcal{D}_n$ and the set of paths in G from $q_{0,0}$ to $q_{0,0}$ with n steps. Let $D \in \mathcal{D}_n$. Read the steps of D two by two, starting with the two middle steps n and $n+1$, then $n-1$ and $n+2$, and progressively moving away from the middle, finishing with the pair 1 and $2n$. For k from 1 to n, let D_k be the subpath of D consisting of the steps at distance at most k from the middle. Let a_k (resp. b_k) be the height difference between the leftmost (resp. rightmost) point of D_k and its lowest point (the height is given by the y-coordinate). Equivalently, a_k (resp. b_k) is the number of left (resp. right) tunnels of D with exactly one of its two delimiting steps belonging to D_k. Define the k-th node of our path in G to be q_{a_k,b_k}.

Note that $q_{a_0,b_0} = q_{a_n,b_n} = q_{0,0}$, and that for every k there is an edge from q_{a_k,b_k} to $q_{a_{k+1},b_{k+1}}$ in G, by the way the numbers a_k and b_k change every time that a pair of steps is read from D. It is not hard to see that this defines a bijection between $\mathcal{D}_n$ and paths in G with n steps starting and ending at $q_{0,0}$. Indeed, the numbers a_k and b_k encode enough information to reconstruct the Dyck path. This proves (1).

To show parts (2) and (3), observe that the node $q_{0,0}$ is used whenever $a_k = b_k = 0$, which means that there is a centered multitunnel between the two endpoints of D_k. Similarly, the edge $(q_{0,0} \to q_{0,0})$ is used when $a_k = b_k = a_{k+1} = b_{k+1} = 0$, and this condition is equivalent to D having a centered tunnel between the endpoints of D_{k+1}. To prove (2), we use the fact from Corollary 4.5 that the number of Dyck paths of length $2n$ with no centered tunnels is $\mathbf{F}_n$. Part (3) follows now from Proposition 3.12, and in fact is also a direct consequence of part (1). □

Combining the bijection just defined with Φ, an alternative and perhaps simpler bijection between $\mathcal{D}_n$ and the set of paths in G from $q_{0,0}$ to $q_{0,0}$ with n steps can be defined. It will be convenient to describe the path in G backwards. Equivalently, we will give a path P in the graph obtained from G by reversing all the edges. Given $D \in \mathcal{D}_n$, read the steps from left to right two at a time, and construct P as follows. Let $q_{i,j}$ be the current node in P. If a **uu** is read, add an edge $(q_{i,j} \to q_{i+1,j+1})$ to the path. If a pair **ud** is encountered, add an edge $(q_{i,j} \to q_{i+1,j-1})$ if $j > 0$, or a loop $(q_{i,j} \to q_{i,j})$ otherwise. For each pair **du**, add an edge $(q_{i,j} \to q_{i-1,j+1})$ if $i > 0$, or a loop $(q_{i,j} \to q_{i,j})$ otherwise. Finally, for each pair **dd**, add an edge $(q_{i,j} \to q_{i-1,j-1})$ if $i,j > 0$, $(q_{i,j} \to q_{i-2,j})$ if $j = 0$, or $(q_{i,j} \to q_{i,j-2})$ if $i = 0$. It can be checked that this is a bijection as well. Note that if at a given point of the construction the current node in P is $q_{i,j}$, then the fragment of D that has been read so far ends at height $i + j$.

Our last interpretation of the Catalan numbers is joint work with Emeric Deutsch and Astrid Reifegerste.

Proposition 3.15 *The following quantities are equal to* $\mathbf{C}_n$*:*

(1) *The number of permutations* $\pi \in \mathcal{S}_{2n+1}(321)$ *such that* $\psi_{\mathsf{L}}(\pi) = \phi_{\mathsf{L}}(\pi)$*.*

(2) *The number of symmetric parallelogram polyominoes*[1] *of perimeter* $4(2n+1)$ *having exactly one horizontal (equivalently, vertical) boundary segment at each level.*

Proof. The equivalency between (1) and (2) is clear when we draw $\psi_{\mathsf{L}}(\pi)$ (as in Figure 2-2) and $\phi_{\mathsf{L}}(\pi)$ (as in Figure 2-8) as lattice paths from the top-left to the bottom-right corner of the array of π. The two paths form a parallelogram polyomino which is symmetric (and satisfies the conditions of (2)) exactly when $\psi_{\mathsf{L}}(\pi) = \phi_{\mathsf{L}}(\pi)$ as Dyck paths.

Now we show that the permutations in (1) are counted by the Catalan numbers. Let $\pi \in \mathcal{S}_{2n+1}(321)$, let $i_1 < i_2 < \cdots < i_e$ be the positions of the excedances of π and let $j_1 < j_2 < \cdots < j_{2n+1-e}$ be the remaining positions. Then, $\psi_{\mathsf{L}}(\pi) = \phi_{\mathsf{L}}(\pi)$ if and only if $e = n$ (π has n excedances) and $\pi_{i_k} = j_k + 1$ for all $1 \leq k \leq n$. Each permutation satisfying these conditions is uniquely determined by its excedance set $\{i_1, i_2, \ldots, i_n\}$. Now, these sets are in bijection with Dyck paths of length $2n$: given such a set, construct a Dyck path having up-steps in positions $\{i_1, i_2, \ldots, i_n\}$ and down-steps everywhere else. Figure 3-6 shows an example for $\pi = 4512736$, whose excedance set is $\{1, 2, 5\}$. $\square$

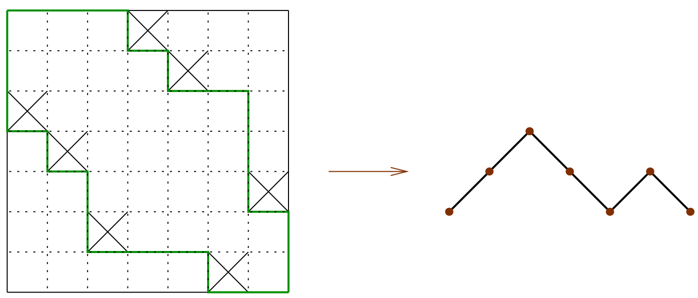


Figure 3-6: A permutation satisfying $\psi_{\mathsf{L}}(\pi) = \phi_{\mathsf{L}}(\pi)$, its symmetric parallelogram polyomino, and the corresponding Dyck path.

[1] Parallelogram polyominoes are unordered pairs of lattice paths starting at $(0,0)$, using steps $(1,0)$ and $(0,-1)$, ending at the same point, and only intersecting at the beginning and at the end.

Chapter 4

A direct bijection preserving fixed points and excedances

In this chapter we give a direct combinatorial proof of Theorem 2.1. We present a bijection between 321- and 132-avoiding permutations that preserves the number of fixed points and the number of excedances. Our bijection is a composition of two slightly modified known bijections into Dyck paths, and the result follows from a new analysis of these bijections. One of them is the bijection φ from Chapter 2. The other one is based on the Robinson-Schensted-Knuth correspondence, and from it stems the difficulty of the analysis.

As a new application of our bijections, we show that the length of the longest increasing subsequence in 321-avoiding permutations corresponds to a statistic in 132-avoiding permutations that we call *rank*, which further refines Theorem 2.1. We also apply our bijections to refined restricted involutions (see Section 4.4). Many of the results in this chapter are joint work with Igor Pak [32].

This chapter is structured as follows. The description of the main bijection is done in Section 4.1, where the new part is a bijection from 321-avoiding permutations to Dyck paths. In Section 4.2 we establish properties of this bijection. Section 4.3 contains proofs of two technical lemmas. We conclude with extensions of our results to refined restricted involutions, and other applications.

4.1 A composition of bijections into Dyck paths

Here is the generalization of Theorem 2.1 that we prove combinatorially in this chapter:

Theorem 4.1 *The number of* 321*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ *and* $\mathrm{lis}(\pi) = k$ *equals the number of* 132*-avoiding permutations* $\pi \in \mathcal{S}_n$ *with* $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ *and* $\mathrm{rank}(\pi) = n - k$, *for any* $0 \leq i, j, k \leq n$.

To prove this theorem, we establish a bijection $\Theta : \mathcal{S}_n(321) \longrightarrow \mathcal{S}_n(132)$ which respects the statistics as above. While Θ is not hard to define, its analysis is less straightforward and will occupy much of the chapter. The bijection Θ is the composition of two bijections, one from $\mathcal{S}_n(321)$ to $\mathcal{D}_n$, and another one from $\mathcal{D}_n$ to $\mathcal{S}_n(132)$. The second one is just the inverse of the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ presented in Section 2.2. The first one is described next.

4.1.1 The bijection Ψ

We define the bijection $\Psi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ in two steps. Given $\pi \in \mathcal{S}_n(321)$, we start by applying the Robinson-Schensted-Knuth correspondence to π [86, Section 7.11] (see also [51, 77]). This correspondence gives a bijection between the symmetric group $\mathcal{S}_n$ and pairs (P, Q) of *standard Young tableaux* of the same shape $\lambda \vdash n$. For $\pi \in \mathcal{S}_n(321)$ the algorithm is particularly easy because in this case the tableaux P and Q have at most two rows. The *insertion tableau* P is obtained by reading π from left to right and, at each step, inserting π_i to the partial tableau obtained so far. Assume that $\pi_1, \dots, \pi_{i-1}$ have already been inserted. If π_i is larger than all the elements on the first row of the current tableau, place π_i at the end of the first row. Otherwise, let m be the leftmost element on the first row that is larger than π_i. Place π_i in the square that m occupied, and place m at the end of the second row (in this case we say that π_i *bumps* m). The *recording tableau* Q has the same shape as P and is obtained by placing i in the position of the square that was created at step i (when π_i was inserted) in the construction of P, for all i from 1 to n. We write $\mathrm{RSK}(\pi) = (P, Q)$.

$$
\begin{array}{llllllll}
\begin{array}{|c|}\hline 2\\\hline\end{array} &
\begin{array}{|c|c|}\hline 2&3\\\hline\end{array} &
\begin{array}{|c|c|c|}\hline 2&3&5\\\hline\end{array} &
\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|c|c|}\hline 1&3&4&6&8\\\hline 2&5\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|c|c|}\hline 1&3&4&6&7\\\hline 2&5&8\\\cline{1-3}\end{array}
\\[2ex]
\begin{array}{|c|}\hline 1\\\hline\end{array} &
\begin{array}{|c|c|}\hline 1&2\\\hline\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} &
\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|c|c|}\hline 1&2&3&6&7\\\hline 4&5\\\cline{1-2}\end{array} &
\begin{array}{|c|c|c|c|c|}\hline 1&2&3&6&7\\\hline 4&5&8\\\cline{1-3}\end{array}
\end{array}
$$

$$
\mathrm{P} = \begin{array}{|c|c|c|c|c|}\hline 1&3&4&6&7\\\hline 2&5&8\\\cline{1-3}\end{array}
\qquad\qquad
\mathrm{Q} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&6&7\\\hline 4&5&8\\\cline{1-3}\end{array}
$$

Figure 4-1: Construction of the RSK correspondence $\mathrm{RSK}(\pi) = (P, Q)$ for $\pi = 23514687$.

Now, the first half of the Dyck path $\Psi(\pi)$ is obtained by adjoining, for i from 1 to n, an up-step if i is on the first row of P, and a down-step if it is on the second row. Let A be the corresponding word of $\mathbf{u}$'s and $\mathbf{d}$'s. Similarly, let B be the word obtained from Q in the same way. We define $\Psi(\pi)$ to be the Dyck path obtained by the concatenation of the word A and the word B written backwards. For example, from the tableaux P and Q as in Figure 4-1 we get the Dyck path shown in Figure 1-2. The following proposition, which is proved in Section 4.2, summarizes some properties of this bijection Ψ:

Proposition 4.2 *The bijection* $\Psi : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ *satisfies*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\Psi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\Psi(\pi))$,

(3) $\mathrm{lis}(\pi) = \frac{1}{2}\big(n + \nu(\Psi(\pi))\big)$,

for all $\pi \in \mathcal{S}_n(321)$.

Suppose $\mathrm{RSK}(\pi) = (P, Q)$ for any $\pi \in \mathcal{S}_n$. A fundamental and highly nontrivial property of the RSK correspondence is the *duality*: $\mathrm{RSK}(\pi^{-1}) = (Q, P)$ [86, Section 7.13]. The classical *Schensted's Theorem* states that $\mathrm{lis}(\pi)$ is equal to the length of the first row of the tableau P (and Q). Both results are used in the proof of Proposition 4.2.

Now the main result of this chapter follows easily from this proposition together with Proposition 2.4.

Proof of Theorem 4.1. Propositions 4.2 and 2.4 imply that $\Theta = \varphi^{-1} \circ \Psi$ is a bijection from $\mathcal{S}_n(321)$ to $\mathcal{S}_n(132)$ which satisfies

$$\mathrm{fp}(\Theta(\pi)) = \mathrm{ct}(\Psi(\pi)) = \mathrm{fp}(\pi),$$

$$\mathrm{exc}(\Theta(\pi)) = \mathrm{rt}(\Psi(\pi)) = \mathrm{exc}(\pi),$$

$$\mathrm{rank}(\Theta(\pi)) = \frac{1}{2}\big(n - \nu(\Psi(\pi))\big) = n - \frac{1}{2}\big(n + \nu(\Psi(\pi))\big) = n - \mathrm{lis}(\pi)\,.$$

This implies the result. □

4.2 Properties of Ψ

In this section we prove Proposition 4.2, which describes the properties of Ψ that we need.

Let us first consider only fixed points in a permutation $\pi \in \mathcal{S}_n$. Let $\pi \in \mathcal{S}_n(321)$ and assume that $\pi_i = i$. Then $\pi_1\pi_2\cdots\pi_{i-1}$ is a permutation of $\{1, 2, \ldots, i-1\}$, and $\pi_{i+1}\pi_{i+2}\cdots\pi_n$ is a permutation of $\{i+1, i+2, \ldots, n\}$. Indeed, if $\pi_j > i$ for some $j < i$, then necessarily $\pi_k < i$ for some $k > i$, and $\pi_j\pi_i\pi_k$ would be an occurrence of 321.

Therefore, when we apply RSK to π, the elements $\pi_i, \pi_{i+1}, \ldots, \pi_n$ will never bump any of the elements $\pi_1, \pi_2, \ldots, \pi_{i-1}$. In particular, the subtableaux of P and Q determined by the entries that are smaller than i will have the same shape. Furthermore, when the elements greater than i are placed in P and Q, the rows in which they are placed do not depend on the subpermutation $\pi_1\pi_2\cdots\pi_{i-1}$. Note also that $\pi_i = i$ will never be bumped, and it will occupy the same position in the first row of P and Q.

When the Dyck path $\Psi(\pi)$ is built from P and Q, this translates into the fact that the steps corresponding to π_i in P and to i in Q will be respectively an up-step in the first half and a down-step in the second half, both at the same height and at the same distance from the center of the path. Besides, the part of the path between them will be itself the Dyck path corresponding to $(\pi_{i+1}-i)(\pi_{i+2}-i)\cdots(\pi_n-i)$. So, the fixed point $\pi_i = i$ determines a centered tunnel in $\Psi(\pi)$. It is clear that the converse is also true, that is, every centered tunnel comes from a fixed point. This shows that $\mathrm{fp}(\pi) = \mathrm{ct}(\Psi(\pi))$, proving the first part of Proposition 4.2.

Let us now consider excedances in a permutation $\pi \in \mathcal{S}_n(321)$. Our goal is to show that the excedances of π correspond to right tunnels of $\Psi(\pi)$. The first observation is that we can assume without loss of generality that π has no fixed points. Indeed, if $\pi_i = i$ is a fixed point of π, then the above reasoning shows that we can decompose $\Psi(\pi) = A\mathbf{u}B\mathbf{d}C$, where AC is the Dyck path $\Psi(\pi_1\pi_2\cdots\pi_{i-1})$ and B is a translation of the Dyck path $\Psi((\pi_{i+1} - i)\cdots(\pi_n - i))$. But we have that $\mathrm{exc}(\pi) = \mathrm{exc}(\pi_1\pi_2\cdots\pi_{i-1}) + \mathrm{exc}((\pi_{i+1}-i)\cdots(\pi_n-i))$ and $\mathrm{rt}(A\mathbf{u}B\mathbf{d}C) = \mathrm{rt}(AC) + \mathrm{rt}(B)$, so in this case the result holds by induction on the number of fixed points. Note also that the above argument showed that $\mathrm{fp}(\pi) = \mathrm{fp}(\pi_1\pi_2\cdots\pi_{i-1}) + \mathrm{fp}((\pi_{i+1} - i)\cdots(\pi_n - i)) + 1$ and $\mathrm{ct}(A\mathbf{u}B\mathbf{d}C) = \mathrm{ct}(AC) + \mathrm{ct}(B) + 1$.

Suppose that $\pi \in \mathcal{S}_n(321)$ has no fixed points. We will use the fact that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements (in this case, the deficiencies) are increasing (see e.g. [69]). Denote by $X_i := (i, \pi_i)$ the crosses of the array representation of π. To simplify the presentation, we will refer indistinctively to i or X_i, hoping this does not lead to confusion. For example, we will say "X_i is an excedance", etc.

Define a matching between the excedances and the deficiencies of π by the following algorithm. Let $i_1 < i_2 < \cdots < i_k$ be the positions of the excedances of π and let $j_1 < j_2 < \cdots < j_{n-k}$ be the deficiencies. Note that from the previous paragraph we know that $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k}$ and $\pi_{j_1} < \pi_{j_2} < \cdots < \pi_{j_{n-k}}$.

Matching Algorithm

(1) Initialize $a := 1$, $b := 1$.

(2) Repeat until $a > k$ or $b > n - k$:

  (a) If $i_a > j_b$, then $b := b + 1$. (X_{j_b} is not matched.)

  (b) Else if $\pi_{i_a} < \pi_{j_b}$, then $a := a + 1$. (X_{i_a} is not matched.)

  (c) Else, match X_{i_a} with X_{j_b}; $a := a + 1$, $b := b + 1$.

(3) Output the matching sequence.

Example. Let $\pi = (4, 1, 2, 5, 7, 8, 3, 6, 11, 9, 10)$ as in Figure 4-2 below. We have $i_1 = 1$, $i_2 = 4$, $i_3 = 5$, $i_4 = 6$, $i_5 = 9$, and $j_1 = 2$, $j_2 = 3$, $j_3 = 7$, $j_4 = 8$, $j_5 = 10$, $j_6 = 11$. In the first execution of the loop in step (2) of the algorithm, neither $i_1 > j_1$ nor $\pi_{i_1} < \pi_{j_1}$ hold, so $X_{i_1} = (1, 4)$ and $X_{j_1} = (2, 1)$ are matched. Now we repeat the loop with $a = b = 2$, and since $i_2 > j_2$, we are in the case given by (2a) ($X_{j_2} = (3, 2)$ is not matched). In the next iteration, $a = 2$ and $b = 3$, so we match $X_{i_2} = (4, 5)$ and $X_{j_3} = (7, 3)$. Now we have $a = 3$ and $b = 4$, so we match $X_{i_3} = (5, 7)$ and $X_{j_4} = (8, 6)$. The values of a and b in the next iteration are 4 and 5 respectively, so we are in the case of (2b), $\pi_{i_4} = 8 < 9 = \pi_{j_5}$, and $X_{i_4} = (6, 8)$ is unmatched. Now $a = b = 5$, and we match $X_{i_5} = (9, 11)$ and $X_{j_5} = (10, 9)$. The matching algorithm ends here because now $a = 6 > 5 = k$.

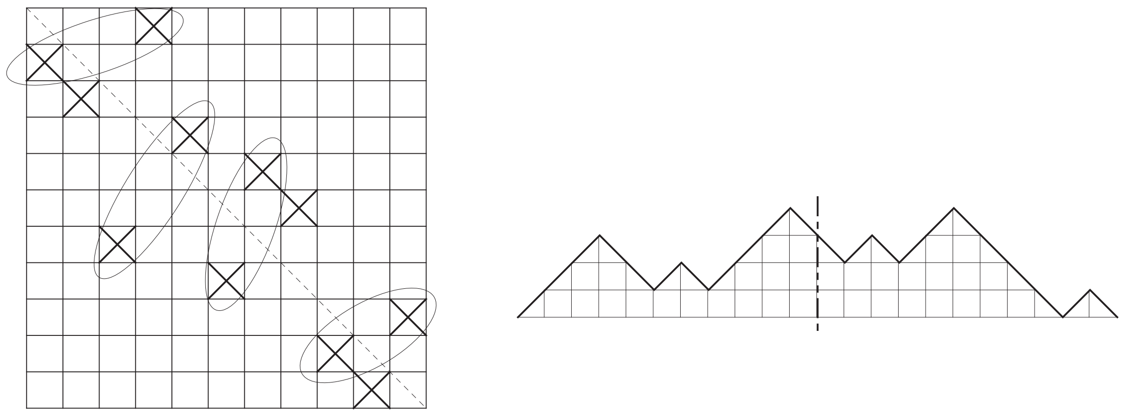


Figure 4-2: Example of the matching for $\pi = (4, 1, 2, 5, 7, 8, 3, 6, 11, 9, 10)$, and $\Psi(\pi)$.

An informal, more geometrical description of the matching algorithm is the following. For each pair of crosses of the array (seen as embedded in the plane), consider the line that their centers determine. If one of these lines has positive slope and leaves all the remaining crosses to the right, match the two crosses that determine it, and delete them from the array. If there is no line with these properties, delete the cross that is closer to the upper-left corner of the array (it is unmatched). Repeat the process until no crosses are left.

Now we consider the matched excedances on one hand and the unmatched ones on the other. We summarize rather technical results in the following two lemmas, which are proved in Section 4.3. Recall the definitions of right-side and left-side tunnels from Section 1.2.2.

Lemma 4.3 *The following quantities are equal:*

(1) *the number of matched pairs* (X_i, X_j)*, where* X_i *is an excedance and* X_j *a deficiency;*

(2) *the length of the second row of* P *(or* Q*);*

(3) *the number of right-side tunnels of* $\Psi(\pi)$*;*

(4) *the number of left-side tunnels of* $\Psi(\pi)$*;*

(5) $\frac{1}{2}\big(n - \nu(\Psi(\pi))\big)$*;*

(6) $n - \mathrm{lis}(\pi)$*.*

Note that (5)=(6) implies that $\mathrm{lis}(\pi) = \frac{1}{2}\big(n + \nu(\Psi(\pi))\big)$, which is the third part of Proposition 4.2.

Lemma 4.4 *The number of unmatched excedances (resp. deficiencies) of* π *equals the number of right-across (resp. left-across) tunnels of* $\Psi(\pi)$*.*

Since each excedance of π either is part of a matched pair (X_i, X_j) or is unmatched, Lemmas 4.3 and 4.4 imply that the total number $\mathrm{exc}(\pi)$ of excedances equals the number of right-side tunnels of $\Psi(\pi)$ plus the number of right-across tunnels, which is $\mathrm{rt}(\Psi(\pi))$. This implies the second part of Proposition 4.2.

To summarize, we will have shown after proving these lemmas that the bijection Ψ satisfies all three properties described in Proposition 4.2, which completes the proof. □

4.3 Properties of the matching algorithm

In this section we prove the two lemmas above. We also give a more direct description of the bijection Ψ using the matching algorithm, without referring explicitly to RSK.

Proof of Lemma 4.3. From the descriptions of the RSK algorithm and the matching, it follows that an excedance X_i and a deficiency X_j are matched with each other precisely when π_j bumps π_i when RSK is performed on π, and that these are the only bumpings that take place. Indeed, an excedance never bumps anything because it is larger than the elements inserted before. On the other hand, when a deficiency X_j is inserted, it bumps the smallest element larger than π_j which has not been bumped yet (which corresponds to an excedance that has not been matched yet), if such an element exists. This proves the equality (1)=(2).

To see that (2)=(3), observe that right-side tunnels correspond to up-steps in the right half of $\Psi(\pi)$, which by the construction of the bijection Ψ correspond to elements on the second row of Q. The equality (3)=(5) follows easily by counting the number of up-steps and down-steps of the right half of the path. The equality (4)=(5) is analogous.

Finally, Schensted's Theorem states that the size of the first row of P equals the length of a longest increasing subsequence of π (see [76] or [86, Section 7.23]). This implies that (2)=(6), which completes the proof. $\square$

The reasoning used in the above proof gives a nice equivalent description of the recording tableau Q in terms of $\mathrm{arr}(\pi)$ and the matching. Read the rows of the array from top to bottom. For i from 1 to n, place i on the first row of Q if X_i is an excedance or it is unmatched, and place i on the second row if X_i is a matched deficiency. In the construction of the right half of $\Psi(\pi)$, this translates into drawing the path from right to left while reading the array from top to bottom, adjoining an up-step for each matched deficiency and a down-step for each other kind of cross.

To get a similar description of the tableau P, we use duality. By construction of the matching algorithm, the matching in the output is invariant under transposition of the array (reflection along the main diagonal). Recall the duality of the RSK correspondence: if $\mathrm{RSK}(\pi) = (P, Q)$, then $\mathrm{RSK}(\pi^{-1}) = (Q, P)$ (see e.g. [86, Section 7.13]). Therefore, the tableau P can be obtained by reading the columns of the array of π from left to right and placing integers in P according to the following rule. For each column j, place j on the first row of P if the cross in column j is a deficiency or it is unmatched. Similarly, place j on the second row if the cross is a matched excedance. Equivalently, the left half of $\Psi(\pi)$, from left to right, is obtained by reading the array from left to right and adjoining a down-step for each matched excedance, and an up-step for each of the remaining crosses.

In particular, when the left half of the path is constructed in this way, every matched pair (X_i, X_j) produces an up-step and a down-step, giving the latter a left-side tunnel. Similarly, in the construction of the right half of the path, a matched pair gives a right-side tunnel. Observe that these are again the equalities (1)=(3)=(4) in Lemma 4.3.

Proof of Lemma 4.4. It is enough to prove it only for the case of excedances. The case of deficiencies follows from it by considering π^{-1} and noticing that $\Psi(\pi^{-1}) = \Psi(\pi)^*$. Indeed, by duality $\mathrm{RSK}(\pi^{-1}) = (Q, P)$, so Q gives rise to the first half of $\Psi(\pi^{-1})$ and P to the second, so the path that we obtain is the reflection of $\Psi(\pi)$ in a vertical axis through the middle of the path. Let X_k be an unmatched excedance of π. We use the above description of $\Psi(\pi)$ in terms of the array and the matching. Each cross X_i produces a step r_i in the right half of the Dyck path and another step ℓ_i in the left half. Crosses above X_k produce steps to the right of r_k, and crosses to the left of X_k produce steps to the left of ℓ_k. In particular, there are $k-1$ steps to the right of r_k, and $\pi_k - 1$ steps to the left of ℓ_k. Note that since X_k is an excedance and π is 321-avoiding, all the crosses above it are also to the left of it. Consider the crosses that lie to the left of X_k. They can be of the following four kinds:

- *Unmatched excedances X_i.* They will necessarily lie above X_k, because the subsequence of excedances of π is decreasing. Each one of these crosses contributes an up-step to the left of ℓ_k and down-step to the right of r_k.

- *Unmatched deficiencies X_j.* They also have to lie above X_k, otherwise X_k would be matched with one of them. So, each such X_j contributes an up-step to the left of ℓ_k and down-step to the right of r_k.

- *Matched pairs (X_i, X_j)* (i.e. X_i is an excedance and X_j a deficiency)*, where both X_i and X_j lie above X_k.* Both crosses together will contribute an up-step and a down-step to the left of ℓ_k, and an up-step and a down-step to the right of r_k.

- *Matched pairs* (X_i, X_j) (i.e. X_i is an excedance and X_j a deficiency), *where* X_j *lies below* X_k. The pair will contribute an up-step and a down-step to the left of ℓ_k. However, to the right of r_k, the only contribution will be a down-step produced by X_i.

Note that there cannot be a deficiency X_j to the left of X_k matched with an excedance to the right of X_k, because in this case X_j would have been matched with X_k by the algorithm. In the first three cases, the contribution to both sides of the Dyck path is the same, so that the heights of r_k and ℓ_k are equally affected. But since $\pi_k > k$, at least one of the crosses to the left of X_k must be below it, and this must be a matched deficiency as in the fourth case. This implies that the step r_k is at a higher y-coordinate than ℓ_k. Let h_k be the height of ℓ_k. We now show that $\Psi(\pi)$ has a right-across tunnel at height h_k.

Observe that h_k is the number of unmatched crosses to the left of X_k, and that the height of r_k is the number of unmatched crosses above X_k (which equals h_k) plus the number of excedances above X_k matched with deficiencies below X_k. The part of the path between ℓ_k and the middle always remains at a height greater than h_k. This is because the only possible down-steps in this part can come from matched excedances X_i to the right of X_k, but each such X_i is matched with a deficiency X_j to the right of X_k but to the left of X_i, which produces an up-step compensating the down-step associated to X_i. Similarly, the part of the path between r_k and the middle remains at a height greater than h_k. This is because the h_k down-steps to the right of r_k that come from unmatched crosses above X_k do not have a corresponding up-step in the part of the path between r_k and the middle. Hence, ℓ_k is the left end of a right-across tunnel, since the right end of this tunnel is to the right of r_k, which in turn is closer to the right end of $\Psi(\pi)$ than ℓ_k is to its left end (see Figure 4-3).

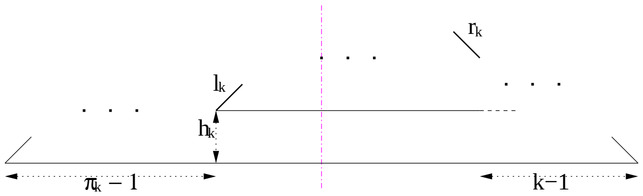


Figure 4-3: An unmateched excedance produces a right-across tunnel.

It can easily be checked that the converse is also true, namely that in every right-across tunnel of $\Psi(\pi)$, the step at its left end corresponds to an unmatched excedance of π. $\square$

4.4 Further applications

Fine numbers

In [73] it is proved that the number of permutations $\pi \in \mathcal{S}_n(132)$ (or $\pi \in \mathcal{S}_n(321)$) with no fixed points is the *Fine number* $\mathbf{F}_n$. This sequence is most easily defined by its relation to Catalan numbers:

$$\mathbf{C}_n = 2\,\mathbf{F}_n + \mathbf{F}_{n-1} \quad \text{for } n \geq 2, \quad \text{and } \mathbf{F}_1 = 0, \mathbf{F}_2 = 1.$$

Although defined some time ago, Fine numbers have received much attention in recent years (see a survey [25]). Special cases of the bijections in this chapter give simple bijections between these two combinatorial interpretations of Fine numbers and a new one: the set of Dyck paths without centered tunnels. In particular, we obtain a bijective proof of the following result.

Corollary 4.5 *The number of Dyck paths $D \in \mathcal{D}_n$ without centered tunnels is equal to $\mathbf{F}_n$.*

Yet another bijective proof of this corollary follows from the bijections in Chapter 3. The bijection Φ maps Dyck paths without centered tunnels to Dyck paths without hills, which in turn correspond through the bijection $\psi_{\llcorner}$ to 321-avoiding derangements.

More statistics

We can extend Propositions 4.2 and 2.4 to statistics $\nu_c(D)$ defined as the height at $x = n-c$ of the Dyck path $D \in \mathcal{D}_n$, for any $c \in \{0, \pm 1, \pm 2, \ldots, \pm(n-1)\}$. The corresponding statistics in $\mathcal{S}_n(132)$ and in $\mathcal{S}_n(321)$ are generalizations of the rank of a permutation and the length of the longest increasing subsequence in a certain subpermutation of π. A generalization of Theorem 4.1 follows, but we omit the details.

Limiting distribution

Let us also note that the limiting distribution of the length of the longest increasing subsequence in $\mathcal{S}_n(321)$ has been studied in [23]. From Theorem 4.1, the results in [23] can be translated into results on the limiting distribution of the statistic rank in $\mathcal{S}_n(132)$.

Refined restricted involutions

In a recent paper [24], Deutsch, Robertson and Saracino introduce the notion of *refined restricted involutions* by considering the statistic fp on involutions avoiding different patterns $\sigma \in \mathcal{S}_3$. They prove the following result:

Theorem 4.6 ([24]) *The number of 321-avoiding involutions $\pi \in \mathcal{S}_n$ with $\mathrm{fp}(\pi) = i$ equals the number of 132-avoiding involutions $\pi \in \mathcal{S}_n$ with $\mathrm{fp}(\pi) = i$, for any $0 \leq i \leq n$.*

Let us show that Theorem 4.6 follows easily from the work in this chapter. Recall from Section 1.2.1 that if $D \in \mathcal{D}_n$, D^* denotes the path obtained by reflection of D from a vertical line $x = n$. Now observe that if $\varphi(\pi) = D$, then $\varphi(\pi^{-1}) = D^*$ (see Lemma 5.27). Similarly, if $\Psi(\pi) = D$, then $\Psi(\pi^{-1}) = D^*$ (by the duality of RSK). Therefore, $\pi \in \mathcal{S}_n(321)$ is an involution if and only if so is $\Theta(\pi) \in \mathcal{S}_n(132)$, which implies the result. Furthermore, restricting Θ to involutions we obtain the following extension of Theorem 4.6:

Theorem 4.7 *The number of 321-avoiding involutions $\pi \in \mathcal{S}_n$ with $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ and $\mathrm{lis}(\pi) = k$ equals the number of 132-avoiding involutions $\pi \in \mathcal{S}_n$ with $\mathrm{fp}(\pi) = i$, $\mathrm{exc}(\pi) = j$ and $\mathrm{rank}(\pi) = n - k$, for any $0 \leq i, j, k \leq n$.*

Chapter 5

Avoidance of subsets of patterns of length 3

In the previous chapters we have focused only on 321-avoiding and on 132-avoiding permutations, and in the distribution of the statistics 'number of fixed points' and 'number of excedances' on them. Here we study the distribution of these statistics on permutations avoiding the other patterns of length 3, and, more generally, avoiding any subset of patterns of length 3. A systematic enumeration (with no statistics) of permutations avoiding any subset of patterns of length 3 was done in [79]. Here we give refinements of these results, by enumerating the same permutations with respect to the statistics fp and exc.

The main technique that we use are bijections between pattern-avoiding permutations and certain kinds of Dyck paths with some restrictions, in such a way that the statistics in permutations that we study correspond to statistics on Dyck paths that are easy to enumerate.

In Section 5.1 we introduce some more properties of the bijection φ that we will need in this chapter. In Section 5.2 we consider permutations with a single restriction. We study the distribution of the statistics fp and exc for each pattern of length 3, except for the pattern 123, for which we can only give partial results regarding fp. In Section 5.3 we solve completely the case of permutations avoiding simultaneously any two patterns of length 3, giving generating functions counting the number of fixed points and the number of excedances. For some particular instances we can generalize the results, allowing one pattern of the pair to have arbitrary length. In Section 5.4 we give the analogous generating functions for permutations avoiding simultaneously any three patterns of length 3 or more. Section 5.5 is concerned with the study of the distribution of these statistics in involutions avoiding any subset of patterns of length 3. In Section 5.6 we compute the expected number of fixed points in permutations avoiding patterns of length 3. We conclude with a few final remarks and a discussion of possible extensions of our work.

5.1 More properties of φ

The bijection φ defined in Section 2.2 will be one of our main tools in this chapter. We will also use repeatedly the array representation of a permutation π as described in Section 1.1.3, as well as the operations $\bar{\pi}$, $\widehat{\pi}$, and the lemmas proved in that section. Recall from Section 1.2.2 that the depth of a tunnel T is defined as $\text{depth}(T) := \frac{1}{2}\text{length}(T) - \text{height}(T) - 1$.

In order to enumerate fixed points and excedances in permutations, we analyze what these statistics are mapped to by φ. Table 5.1 summarizes the correspondences of φ that we will use.

In the permutation π	In the array of π	In the Dyck path $\varphi(\pi)$
fixed points of π	crosses on the main diagonal	centered tunnels
excedances of π	crosses to the right of the main diagonal	right tunnels
fixed points of $\bar{\pi}$	crosses on the secondary diagonal	tunnels of depth 0
excedances of $\bar{\pi}$	crosses to the left of the secondary diagonal	tunnels of negative depth

Table 5.1: Behavior of φ on fixed points and excedances.

The correspondences between the first two columns have been discussed in Section 1.1.3. In Proposition 2.4 we showed that the first two rows of the table describe some properties of φ. Here we repeat the same reasoning from the proof of that proposition to show how φ maps crosses on the secondary diagonal to tunnels of depth 0, and crosses to the left of the secondary diagonal to tunnels of negative depth.

Again, instead of using $D = \varphi(\pi)$, it will be convenient to consider the path U from the lower-left corner to the upper-right corner of the array of π, and to talk about tunnels of U to refer to the corresponding tunnels of D under this trivial transformation.

Recall how in the proof of Proposition 2.4 we associated a unique tunnel T of D to each cross X of $\mathrm{arr}(\pi)$. Given a cross $X = (i, j)$, U has a north step in row i and an east step in column j. These two steps in U correspond to steps $\mathbf{u}$ and $\mathbf{d}$ in D, respectively, so they determine a decomposition $D = A\mathbf{u}B\mathbf{d}C$ (see Figure 2-4), and therefore a tunnel T of D.

The distance between these two steps determines the length of T, and the distance from these steps to the secondary diagonal of the array determines the height of T. In order for the corresponding cross to lie on the secondary diagonal, the relation between these two quantities must be $\frac{1}{2}\mathrm{length}(T) = \mathrm{height}(T) + 1$, which is equivalent to $\mathrm{depth}(T) = 0$, by the definition of depth. The depth of T indicates how far from the secondary diagonal X is. The cross lies to the left of the secondary diagonal exactly when $\mathrm{depth}(T) < 0$. This justifies the last two rows of the table.

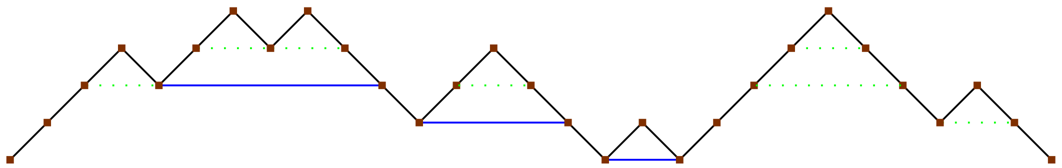


Figure 5-1: Three tunnels of depth 0 and seven tunnels of negative depth.

We define two new statistics on Dyck paths. For $D \in \mathcal{D}$, let $\mathrm{td}_0(D)$ be the number of tunnels of depth 0 of D, and let $\mathrm{td}_{<0}(D)$ be the number of tunnels of negative depth of D. In Figure 5-1, there are three tunnels of depth 0 drawn with a solid line, and seven tunnels of negative depth drawn with dotted lines. Let us state these results as a lemma, which partially overlaps with Proposition 2.4.

Lemma 5.1 *Let $\pi \in \mathcal{S}_n(132)$, $\rho \in \mathcal{S}_n(312)$. We have*

(1) $\mathrm{fp}(\pi) = \mathrm{ct}(\varphi(\pi))$,

(2) $\mathrm{exc}(\pi) = \mathrm{rt}(\varphi(\pi))$,

(3) $\mathrm{fp}(\rho) = \mathrm{td}_0(\varphi(\bar{\rho}))$,

(4) $\mathrm{exc}(\rho) = \mathrm{td}_{<0}(\varphi(\bar{\rho}))$.

5.2 Single restrictions

When studying statistics on permutations avoiding subsets of patterns of length 3, the most difficult case appears to be that of permutations avoiding one single pattern. It is well known that for any $\sigma \in \mathcal{S}_3$, $|\mathcal{S}_n(\sigma)| = \mathbf{C}_n$. By Lemma 1.2, we have that $132 \approx 213$, and that $231 \sim 312$. These are the only equivalences that follow from the trivial bijections. In Chapter 2 we showed that in fact $321 \approx 132$. So, we have the following equivalence classes of patterns of length 3 with respect to fixed points and excedances:

$$\begin{array}{c} \mathbf{a)}\ 123 \\ \mathbf{b)}\ 132 \approx 213 \approx 321 \\ \mathbf{c)}\ 231 \quad \sim \quad \mathbf{c')}\ 312 \end{array}$$

5.2.1 a) 123

For this case we have not been able to find a satisfactory expression for $F_{123}(x, q, z)$. We can nevertheless give summation formulas for the number of permutations in $\mathcal{S}_n(123)$ with a given number of fixed points. The first trivial observation is that if π avoids 123, then it can have at most two fixed points. If $\pi_i = i$, we say that i is a *big fixed point* of π if $i \geq \frac{n+1}{2}$, and that it is a *small fixed point* if $i < \frac{n+1}{2}$.

We already mentioned that a permutation is 321-avoiding if and only if both the subsequence determined by its excedances and the one determined by the remaining elements are increasing. Using the fact that π avoids 123 if and only if $\bar{\pi}$ avoids 321, we obtain a characterization of 123-avoiding permutations as those with the following property: the elements π_i such that $\pi_i < n + 1 - i$ form a decreasing subsequence, and so do the remaining elements. In particular, since no two fixed points can be in the same decreasing subsequence, this implies that π can have at most one big fixed point and one small fixed point.

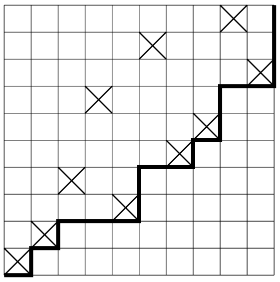
Recall the bijection $\psi_{\llcorner} : \mathcal{S}_n(321) \longrightarrow \mathcal{D}_n$ that we defined in Section 2.1. Composing it with the complementation operation sending $\pi \in \mathcal{S}_n(123)$ to $\bar{\pi} \in \mathcal{S}_n(321)$, we obtain a bijection between $\mathcal{S}_n(123)$ and $\mathcal{D}_n$, which we denote by $\psi_{\lrcorner}$. Figure 5-2 shows an example when $\pi = (9, 6, 10, 4, 8, 7, 3, 5, 2, 1)$.

Note that the peaks of the path are determined by the crosses of elements π_i such that $\pi_i \geq n+1-i$, which form a decreasing subsequence. Now it is easy to determine how many permutations have a big (resp. small) fixed point.

Proposition 5.2 *Let $n \geq 1$. We have*

(1) $|\{\pi \in \mathcal{S}_n(123) : \pi \textit{ has a big fixed point}\}| = \mathbf{C}_{n-1}$,

(2) $|\{\pi \in \mathcal{S}_n(123) : \pi \textit{ has a small fixed point}\}| = \begin{cases} \mathbf{C}_{n-1} & \textit{if } n \textit{ is even,} \\ \mathbf{C}_{n-1} - \mathbf{C}^2_{\frac{n-1}{2}} & \textit{if } n \textit{ is odd.} \end{cases}$



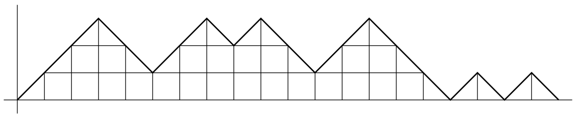


Figure 5-2: The bijection $\psi_\lrcorner$.

Proof. (1) It is clear from the definition of $\psi_\lrcorner$ that π has a big fixed point if and only if $\psi_\lrcorner(\pi)$ has a peak in the middle. Now, we can easily define a bijection from the subset of elements of $\mathcal{D}_n$ with a peak in the middle and $\mathcal{D}_{n-1}$, by removing the two middle steps **ud**.

(2) Clearly, $\pi \in \mathcal{S}_n(123)$ if and only if $\widehat{\pi} \in \mathcal{S}_n(123)$. This involution switches big and small fixed points, except for the possible big fixed point in position $\frac{n+1}{2}$, which remains unchanged. Applying now $\psi_\lrcorner$, a small fixed point of π is transformed into a peak in the middle of $\psi_\lrcorner(\widehat{\pi})$ of height at least two (indeed, a hill would correspond to the big fixed point $\frac{n+1}{2}$). Knowing that the number of paths in $\mathcal{D}_n$ with a peak in the middle is $\mathbf{C}_{n-1}$, we just have to subtract those where this middle peak has height 1. If n is even, paths in $\mathcal{D}_n$ cannot have a hill in the middle. If n is odd, such paths have the form $A\mathbf{ud}B$, where $A, B \in \mathcal{D}_{\frac{n-1}{2}}$, so the formula follows. $\square$

For $k \geq 0$, let $s_n^k(123) := |\{\pi \in \mathcal{S}_n(123) : \mathrm{fp}(\pi) = k\}|$. We have mentioned that $s_n^k(123) = 0$ for $k \geq 3$. The following corollary reduces the problem of studying the distribution of fixed points in $\mathcal{S}_n(123)$ to that of determining $s_n^2(123)$.

Corollary 5.3 *Let $n \geq 1$. We have*

$$(1)\ \ s_n^1(123) = \begin{cases} 2(\mathbf{C}_{n-1} - s_n^2(123)) & \textit{if } n \textit{ even,} \\ 2(\mathbf{C}_{n-1} - s_n^2(123)) - \mathbf{C}^2_{\frac{n-1}{2}} & \textit{if } n \textit{ odd.} \end{cases}$$

$$(2)\ \ s_n^0(123) = \begin{cases} \mathbf{C}_n - 2\mathbf{C}_{n-1} + s_n^2(123) & \textit{if } n \textit{ even,} \\ \mathbf{C}_n - 2\mathbf{C}_{n-1} + s_n^2(123) + \mathbf{C}^2_{\frac{n-1}{2}} & \textit{if } n \textit{ odd.} \end{cases}$$

Proof. (1) By inclusion-exclusion, $s_n^1(123) = |\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a big fixed point}\}| + |\{\pi \in \mathcal{S}_n(123) : \pi \text{ has a small fixed point}\}| - 2s_n^2(123)$. Now we apply Proposition 5.2.

(2) Clearly, $s_n^0(123) = \mathbf{C}_n - s_n^1(123) - s_n^2(123)$. $\square$

The next theorem, together with the previous corollary, gives a formula for the distribution of fixed points in 123-avoiding permutations.

Theorem 5.4

$$s_n^2(123) = \sum_{i=1}^{n-1} \sum_{r,s=1}^{i} \left[\left(\binom{2i-r-1}{i-1} - \binom{2i-r-1}{i} \right) \cdot \left(\binom{2i-s-1}{i-1} - \binom{2i-s-1}{i} \right) \right.$$
$$\left. \cdot \sum_{\substack{h=1 \\ n-h \text{ even}}}^{n} \sum_{k=0}^{n-2i} f(k,r,h,n-2i+r) f(n-2i-k,s,h,n-2i+s) \right],$$

where

$$f(k,r,h,\ell) = \begin{cases} \binom{\frac{\ell+h-r}{2}-1}{k}\binom{\frac{\ell-h+r}{2}-1}{k-1} - \binom{\frac{\ell-h-r}{2}-1}{k}\binom{\frac{\ell+h+r}{2}-1}{k-1} & \text{if } k \geq 1, \\ 1 & \text{if } k=0 \text{ and } \ell = h-r, \\ 0 & \text{otherwise,} \end{cases}$$

with the convention $\binom{a}{b} := 0$ *if* $a < 0$.

Proof. Recall that $s_n^2(123)$ counts permutations with both a big and a small fixed point. We have seen already that $\psi_\lrcorner$ maps a big fixed point of the permutation into a peak in the middle of the Dyck path. Now we look at how a small fixed point of the permutation is transformed by $\psi_\lrcorner$. We claim that $\pi \in \mathcal{S}_n(123)$ has a small fixed point if and only if $D = \psi_\lrcorner(\pi)$ satisfies the following condition (which we call condition C1): there exists i such that the i-th and $(i+1)$-st up-steps of D are consecutive, the i-th and $(i+1)$-st down-steps from the end are consecutive, and there are exactly $n+1-2i$ peaks of D between them. To see this, assume that i is a small fixed point of π (see Figure 5-3). Then, the path from the upper-right to the lower-left corner of the array of π, used to define $\psi_\lrcorner(\pi)$, has two consecutive vertical steps in rows i and $i+1$, and two consecutive horizontal steps in columns i and $i+1$. Besides, there are $n+1-2i$ crosses below and to the right of cross (i,i), each one of which produces a peak in the Dyck path $\psi_\lrcorner(\pi)$. Reciprocally, it can be checked that if $\psi_\lrcorner(\pi)$ satisfies condition C1 then π has a small fixed point.

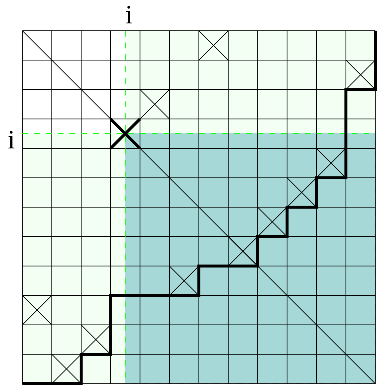


Figure 5-3: A small fixed point i has $n+1-2i$ crosses below and to the right.

All we have to do is count how many paths $D \in \mathcal{D}_n$ with a peak in the middle satisfy condition C1. For such a Dyck path D, define the following parameters: let i be the value such that condition C1 holds, let $h = \nu(D)$ be the height of D in the middle, r the height at which the i-th up-step ends, and s the height at which the i-th down-step from the end begins. In the example of Figure 5-4, $n = 12$, $i = 4$, $h = 4$, $r = 3$, and $s = 1$.

Fix n, i, h, r and s. We will count the number of Dyck paths D with these given parameters. We can write $D = A\mathbf{uu}B_1B_2\mathbf{dd}C$, where the distinguished $\mathbf{u}$'s are the i-th and $(i+1)$-st up-steps, the two $\mathbf{d}$'s are the i-th and $(i+1)$-st down-steps from the end, and the middle of D is between B_1 and B_2. Then A is a path from $(0,0)$ to $(2i-r-1, r-1)$ not going below $y=0$. It is easy to see that there are $\binom{2i-r-1}{i-1} - \binom{2i-r-1}{i}$ such paths A. By symmetry, there are $\binom{2i-s-1}{i-1} - \binom{2i-s-1}{i}$ possibilities for C.

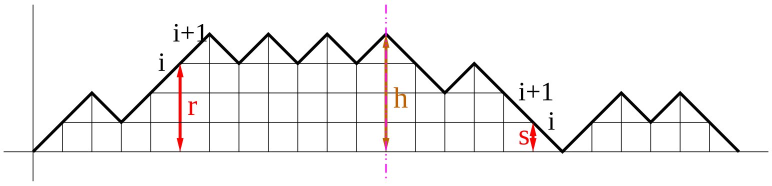


Figure 5-4: The parameters i, h, r and s in a Dyck path.

Now we count the possibilities for B_1 and B_2. It can be checked that $f(k, r, h, \ell)$ counts the number of paths from $(0, r)$ to (ℓ, h) having exactly k peaks, starting and ending with an up-step, and never going below $y = 0$. The fragment $\mathbf{u}B_1$ is a path from $(2i - r, r)$ to (n, h) not going below $y = 0$, and ending with an up-step (since D has a peak in the middle). If we fix k as the number of peaks of this fragment, then there are $f(k, r, h, n - 2i + r)$ such paths $\mathbf{u}B_1$. Similarly, there are $f(n - 2i - k, s, h, n - 2i - s)$ possibilities for $B_2\mathbf{d}$ with $n - 2i - k$ peaks.

Summing over all possible values of k, h, r, s and i we obtain the expression in the theorem. □

Using Corollary 5.3, we can prove that among the derangements of length n, the number of 123-avoiding ones is at least the number of 132-avoiding ones. This inequality was conjectured by Miklós Bóna and Olivier Guibert.

Theorem 5.5 ([16]) *For all $n \geq 4$, $s_n^0(132) < s_n^0(123)$.*

Proof. For $n \leq 12$ the result can be checked by exhaustive enumeration of all derangements by computer. Let us assume that $n \geq 13$.

From part (2) of Corollary 5.3, we have that

$$s_n^0(123) \geq \mathbf{C}_n - 2\mathbf{C}_{n-1}.$$

It is known [73] that $s_n^0(132) = \mathbf{F}_n$, the n-th *Fine number*. Therefore, the theorem will be proved if we show that

$$\mathbf{F}_n < \mathbf{C}_n - 2\mathbf{C}_{n-1} \tag{5.1}$$

for $n \geq 13$. Using the identity $\mathbf{F}_n = \frac{1}{2}\sum_{i=0}^{n-2}(\frac{-1}{2})^i \mathbf{C}_{n-i}$, we get the inequality $\mathbf{F}_n < \frac{1}{2}\mathbf{C}_n - \frac{1}{4}\mathbf{C}_{n-1} + \frac{1}{8}\mathbf{C}_{n-2}$, which reduces (5.1) to showing that $\mathbf{C}_n > \frac{7}{2}\mathbf{C}_{n-1} + \frac{1}{4}\mathbf{C}_{n-2}$. This inequality certainly holds asymptotically, because $\mathbf{C}_n$ grows like $\frac{1}{\sqrt{\pi}} n^{-\frac{3}{2}} 4^n$ as n tends to infinity, and it is not hard to see that in fact it holds for all $n \geq 13$. □

5.2.2 b) $132 \approx 213 \approx 321$

We already studied this case in Chapter 2. The corresponding GF with variables x and q marking fixed points and excedances respectively is the following.

Theorem 5.6

$$F_{132}(x, q, z) = F_{213}(x, q, z) = F_{321}(x, q, z) =$$
$$= \frac{2}{1 + (1 + q - 2x)z + \sqrt{1 - 2(1 + q)z + (1 - q)^2 z^2}}.$$

A generalization of this formula is Theorem 2.3, which gives the GF for 321-avoiding permutations with respect to fixed points, excedances and descents. Similarly, the generalization of Theorem 5.6 which enumerates 132-avoiding permutations with respect to these three statistics is given in Theorem 3.10.

Longest increasing subsequence

Here we make a small digression and consider two other statistics in 132-avoiding permutations. Recall that $\mathrm{nlis}(\pi)$ denotes the number of increasing subsequences of π of maximum length $\mathrm{lis}(\pi)$. Answering a question of Emeric Deutsch, here we describe the joint distribution of the pair of statistics lis and nlis. For $k \geq 0$, let

$$H_k(u,z) := \sum_{n\geq 0} \sum_{\substack{\pi\in\mathcal{S}_n(132)\\ \mathrm{lis}(\pi)=k}} u^{\mathrm{nlis}(\pi)} z^n.$$

Parts (4) and (5) of Proposition 2.4 show that the bijection φ maps the statistics lis and nlis on 132-avoiding permutations to the statistics 'height' and 'number of peaks at maximum height' on Dyck paths respectively. Denoting by $\mathcal{D}^k$ the set of Dyck paths of height k, we can express H_k as

$$H_k(u,z) = \sum_{D\in\mathcal{D}^k} u^{\#\{\text{peaks of } D \text{ at height } k\}} z^{|D|}.$$

In other words, $H_k(u,z)$ is the GF for Dyck paths of height k where u marks the number of peaks of maximum height (i.e., k). We have the following expression for H_k.

Theorem 5.7

$$H_k(u,z) = \frac{uz^k}{V_{k-1}(V_{k-1} - uzV_{k-2})},$$

where the V_k are polynomials in z defined by $V_k = V_{k-1} - zV_{k-2}$, $V_{-1} = V_0 = 1$. Equivalently, H_k can be expressed as

$$H_k(u,z) = \frac{uz^k(1-4z)}{\alpha^{2k+1}(\alpha - uz) + \overline{\alpha}^{2k+1}(\overline{\alpha} - uz) + z^{k+1}(u-2)},$$

where $\alpha = \frac{1+\sqrt{1-4z}}{2}$, $\overline{\alpha} = \frac{1-\sqrt{1-4z}}{2}$.

Note that the V_k are related to the Chebyshev polynomials of the second kind U_k, defined in Section 1.3.2, by

$$V_k(z) = (\sqrt{z})^k U_k(\frac{1}{2\sqrt{z}}),$$

$$U_k(x) = (2x)^k V_k(\frac{1}{4x^2}).$$

Solving the recurrence that defined the V_k, we get that

$$V_k = \frac{1}{\sqrt{1-4z}}(\alpha^{k+2} - \overline{\alpha}^{k+2}),$$

with α and $\overline{\alpha}$ as defined above.

Proof. For $k \geq 0$, let

$$\widehat{H}_k(z) := H_k(1,z) = \sum_{D \in \mathcal{D}^k} z^{|D|}.$$

If $k \geq 1$, any $D \in \mathcal{D}^k$ can be written uniquely as $D = \mathbf{u}A_1\mathbf{d}\mathbf{u}A_2\mathbf{d}\cdots$ where each $A_i \in \mathcal{D}^{k-1}$. So, we have the recurrence

$$\widehat{H}_k(z) = \frac{1}{1 - z\widehat{H}_{k-1}(z)}, \quad \widehat{H}_0(z) = 1,$$

from where it follows using induction that $\widehat{H}_k(z) = \dfrac{V_{k-1}}{V_k}$.

Now let

$$R_k(u,z) = \sum_{n \geq 0} \sum_P u^{\#\{\text{peaks of } P \text{ at height } k\}} z^n,$$

where P ranges over all paths from $(0,0)$ to $(2n+k, k)$ with steps $\mathbf{u} = (1,1)$ and $\mathbf{d} = (1,-1)$ that always stay between the lines $y = 0$ and $y = k$, and peaks are defined in the natural way as occurrences of $\mathbf{ud}$.

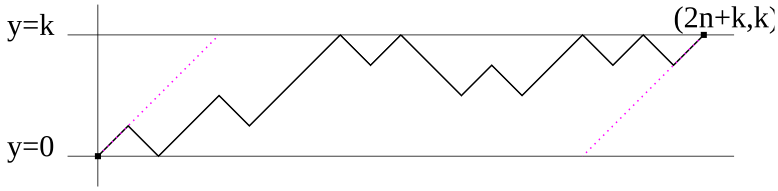


Figure 5-5: A path from $(0,0)$ to $(2n+k, k)$.

Denote by $\mathcal{R}_k$ the set of such paths P for fixed k and varying n. If such a path P hits $y = k$ only at the end, then we can write $P = X\mathbf{u}$, where $X \in \mathcal{R}_{k-1}$. If P hits $y = k$ more than once, we can decompose it uniquely as $P = Y\mathbf{d}Z\mathbf{u}$, where $Y \in \mathcal{R}_k$ and Z can be seen as a Dyck path of height at most $k-1$ drawn upside-down.

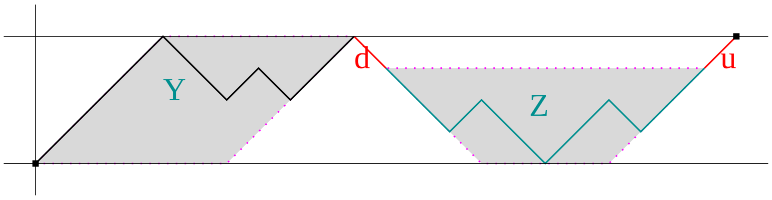


Figure 5-6: The decomposition $P = Y\mathbf{d}Z\mathbf{u}$.

This decomposition gives

$$R_k(u,z) = uR_{k-1}(1,z) + uzR_k(u,z)\widehat{H}_{k-1}(z),$$

which implies that

$$R_k(u,z) = \frac{uR_{k-1}(1,z)}{1 - uz\widehat{H}_{k-1}(z)}. \tag{5.2}$$

Substituting $u = 1$ in (5.2) and using that $\widehat{H}_{k-1} = \frac{V_{k-2}}{V_{k-1}}$, it can be checked by induction that $R_k(1,z) = \frac{1}{V_k}$.

Now, a Dyck path D of height k can be decomposed uniquely as $D = X\mathbf{u}Y$, where $X \in \mathcal{R}_{k-1}$, and Y is the reflection over a vertical line of a path in $\mathcal{R}_k$.

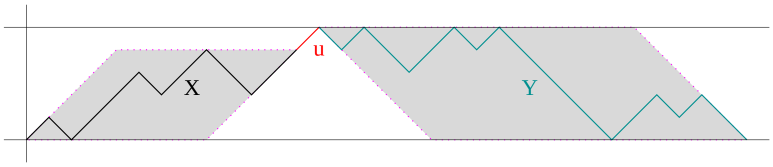


Figure 5-7: The decomposition $D = X\mathbf{u}Y$.

By this decomposition, H_k can be written as

$$H_k(u,z) = R_{k-1}(1,z)R_k(u,z)z^k.$$

Thus,

$$H_k(u,z) = \frac{uz^k R_{k-1}(1,z)^2}{1 - uz\widehat{H}_{k-1}(z)} = \frac{uz^k}{V_{k-1}(V_{k-1} - uzV_{k-2})}.$$

□

For the first values of k, the theorem gives

$$H_1(u,z) = \frac{uz}{1-uz},$$

$$H_2(u,z) = \frac{uz^2}{(1-z)(1-(u+1)z)}.$$

5.2.3 c, c') 231 ∼ 312

Using the bijection

$$\begin{array}{ccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \varphi(\bar{\pi}), \end{array}$$

Lemma 5.1 implies that

$$F_{312}(x,q,z) = \sum_{D \in \mathcal{D}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

To enumerate tunnels of depth 0, we will separate them according to their height. For every $h \geq 0$, a tunnel at height h must have length $2(h+1)$ in order to have depth 0. It is important to notice that if a tunnel of depth 0 of D corresponds to a decomposition $D = A\mathbf{u}B\mathbf{d}C$, then D has no tunnels of depth 0 in the part given by B. In other words, the projections on the x-axis of all the tunnels of depth 0 of a given Dyck path are disjoint. This observation allows us to give a continued fraction expression for $F_{312}(x,1,z)$.

Theorem 5.8 *$F_{312}(x,1,z)$ is given by the following continued fraction.*

$$F_{312}(x,1,z)=\cfrac{1}{1-(x-1)z-\cfrac{z}{1-(x-1)z^2-\cfrac{z}{1-2(x-1)z^3-\cfrac{z}{1-5(x-1)z^4-\cfrac{z}{\ddots}}}}},$$

where at the n-th level, the coefficient of $(x-1)z^{n+1}$ is the Catalan number $\mathbf{C}_n$.

Proof. For every $h\geq 0$, let $\mathrm{td}_0^h(D)$ be the number of tunnels of D of height h and length $2(h+1)$. Note that $\mathrm{td}_0(D)=\sum_{h\geq 0}\mathrm{td}_0^h(D)$. We will show now that for every $h\geq 1$, the generating function for Dyck paths where x marks the statistic $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^{h-1}$ is given by the continued fraction of the theorem truncated at level h, with the $(h+1)$-st level replaced with $\mathbf{C}(z)$.

A Dyck path D can be written uniquely as a sequence of elevated Dyck paths, that is, as $D=\mathbf{u}A_1\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each $A_i\in\mathcal{D}$. In terms of the GF $\mathbf{C}(z)=\sum_{D\in\mathcal{D}}z^{|D|}$, this translates into the equation $\mathbf{C}(z)=\frac{1}{1-z\mathbf{C}(z)}$. A tunnel of height 0 and length 2 (i.e., a hill) appears in D for each empty A_i. Therefore, the GF enumerating hills is

$$\sum_{D\in\mathcal{D}}x^{\mathrm{td}_0^0(D)}z^{|D|}=\frac{1}{1-z[x-1+\mathbf{C}(z)]},\tag{5.3}$$

since an empty A_i has to be counted as x, not as 1.

Let us enumerate simultaneously hills (as above), and tunnels of height 1 and length 4. The GF (5.3) can be written as

$$\frac{1}{1-z\left[x-1+\frac{1}{1-z\mathbf{C}(z)}\right]}.$$

Combinatorially, this corresponds to expressing each A_i as a sequence $\mathbf{u}B_1\mathbf{d}\cdots\mathbf{u}B_s\mathbf{d}$, where each $B_j\in\mathcal{D}$. Notice that since each $\mathbf{u}B_j\mathbf{d}$ starts at height 1, a tunnel of height 1 and length 4 is created by each $B_j=\mathbf{ud}$ in the decomposition. Thus, if we want x to mark also these tunnels, such a B_j has to be counted as xz, not z. The corresponding GF is

$$\sum_{D\in\mathcal{D}}x^{\mathrm{td}_0^0(D)+\mathrm{td}_0^1(D)}z^{|D|}=\frac{1}{1-z\left[x-1+\frac{1}{1-z[(x-1)z+\mathbf{C}(z)]}\right]}.$$

Now it is clear how iterating this process indefinitely we obtain the continued fraction of the theorem. From the GF where x marks $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^{h-1}$, we can obtain the one where x marks $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^h$ by replacing the $\mathbf{C}(z)$ at the lowest level with

$$\frac{1}{1-z[(x-1)\mathbf{C}_h z^h+\mathbf{C}(z)]},$$

to account for tunnels of height h and length $2(h+1)$, which in the decomposition correspond to elevated Dyck paths at height h. $\square$

The same technique can be used to enumerate excedances in 312-avoiding permutations, which correspond to tunnels of negative depth in the Dyck path. Recall that

$$\mathbf{C}_{<i}(z) = \sum_{j=0}^{i-1} \mathbf{C}_j z^j$$

denotes the series for the Catalan numbers truncated at degree i.

Theorem 5.9 *$F_{312}(x,q,z)$ is given by the following continued fraction.*

$$F_{312}(x,q,z) = \cfrac{1}{1 - zK_0 + \cfrac{z}{1 - zK_1 + \cfrac{z}{1 - zK_2 + \cfrac{z}{1 - zK_3 + \cfrac{z}{\ddots}}}}},$$

where $K_n = (x-1)\mathbf{C}_n q^n z^n + (q-1)\mathbf{C}_{<n}(qz)$ for $n \geq 0$.

Note that the first values of K_n are

$$K_0 = x - 1, \qquad K_1 = (x-1)qz + q - 1,$$
$$K_2 = 2(x-1)q^2z^2 + (q-1)(1+qz), \quad K_3 = 5(x-1)q^3z^3 + (q-1)(1+qz+2q^2z^2).$$

Proof. We use the same decomposition as above, now keeping track of tunnels of negative depth as well. For every $h \geq 0$, let $\mathrm{td}^h_{<0}(D)$ be the number of tunnels of D of height h and length less than $2(h+1)$. Note that $\mathrm{td}_{<0}(D) = \sum_{h\geq 0} \mathrm{td}^h_{<0}(D)$. To follow the same structure as in the previous proof, counting tunnels height by height, it will be convenient that at the h-th step of the iteration, q marks not only tunnels of negative depth up to height h but also all the tunnels at higher levels. Denote by $\mathrm{alltun}^{>h}(D)$ the number of tunnels of D of height strictly greater than h.

We will show now that for every $h \geq 1$, the generating function for Dyck paths where x marks the statistic $\mathrm{td}^0_0 + \cdots + \mathrm{td}^{h-1}_0$ and q marks $\mathrm{td}^0_{<0} + \cdots + \mathrm{td}^{h-1}_{<0} + \mathrm{alltun}^{>h-1}$ is given by the continued fraction of the theorem truncated at level h, with the $(h+1)$-st level replaced with $\mathbf{C}(qz)$.

The analogous to equation (5.3) is now

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}^0_0(D)} q^{\mathrm{td}^0_{<0}(D)+\mathrm{alltun}^{>0}(D)} z^{|D|} = \frac{1}{1 - z[x - 1 + \mathbf{C}(qz)]}. \tag{5.4}$$

Indeed, decomposing D as $\mathbf{u}A_1\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, q counts all the tunnels that appear in any A_i, and whenever an A_i is empty we must mark it as x.

Let us enumerate now tunnels of depth 0 and negative depth at both height 0 and height 1. Modifying (5.4) so that q no longer counts tunnels at height 1, we get

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}^0_0(D)} q^{\mathrm{td}^0_{<0}(D)+\mathrm{alltun}^{>1}(D)} z^{|D|} = \frac{1}{1 - z\left[x - 1 + \dfrac{1}{1 - z\mathbf{C}(qz)}\right]}, \tag{5.5}$$

which corresponds to writing each A_i as $A_i = \mathbf{u}B_1\mathbf{d}\cdots\mathbf{u}B_s\mathbf{d}$, and having q count all tunnels in each B_j. Now, in order for x to mark tunnels of depth 0 at height 1, each $B_j = \mathbf{ud}$, that in (5.5) is counted as qz, has to be now counted as xqz instead. Similarly, to have q mark tunnels of negative depth at height 1, we must count each empty B_j as q, not as 1. This gives us the following GF:

$$\sum_{D\in\mathcal{D}} x^{\mathrm{td}_0^0(D)+\mathrm{td}_0^1(D)} q^{\mathrm{td}_{<0}^0(D)+\mathrm{td}_{<0}^1(D)+\mathrm{alltun}^{>1}(D)} z^{|D|}$$

$$= \frac{1}{1-z\left[x-1+\dfrac{1}{1-z[(x-1)qz+q-1+\mathbf{C}(qz)]}\right]}.$$

Iterating this process level by level indefinitely we obtain the continued fraction of the theorem. At each step, from the GF where x marks $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^{h-1}$, and q marks $\mathrm{td}_{<0}^0+\cdots+\mathrm{td}_{<0}^{h-1}+\mathrm{alltun}^{>h-1}$, we can obtain the one where x marks $\mathrm{td}_0^0+\cdots+\mathrm{td}_0^h$ and q marks $\mathrm{td}_{<0}^0+\cdots+\mathrm{td}_{<0}^h+\mathrm{alltun}^{>h}$ by replacing the $\mathbf{C}(qz)$ at the lowest level with

$$\frac{1}{1-z[(x-1)\mathbf{C}_h q^h z^h+(q-1)\mathbf{C}_{<h}(qz)+\mathbf{C}(qz)]}. \tag{5.6}$$

This change makes x account for tunnels of depth 0 at height h, which in the decomposition correspond to the $\mathbf{C}_h$ possible elevated Dyck paths of length $2(h+1)$ when they occur at height h. It also makes q count tunnels of negative depth at height h, which in the decomposition correspond to elevated Dyck paths at height h of length less than $2(h+1)$. The GF for these ones becomes $q\mathbf{C}_{<h}(qz)$ instead of $\mathbf{C}_{<h}(qz)$, since for every $j<h$, an elevated path $\mathbf{u}C\mathbf{d}$ with $C\in\mathcal{D}_j$ contributes to one extra tunnel of negative depth at height h, aside from the j tunnels of height more than h that it contains. □

For 231-avoiding permutations we get the following GF.

Corollary 5.10 *$F_{231}(x,q,z)$ is given by the following continued fraction.*

$$F_{231}(x,q,z)=\cfrac{1}{1-zK'_0+\cfrac{qz}{1-zK'_1+\cfrac{qz}{1-zK'_2+\cfrac{qz}{1-zK'_3+\cfrac{qz}{\ddots}}}}},$$

where $K'_n=(x-q)\mathbf{C}_n z^n+(1-q)\mathbf{C}_{<n}(z)$.

The first values of K'_n are

$$K'_0=x-q, \qquad K'_1=(x-q)z+1-q,$$
$$K'_2=2(x-q)z^2+(1-q)(1+z), \qquad K'_3=5(x-q)z^3+(1-q)(1+z+2z^2).$$

Proof. By Lemma 1.2, we have that $F_{231}(x,q,z)=F_{312}(x/q,1/q,qz)$, so the expression follows from Theorem 5.9. □

5.3 Double restrictions

In this section we consider simultaneous avoidance of any two patterns of length 3. Using Lemma 1.2, the pairs of patterns fall into the following equivalence classes.

a) $\{123, 132\} \approx \{123, 213\}$
b) $\{231, 321\}$ $\sim$ **b')** $\{312, 321\}$
c) $\{132, 213\}$
d) $\{231, 312\}$
e) $\{132, 231\} \approx \{213, 231\}$ $\sim$ **e')** $\{132, 312\} \approx \{213, 312\}$
f) $\{132, 321\} \approx \{213, 321\}$
g) $\{123, 231\}$ $\sim$ **g')** $\{123, 312\}$
h) $\{123, 321\}$

In [79] it is shown that the number of permutations in $\mathcal{S}_n$ avoiding any of the pairs in the classes **a)**, **b)**, **b')**, **c)**, **d)**, **e)**, and **e')** is 2^{n-1}, and that for the pairs in **f)**, **g)** and **g')**, the number of permutations avoiding any of them is $\binom{n}{2}+1$. The case **h)** is trivial because this pair is avoided only by permutations of length at most 4.

In terms of generating functions, this means that when we substitute $x = q = 1$ in $F_\Sigma(x, q, z)$, where Σ is any of the pairs in the classes from **a)** to **e')**, we get

$$F_\Sigma(1, 1, z) = \sum_{n\geq 0} 2^{n-1} z^n = \frac{1-z}{1-2z}.$$

If Σ is a pair from the classes **f)**, **g)**, **g')**, we get

$$F_\Sigma(1, 1, z) = \sum_{n\geq 0} (\binom{n}{2} + 1) z^n = \frac{1-2z+2z^2}{(1-z)^3}.$$

5.3.1 a) $\{123, 132\} \approx \{123, 213\}$

Proposition 5.11

$$\begin{aligned} &F_{\{123,132\}}(x, q, z) = F_{\{123,213\}}(x, q, z) \\ &\quad = \frac{1 + xz + (x^2 - 4q)z^2 + (-3xq + q + q^2)z^3 + (xq + xq^2 - 3x^2q + 3q^2)z^4}{(1 - qz^2)(1 - 4qz^2)}. \end{aligned} \tag{5.7}$$

Proof. Consider the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ described in Section 2.2. Part (4) of Proposition 2.4 says that the height of the Dyck path $\varphi(\pi)$ is the length of the longest increasing subsequence of π. In particular, $\pi \in \mathcal{S}_n(12\cdots(k+1), 132)$ if and only if $\varphi(\pi)$ has height at most k. Thus, by Lemma 5.1, $F_{\{123,132\}}(x, q, z)$ can be written in terms of Dyck paths as

$$\sum_{D\in\mathcal{D}^{\leq 2}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}. \tag{5.8}$$

Let us first find the univariate GF for paths of height at most 2 (with no statistics). Clearly, the GF for Dyck paths of height at most 1 is $\frac{1}{1-z}$, since such paths are just sequences of hills. A path D of height at most 2 can be written uniquely as $D = \mathbf{u}A_1\mathbf{d}\mathbf{u}A_2\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each A_i is a path of height at most 1. The GF for each $\mathbf{u}A_i\mathbf{d}$ is $\frac{z}{1-z}$. Thus,

$$\sum_{D\in\mathcal{D}^{\leq 2}} z^{|D|} = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = \sum_{n\geq 0} 2^{n-1}z^n.$$

In the rest of this proof, we assume that all Dyck paths that appear have height at most 2 unless otherwise stated. To compute (5.8), we will separate paths according to their height at the middle. Consider first paths whose height at the middle is 0. Splitting such a path at its midpoint we obtain a pair of paths of the same length. Thus, the corresponding GF is

$$\sum_{m\geq 0} 2^{m-1}z^m \cdot 2^{m-1}q^m z^m = \frac{1-3qz^2}{1-4qz^2}, \tag{5.9}$$

since the number of right tunnels of such a path is the semilength of its right half.

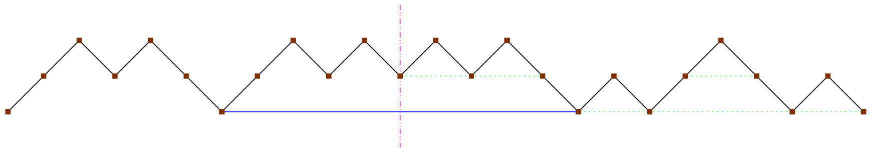


Figure 5-8: A path of height 2 with a centered tunnel.

Now we consider paths whose height at the middle is 1. It is easy to check that without the variables x and q, the GF for such paths is

$$\frac{z}{1-4z^2}. \tag{5.10}$$

Let us first look at paths of this kind that have a centered tunnel. They must be of the form $D = A\mathbf{u}B\mathbf{d}C$ where $A, C \in \mathcal{D}^{\leq 2}$ have the same length, and B is a sequence of an even number of hills. Thus, their GF is

$$xz \cdot \frac{1}{1-qz^2} \cdot \frac{1-3qz^2}{1-4qz^2}, \tag{5.11}$$

where x marks the centered tunnel, $\frac{1}{1-qz^2}$ corresponds to the sequence of hills B, half of which give right tunnels, and the last fraction comes from the pair AC, which is counted as in (5.9). From (5.10) and (5.11) it follows that the univariate GF (with just variable z) for paths with height at the middle 1, not having a centered tunnel, is

$$\frac{z}{1-4z^2} - \frac{z(1-3z^2)}{(1-z^2)(1-4z^2)} = \frac{2z^3}{(1-z^2)(1-4z^2)}.$$

By symmetry, in half of these paths, the tunnel of height 0 that goes across the middle is a right tunnel. Thus, the multivariate GF for all paths with height 1 at the middle is

$$\frac{xz(1-3qz^2)}{(1-qz^2)(1-4qz^2)} + \frac{(q+1)qz^3}{(1-qz^2)(1-4qz^2)}. \tag{5.12}$$

Here the right summand corresponds to paths with no centered tunnel: the term $(q+1)$ distinguishes whether the tunnel that goes across the middle is a right tunnel or not, and the other q's mark tunnels completely contained in the right half.

Paths with height 2 at the middle are easy to enumerate now. Indeed, they must have a peak **ud** in the middle, whose removal induces a bijection between these paths and paths with height 1 at the middle. This bijection preserves the number of right tunnels, and decreases the length and the number of centered tunnels by one. Thus, the GF for paths with height 2 at the middle is xz times expression (5.12). Adding up this GF for paths with height 2 at the middle, to the expressions (5.9) and (5.12) for paths whose height at the middle is 0 and 1 respectively, we obtain expression (5.7) for $F_{\{123,132\}}(x,q,z)$. $\square$

Let us see how the same technique used in this proof can be generalized to enumerate fixed points in $\mathcal{S}_n(132, 12\cdots(k+1))$ for an arbitrary $k \geq 0$.

Theorem 5.12 *For $k \geq 0$, let*

$$J_k(x,z) := F_{\{132,12\cdots(k+1)\}}(x,1,z) = \sum_{n\geq 0} \sum_{\pi\in\mathcal{S}_n(132,12\cdots(k+1))} x^{\mathrm{fp}(\pi)} z^n.$$

Then the J_k's satisfy the recurrence

$$J_k(x,z) = \sum_{\ell=0}^{k} I_{k,\ell}(z)(1+(x-1)zJ_{\ell-1}(x,z)), \tag{5.13}$$

where $J_{-1}(x,z) := 0$, and $I_{k,\ell}(z)$ is defined as

$$I_{k,\ell}(z) := \sum_{n\geq 0} g_{k,\ell}^2(n) z^n, \text{ where } \sum_{n\geq 0} g_{k,\ell}(n) z^n = \frac{U_\ell(\frac{1}{2z})}{zU_{k+1}(\frac{1}{2z})},$$

where U_m are the Chebyshev polynomials of the second kind, defined in Section 1.3.2.

Before proving this theorem, let us show how to apply it to obtain the GFs J_k for the first few values of k. For $k=1$, we have $I_{1,0}(z) = \frac{z}{1-z^2}$, $I_{1,1}(z) = \frac{1}{1-z^2}$, so

$$J_1(x,z) = \frac{1+xz}{1-z^2}.$$

For $k=2$, we get $I_{2,0}(z) = \frac{z^2}{1-4z^2}$, $I_{2,1}(z) = \frac{z}{1-4z^2}$, $I_{2,2}(z) = 1 + \frac{z^2}{1-4z^2}$, thus

$$J_2(x,z) = \frac{1+xz+(x^2-4)z^2+(2-3x)z^3+(3+2x-3x^2)z^4}{(1-z^2)(1-4z^2)},$$

which is the expression of Proposition 5.11 for $q=1$.

For $k=3$, we obtain $I_{3,0}(z) = \frac{z^3+z^5}{(1-z^2)(1-7z^2+z^4)}$, $I_{3,1}(z) = \frac{z^2+z^4}{(1-z^2)(1-7z^2+z^4)}$, $I_{3,2}(z) = \frac{z(1-4z^2+z^4)}{(1-z^2)(1-7z^2+z^4)}$, $I_{3,2}(z) = 1 + \frac{z^2(1-4z^2+z^4)}{(1-z^2)(1-7z^2+z^4)}$, so

$$\begin{aligned} J_3(x,z) = {} & [1+xz+(x^2-12)z^2+(x^3-11x+2)z^3+(-10x^2+4x+45)z^4+(-10x^3+4x^2+37x-10)z^5 \\ & +(25x^2-22x-52)z^6+(25x^3-22x^2-41x+16)z^7+(-12x^2+16x+16)z^8+(-12x^3+16x^2 \\ & +12x-8)z^9] \;/\; [(1-z^2)^2(1-4z^2)(1-7z^2+z^4)]. \end{aligned}$$

Proof. As mentioned in the previous proof, φ induces a bijection between $\mathcal{S}_n(132, 12\cdots(k+1))$ and $\mathcal{D}^{\leq k}$, the set of Dyck paths of height at most k. Thus, by Lemma 5.1,

$$J_k(x,z) = \sum_{D\in\mathcal{D}^{\leq k}} x^{\mathrm{ct}(D)} z^{|D|}.$$

In order to find a recursion for this GF, we are going to apply the same technique of marking tunnels that we used in Sections 2.2.1 and 2.2.2. We count pairs (D,S) where $D\in\mathcal{D}^{\leq k}$ and S is a subset of $\mathrm{CT}(D)$, the set of centered tunnels of D. In other words, we are considering Dyck paths where some centered tunnels (namely those in S) are marked. Each such pair is given weight $(x-1)^{|S|}z^{|D|}$, so that for a fixed D, the sum of weights of all pairs (D,S) is $x^{\mathrm{ct}(D)}z^{|D|}$, which is the weight that D has in $J_k(x,z)$.

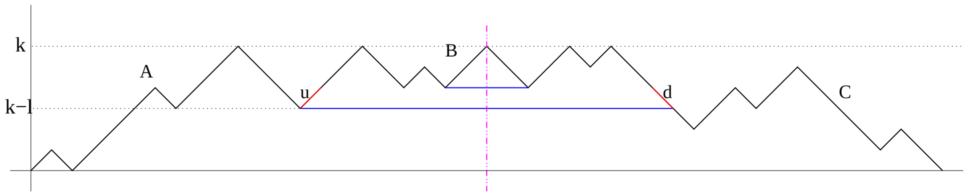


Figure 5-9: A path of height k with two marked centered tunnels.

If $D\in\mathcal{D}^{\leq k}$ has some marked centered tunnel, consider the decomposition $D = A\mathbf{u}B\mathbf{d}C$ given by the longest marked tunnel (i.e., all the other marked tunnels are inside the part B of the path). Let ℓ be the distance between this tunnel and the line $y=k$ (see Figure 5-9). Equivalently, A ends at height $k-\ell$, the same height where C begins. Then, B is an arbitrary Dyck path of height at most $\ell-1$ with possibly some marked centered tunnels, so its corresponding GF is $J_{\ell-1}(x,z)$ (with the convention $J_{-1}(x,z):=0$, since for $\ell=0$ there is no such B). Giving weight $(x-1)$ to the tunnel that determines our decomposition, we have that the part $\mathbf{u}B\mathbf{d}$ of the path contributes $(x-1)zM_{\ell-1}(x,z)$ to the GF.

Now we look at the GF for the part A of the path. Let $g_{k,\ell}(n)$ be the number of paths from $(0,0)$ to $(n,k-\ell)$ staying always between $y=0$ and $y=k$. A path of this type can be decomposed uniquely as $A = E_k\mathbf{u}E_{k-1}\mathbf{u}\cdots\mathbf{u}E_{\ell+1}\mathbf{u}E_\ell$, where each $E_i\in\mathcal{D}^{\leq i}$. The GF of Dyck paths of height at most i is

$$J_i(1,z) = \frac{U_i(\frac{1}{2\sqrt{z}})}{\sqrt{z}U_{i+1}(\frac{1}{2\sqrt{z}})},$$

as shown for example in [52]. Let $w=\sqrt{z}$, which is the weight of a single step of a path, and let $\widetilde{R}_{k,\ell}(w) := \sum_{n\geq 0} g_{k,\ell}(n)w^n$. From the above decomposition of A,

$$\widetilde{R}_{k,\ell}(w) = J_k(1,w^2)wJ_{k-1}(1,w^2)w\cdots wJ_\ell(1,w^2) = \frac{U_\ell(\frac{1}{2w})}{wU_{k+1}(\frac{1}{2w})}.$$

The part C of the path D, flipped over a vertical line, can be regarded as a path with the same endpoints as A, since it must have the same length and end at the same height $k-\ell$. Thus, the GF for pairs (A,C) of paths of the same length from height 0 to height $k-\ell$ and not going above $y=k$ is $\sum_{n\geq 0} g_{k,\ell}^2(n)z^n = I_{k,\ell}(z)$.

Hence, the GF for paths $D \in \mathcal{D}^{\leq k}$ having the longest marked centered tunnel at height $k-\ell$ is $I_{k,\ell}(z)(x-1)zM_{\ell-1}(x,z)$.

If D has no marked tunnel, decompose it as $D = AC$ where A and C have the same length. Letting $k-\ell$ be again the height where A ends and C begins, the situation is the same as above but without any contribution coming from the central part of D. The parameter ℓ can take any value between 0 and k. Thus, summing over all possible decompositions of D, we get

$$J_k(x,z) = \sum_{\ell=0}^{k} I_{k,\ell}(z)(1+(x-1)zJ_{\ell-1}(x,z)).$$

□

5.3.2 b, b') $\{231, 321\} \sim \{312, 321\}$

Proposition 5.13

$$F_{\{312,321\}}(x,q,z) = \frac{1-qz}{1-(x+q)z+(x-1)qz^2}. \tag{5.14}$$

Proof. The length of the longest decreasing subsequence of π equals the height of the Dyck path $\varphi(\bar{\pi})$. In particular, we have a bijection

$$\begin{array}{ccc} \mathcal{S}_n(312,321) & \longleftrightarrow & \mathcal{D}_n^{\leq 2} \\ \pi & \mapsto & \varphi(\bar{\pi}) \end{array}$$

Thus, by Lemma 5.1,

$$F_{\{312,321\}}(x,q,z) = \sum_{D \in \mathcal{D}^{\leq 2}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

But the only tunnels of depth 0 that a Dyck path of height at most 2 can have are hills, and the only tunnels of negative depth that it can have are peaks at height 2. A path $D \in \mathcal{D}^{\leq 2}$ can be written uniquely as $D = \mathbf{u}A_1\mathbf{d}\mathbf{u}A_2\mathbf{d}\cdots\mathbf{u}A_r\mathbf{d}$, where each A_i is a (possibly empty) sequence of hills. An empty A_i creates a tunnel of depth 0 in D, so it contributes as x. An A_i of length $2j > 0$ contributes as $q^j z^j$, since it creates j peaks at height 2 in D. Thus,

$$F_{\{312,321\}}(x,q,z) = \frac{1}{1-z\left(x+\dfrac{qz}{1-qz}\right)},$$

which is equivalent to (5.14). □

Corollary 5.14

$$F_{\{231,321\}}(x,q,z) = \frac{1-z}{1-(x+1)z+(x-q)z^2}.$$

Proof. It follows from Lemma 1.2 that $F_{\{231,321\}}(x,q,z) = F_{\{312,321\}}(x/q, 1/q, qz)$. □

As in the previous section, these results can be generalized to the case when instead of the pattern 321 we have a decreasing pattern $(k+1)k\cdots 21$ of arbitrary length. For $i, h \geq 0$, let $\mathbf{C}_i^{\leq h}$ be the number of Dyck paths of length $2i$ and height at most h. As mentioned before, it is known that

$$\sum_{i\geq 0}\mathbf{C}_i^{\leq h}z^i=\frac{U_h(\frac{1}{2\sqrt{z}})}{\sqrt{z}U_{h+1}(\frac{1}{2\sqrt{z}})},$$

where U_m are the Chebyshev polynomials of the second kind. Let

$$\mathbf{C}_{<i}^{\leq h}(z):=\sum_{j=0}^{i-1}\mathbf{C}_j^{\leq h}z^j.$$

The following theorem deals with fixed points and excedances in $\mathcal{S}_n(312,(k+1)k\cdots 1)$ for any $k\geq 0$.

Theorem 5.15 *Let* $\mathbf{C}_i^{\leq h}=|\mathcal{D}_i^{\leq h}|$ *and* $\mathbf{C}_{<i}^{\leq h}(z)$ *be defined as above. Then, for* $k\geq 0$,

$$F_{\{312,(k+1)k\cdots 21\}}(x,q,z)=A_0^k(x,q,z),$$

where A_i^k *is recursively defined by*

$$A_i^k(x,q,z)=\begin{cases}\dfrac{1}{1-z[(x-1)\mathbf{C}_i^{\leq k-i-1}q^iz^i+(q-1)\mathbf{C}_{<i}^{\leq k-i-1}(qz)+A_{i+1}^k(x,q,z)]} & \text{if } i<k,\\ 1 & \text{if } i=k.\end{cases}$$

For example, for $k=2$ we obtain Proposition 5.13, and for $k=3$ we get

$$\begin{aligned}F_{\{312,4321\}}(x,q,z)&=\frac{1}{1-z\left[x-1+\dfrac{1}{1-z\left[(x-1)qz+q-1+\frac{1}{1-qz}\right]}\right]}\\&=\frac{1-2qz+(q^2-xq)z^2+(xq^2-q^2)z^3}{1-(x+2q)z+(xq+q^2-q)z^2+(x^2q-xq)z^3+(-x^2q^2+2xq^2-q^2)z^4}.\end{aligned}$$

Proof. It is analogous to the proof of Theorem 5.9, with the only difference that here we consider only those paths that do not go above the line $y=k$. □

Making the appropriate substitutions in the statement of Theorem 5.15, we obtain an expression for the generating function $F_{\{231,(k+1)k\cdots 1\}}(x,q,z)=F_{\{312,(k+1)k\cdots 1\}}(\frac{x}{q},\frac{1}{q},qz)$.

5.3.3 c) $\{132,213\}$

Proposition 5.16

$$F_{\{132,213\}}(x,q,z)=\frac{1-(1+q)z-2qz^2+4q(1+q)z^3-(xq^2+xq+5q^2)z^4+2xq^2z^5}{(1-z)(1-xz)(1-qz)(1-4qz^2)}.$$

Proof. We use again the bijection $\varphi:\mathcal{S}_n(132)\longrightarrow\mathcal{D}_n$. From its description given in Section 2.2, it is not hard to see that a permutation $\pi\in\mathcal{S}_n(132)$ avoids 213 if and only if all the valleys of the corresponding Dyck path $\varphi(\pi)$ have their lowest point on the x-axis. A path with such property can be described equivalently as a *sequence of pyramids*. Denote by $\mathcal{P}yr_n\subseteq\mathcal{D}_n$ the set of sequences of pyramids of length $2n$, and let $\mathcal{P}yr:=\bigcup_{n\geq 0}\mathcal{P}yr_n$. We have just seen that φ restricts to a bijection between $\mathcal{S}_n(132,213)$ and $\mathcal{P}yr_n$. By Lemma 5.1, we can write $F_{\{132,213\}}(x,q,z)$ as

$$\sum_{D\in\mathcal{P}yr} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

Since for each $n \geq 1$ there is exactly one pyramid of length $2n$, the univariate GF of sequences of pyramids is just $\sum_{D\in\mathcal{P}yr} z^{|D|} = \frac{1}{1-\frac{z}{1-z}} = \frac{1-z}{1-2z} = 1 + \sum_{n\geq 1} 2^{n-1} z^n$.

Let us first consider elements of $\mathcal{P}yr$ that have height 0 in the middle (equivalently, the two central steps are **du**). Each one of their halves is a sequence of pyramids, both of the same length. They have no centered tunnels, and the number of right tunnels is given by the semilength of the right half. Thus, their multivariate GF is

$$1 + \sum_{m\geq 1} 4^{m-1} q^m z^{2m} = 1 + \frac{qz^2}{1-4qz^2}. \tag{5.15}$$

Now we count elements of $\mathcal{P}yr$ whose two central steps are **ud**. They are obtained uniquely by inserting a pyramid of arbitrary length in the middle of a path with height 0 at the middle. The tunnels created by the inserted pyramid are all centered tunnels, so the corresponding GF is

$$\frac{xz}{1-xz}\left(1 + \frac{qz^2}{1-4qz^2}\right). \tag{5.16}$$

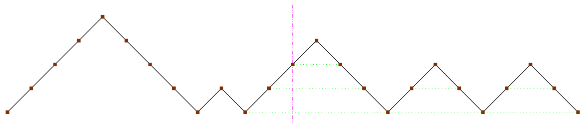


Figure 5-10: A sequence of pyramids.

It remains to count the elements of $\mathcal{P}yr$ that in the middle have neither a peak nor a valley. From a non-empty sequence of pyramids with height 0 in the middle, if we increase the size of the leftmost pyramid of the right half by an arbitrary number of steps, we obtain a sequence of pyramids whose two central steps are **uu**. Reciprocally, by this procedure every such sequence of pyramids can be obtained in a unique way from a sequence of pyramids with height 0 in the middle. Thus, the GF for the elements of $\mathcal{P}yr$ whose two central steps are **uu** is

$$\frac{qz}{1-qz} \cdot \frac{qz^2}{1-4qz^2}. \tag{5.17}$$

By symmetry, the GF for the elements of $\mathcal{P}yr$ whose two central steps are **dd** is

$$\frac{z}{1-z} \cdot \frac{qz^2}{1-4qz^2}, \tag{5.18}$$

where the difference with respect to 5.17 is that now the pyramid across the middle does not create right tunnels. Adding up (5.15), (5.16), (5.17) and (5.18) we get the desired GF. $\square$

5.3.4 d) $\{231, 312\}$

Proposition 5.17

$$F_{\{231,312\}}(x, q, z) = \frac{1 - qz^2}{1 - xz - 2qz^2}.$$

Proof. We have shown in the proof of Proposition 5.16 that φ induces a bijection between $\mathcal{S}_n(132, 213)$ and $\mathcal{P}yr_n$, the set of sequences of pyramids of length $2n$. Composing it with the complementation operation, we get a bijection $\pi \mapsto \varphi(\bar{\pi})$ between $\mathcal{S}_n(231, 312)$ and $\mathcal{P}yr_n$. Together with Lemma 5.1, this allows us to express $F_{\{231,312\}}(x, q, z)$ as

$$\sum_{D \in \mathcal{P}yr} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

All that remains is to observe how many tunnels of zero and negative depth are created by a pyramid according to its size. A pyramid of odd semilength $2m + 1$ creates one tunnel of depth 0 and m tunnels of negative depth. A pyramid of even semilength $2m$ creates only m tunnels of negative depth. Thus, we have that

$$F_{\{231,312\}}(x, q, z) = \frac{1}{1 - \dfrac{xz}{1 - qz^2} - \dfrac{qz^2}{1 - qz^2}},$$

which equals the expression above. □

5.3.5 e, e') $\{132, 231\} \approx \{213, 231\} \sim \{132, 312\} \approx \{213, 312\}$

Proposition 5.18

$$F_{\{132,231\}}(x, q, z) = F_{\{213,231\}}(x, q, z) = \frac{1 - z - qz^2 + xqz^3}{(1 - xz)(1 - z - 2qz^2)}.$$

Proof. As usual, we use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$. Now we are interested in how the condition that π avoids 231 is reflected in the Dyck path $\varphi(\pi)$. It is easy to see from the description of φ and φ^{-1} in Section 2.2 that π is 231-avoiding if and only if $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step (equivalently, all the non-isolated up-steps occur at the beginning of the path). Let $\mathcal{E}_n \subseteq \mathcal{D}_n$ be the set of Dyck paths with this condition, and let $\mathcal{E} := \bigcup_{n \geq 0} \mathcal{E}_n$. Then, φ induces a bijection between $\mathcal{S}_n(132, 231)$ and $\mathcal{E}_n$. By Lemma 5.1, $F_{\{132,231\}}(x, q, z)$ can be written as

$$\sum_{D \in \mathcal{E}} x^{\mathrm{ct}(D)} q^{\mathrm{rt}(D)} z^{|D|}.$$

If $D \in \mathcal{E}$, centered tunnels of D can appear only in the following two places. There can be a centered tunnel produced by a peak in the middle of D. All the other centered tunnels of D must have their endpoints in the initial ascending run and the final descending run of D (that is, in their corresponding decomposition $D = A\mathbf{u}B\mathbf{d}C$, A is a sequence of up-steps and C is a sequence of down-steps). For convenience we call this second kind of tunnels *bottom* tunnels. All the right tunnels of D come from peaks on the right half.

It is an exercise to check that the number of paths in $\mathcal{E}_n$ having a peak in the middle and r peaks on the right half is $\binom{n-r-1}{r} 2^{r-1}$ if $r \geq 1$, and 1 if $r = 0$. Similarly, the number

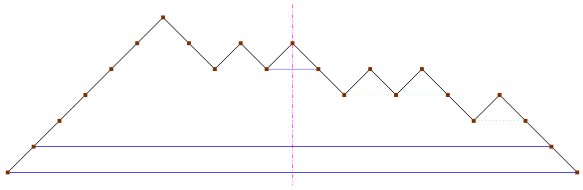


Figure 5-11: A path in $\mathcal{E}$ with a peak in the middle and two bottom tunnels.

of paths in $\mathcal{E}_n$ with no peak in the middle and r peaks on the right half is $\binom{n-r}{r}2^{r-1}$ if $r \geq 1$, and 0 if $r = 0$. Let us ignore for the moment the bottom tunnels. For peaks in the middle and right tunnels we have the following GF.

$$\begin{aligned}
Q(x,q,z) &:= \sum_{D\in\mathcal{E}\setminus\mathcal{E}_0} x^{\#\{\text{peaks in the middle of D}\}} q^{\mathrm{rt}(D)} z^{|D|} \\
&= \sum_{n\geq 1}\left[\sum_{r=1}^{\lfloor n/2\rfloor}\binom{n-r}{r}2^{r-1}q^r + x\left(1+\sum_{r=1}^{\lfloor (n-1)/2\rfloor}\binom{n-r-1}{r}2^{r-1}q^r\right)\right]z^n \\
&= \frac{xz+(q-x)z^2-xqz^3}{(1-z)(1-z-2qz^2)}. \qquad (5.19)
\end{aligned}$$

Now, to take into account all centered tunnels, we use that every $D \in \mathcal{E}$ can be written uniquely as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and $D' \in \mathcal{E}$ has no bottom tunnels. The GF for elements of $\mathcal{E}$ that do have bottom tunnels, where x marks peaks in the middle, is $xz + zQ(x,q,z)$ (the term xz is the contribution of the path $\mathbf{ud}$). Hence, the sought GF where x marks all centered tunnels is

$$F_{\{132,231\}}(x,q,z) = \frac{1}{1-xz}[1+Q(x,q,z)-xz-zQ(x,q,z)] = 1+\frac{1-z}{1-xz}Q(x,q,z),$$

which together with (5.19) implies the proposition. □

Corollary 5.19

$$F_{\{132,312\}}(x,q,z) = F_{\{213,312\}}(x,q,z) = \frac{1-qz-qz^2+xqz^3}{(1-xz)(1-qz-2qz^2)}.$$

Proof. It follows from Lemma 1.2 that $F_{\{132,312\}}(x,q,z) = F_{\{132,231\}}(x/q,1/q,qz)$. □

5.3.6 f) $\{132,321\} \approx \{213,321\}$

Proposition 5.20

$$F_{\{132,321\}}(x,q,z) = F_{\{213,321\}}(x,q,z) = \frac{1-(1+q)z+2qz^2}{(1-z)(1-xz)(1-qz)}.$$

Proof. We saw in part (6) of Proposition 2.4 that the number of peaks of the Dyck path $\varphi(\pi)$ equals the length of the longest decreasing subsequence of π. In particular, π is 321-avoiding if and only if $\varphi(\pi)$ has at most two peaks. By Lemma 5.1, $F_{\{132,321\}}(x,q,z) = \sum x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}z^{|D|}$, where the sum is over Dyck paths D with at most two peaks. Clearly, such a path can be uniquely written as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and D' is either empty or a pair of adjacent pyramids (see Figure 5-12). Therefore,

$$F_{\{132,321\}}(x,q,z) = \frac{1}{1-xz}\left(1 + \frac{z}{1-z}\cdot\frac{qz}{1-qz}\right),$$

since centered tunnels are produced by the steps outside D', and right tunnels are created by the right pyramid of D'. $\square$

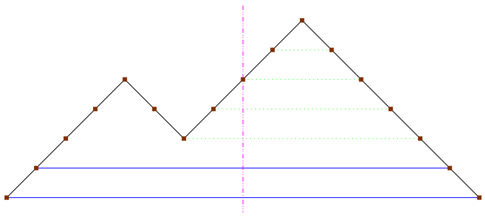


Figure 5-12: A path with two peaks.

This case can be generalized to the situation when instead of 321 we have a decreasing pattern of arbitrary length. Observe that by Lemma 1.2, $F_{\{132,(k+1)k\cdots 21\}}(x,q,z) = F_{\{213,(k+1)k\cdots 21\}}(x,q,z)$ for all k.

Theorem 5.21

$$\sum_{k\geq 0} F_{\{132,(k+1)k\cdots 21\}}(x,q,z)\,p^k = \frac{2(1+xz(p-1))}{(1-p)[1+(1+q-2x)z-qz^2(p-1)^2+\sqrt{f_1(q,z)}]},$$

where $f_1(q,z) = 1-2(1+q)z+[(1-q)^2-2q(p-1)(p+3)]z^2-2q(1+q)(p-1)^2z^3+q^2(p-1)^4z^4$.

Proof. We use again the fact from Proposition 2.4 that the number of peaks of $\varphi(\pi)$ equals the length of the longest decreasing subsequence of π. Thus, φ induces a bijection between $\mathcal{S}_n(132,(k+1)k\cdots 21)$ and the subset of $\mathcal{D}_n$ of paths with at most k peaks. This implies that $\sum_{k\geq 0} F_{\{132,(k+1)k\cdots 21\}}(x,q,z)\,p^k$ can be expressed as

$$\frac{1}{1-p}\sum_{D\in\mathcal{D}} x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}p^{\#\{\text{peaks of } D\}}z^{|D|}.$$

The result now follows from Theorem 3.10 and the expression for the generating function $\sum x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}p^{\#\{\text{peaks of } D\}}z^{|D|}$ given in its proof. $\square$

5.3.7 g, g') $\{123,231\} \sim \{123,312\}$

Proposition 5.22

$$F_{\{123,312\}}(x,q,z) = \frac{1+xz+(x^2-2q)z^2+(-x^2q+xq^2+3q^2)z^4+3q^3z^5-q^3z^6-4q^4z^7-2xq^4z^8}{(1-qz^2)^3(1-q^2z^3)}.$$

Proof. We have seen in the proof of Proposition 5.20 that φ induces a bijection between $\mathcal{S}_n(132,321)$ and the set of paths in $\mathcal{D}_n$ with at most two peaks. Composing it with the complementation operation, we get a bijection $\pi \mapsto \varphi(\bar{\pi})$ between $\mathcal{S}_n(123,312)$ and such set of Dyck paths. Using Lemma 5.1, we can write $F_{\{123,312\}}(x,q,z) = \sum x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}$, where the sum is over Dyck paths D with at most two peaks. Again, such a D can be uniquely written as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and D' is either empty or a pair of adjacent pyramids, i.e., $D' = \mathbf{u}^i \mathbf{d}^i \mathbf{u}^j \mathbf{d}^j$ with $i, j \geq 1$. The idea is to consider cases depending on the relations among i, j and k.

To enumerate Dyck paths with at most two peaks with respect to td_0 and $\mathrm{td}_{<0}$, it is important to look at where the tunnels of depth 0 and depth 1 occur. For convenience in this proof, we call such tunnels *frontier tunnels*, since they determine where tunnels of negative depth are: above them all tunnels have negative depth, and below them tunnels have positive depth. There are four possibilities according to where the frontier tunnels of D occur in the decomposition above:

(1) outside D',

(2) inside one of the pyramids of D',

(3) inside both pyramids of D',

(4) D has no frontier tunnel.

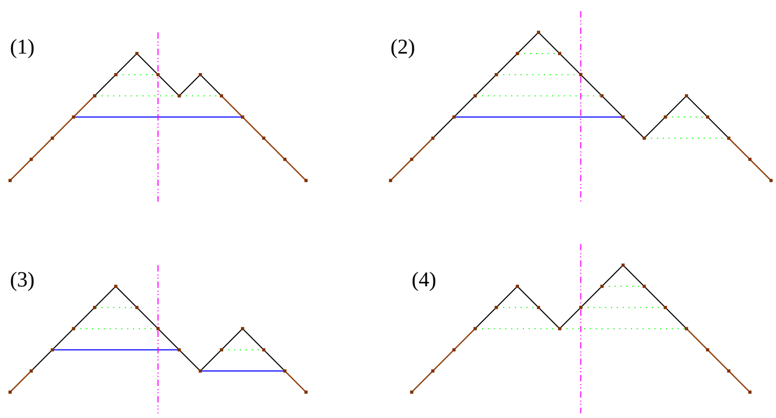


Figure 5-13: Four possible locations of the frontier tunnels.

Figure 5-13 shows an example of each of the four cases. The frontier tunnels (whose depth is 0 in this example) are drawn with a solid line, while the dotted lines are the tunnels of negative depth.

Note that in case (4) the tunnels of negative depth are exactly those in D'. We show as an example how to find the GF in case (1). In this case, the frontier tunnel T gives a decomposition $D = A\mathbf{u}B\mathbf{d}C$ where $A = \mathbf{u}^m$, $C = \mathbf{d}^m$, $m \geq 0$, and B is a Dyck path with at most two peaks, of semilength $|B| = m$ if $\mathrm{depth}(T) = 0$, and $|B| = m + 1$ if $\mathrm{depth}(T) = 1$. It follows from Proposition 5.20 that the GF for Dyck paths with at most two peaks is $\frac{1-2z+2z^2}{(1-z)^3}$. In the situation where $\mathrm{depth}(T) = 0$, we have that $|D| = 2|B| + 1$ and $\mathrm{td}_{<0}(D) = |B|$. Thus, the corresponding GF is

$$xz \cdot \frac{1 - 2qz^2 + 2q^2 z^4}{(1 - qz^2)^3}.$$

Similarly, in the situation where $\mathrm{depth}(T) = 1$, we have that $|D| = 2|B|$ and $\mathrm{td}_{<0}(D) = |B|$, thus the corresponding GF is

$$\frac{1 - 2qz^2 + 2q^2z^4}{(1 - qz^2)^3}.$$

The other cases are similar. Adding up the GFs obtained in each case, we get the desired expression for $F_{\{123,312\}}(x, q, z)$. □

Corollary 5.23

$$F_{\{123,231\}}(x, q, z) = \frac{1 + xz + (x^2 - 2q)z^2 + (-x^2q + xq + 3q^2)z^4 + 3q^2z^5 - q^3z^6 - 4q^3z^7 - 2xq^3z^8}{(1 - qz^2)^3(1 - qz^3)}.$$

Proof. By Lemma 1.2, we have that $F_{\{123,231\}}(x, q, z) = F_{\{123,312\}}(x/q, 1/q, qz)$. □

5.3.8 h) $\{123, 321\}$

Proposition 5.24

$$F_{\{123,321\}}(x, q, z) = 1 + xz + (x^2 + q)z^2 + (2xq + q^2 + q)z^3 + 4q^2z^4.$$

Proof. By a well-known result of Erdös and Szekeres, any permutation of length at least 5 contains an occurrence of either 123 or 321. This reduces the problem to counting fixed points and excedances in permutations of length at most 4, which is trivial. □

5.4 Triple restrictions

Here we consider simultaneous avoidance of any three patterns of length 3. Applying Lemma 1.2, the triplets of patterns fall into the following equivalence classes.

a) $\{123, 132, 213\}$
b) $\{231, 312, 321\}$
c) $\{123, 132, 231\} \approx \{123, 213, 231\}$ $\sim$ **c')** $\{123, 132, 312\} \approx \{123, 213, 312\}$
d) $\{132, 231, 321\} \approx \{213, 231, 321\}$ $\sim$ **d')** $\{132, 312, 321\} \approx \{213, 312, 321\}$
e) $\{132, 213, 231\}$ $\sim$ **e')** $\{132, 213, 312\}$
f) $\{132, 231, 312\} \approx \{213, 231, 312\}$
g) $\{123, 231, 312\}$
h) $\{132, 213, 321\}$
i) $\{123, 132, 321\} \approx \{123, 213, 321\}$
j) $\{123, 231, 321\}$ $\sim$ **j')** $\{123, 312, 321\}$

It is known [79] that the number of permutations in $\mathcal{S}_n$ avoiding the triplets in the classes **a)** and **b)** is the Fibonacci number F_{n+1}. The number of permutations avoiding any of the triplets in the classes **c)**, **c')**, **d)**, **d')**, **e)**, **e')**, **f)**, **g)** and **h)** is n. The cases of the triplets **i)**, **j)** and **j')** are trivial, because they are avoided only by permutations of length at most 4.

In terms of generating functions, when we substitute $x = q = 1$ in $F_\Sigma(x, q, z)$ where Σ is a triplet from one of the classes between **a)** and **g)**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} F_{n+1} z^n = \frac{1}{1-z-z^2}.$$

If Σ is any triplet from the classes between **c)** and **h)**, we get

$$F_\Sigma(1,1,z) = \sum_{n\geq 0} n z^n = \frac{1-z+z^2}{(1-z)^2}.$$

The following theorem gives all the generating functions of permutations avoiding any triplet of patterns of length 3.

Theorem 5.25 a)

$$F_{\{123,132,213\}}(x,q,z) = \frac{1+xz+(x^2-q)z^2+(-xq+q^2+q)z^3-x^2qz^4}{(1+qz^2)(1-3qz^2+q^2z^4)}$$

b)

$$F_{\{231,312,321\}}(x,q,z) = \frac{1}{1-xz-qz^2}$$

c)

$$\begin{aligned} F_{\{123,132,231\}}(x,q,z) &= F_{\{123,213,231\}}(x,q,z) \\ &= \frac{1+xz+(x^2-q)z^2+qz^3+(-x^2q+xq+q^2)z^4}{(1-qz^2)^2} \end{aligned}$$

c')

$$\begin{aligned} F_{\{123,132,312\}}(x,q,z) &= F_{\{123,213,312\}}(x,q,z) \\ &= \frac{1+xz+(x^2-q)z^2+q^2z^3+(-x^2q+xq^2+q^2)z^4}{(1-qz^2)^2} \end{aligned}$$

d)

$$F_{\{132,231,321\}}(x,q,z) = F_{\{213,231,321\}}(x,q,z) = \frac{1-z+qz^2}{(1-z)(1-xz)}$$

d')

$$F_{\{132,312,321\}}(x,q,z) = F_{\{213,312,321\}}(x,q,z) = \frac{1-qz+qz^2}{(1-xz)(1-qz)}$$

e)

$$F_{\{132,213,231\}}(x,q,z) = \frac{1-z-qz^2+2qz^3+(-x^2q+q^2-xq)z^4+(x^2q-2q^2)z^5+xq^2z^6}{(1-z)(1-xz)(1-qz^2)^2}$$

e')

$$F_{\{132,213,312\}}(x,q,z)$$
$$= \frac{1-qz-qz^2+2q^2z^3+(-x^2q-xq^2+q^2)z^4+(x^2q^2-2q^3)z^5+xq^3z^6}{(1-xz)(1-qz)(1-qz^2)^2}$$

f)

$$F_{\{132,231,312\}}(x,q,z) = F_{\{213,231,312\}}(x,q,z) = \frac{1+xqz^3}{(1-xz)(1-qz^2)}$$

g)

$$F_{\{123,231,312\}}(x,q,z) = \frac{1+xz+(x^2-q)z^2+xqz^3+q^2z^4}{(1-qz^2)^2}$$

h)

$$F_{\{132,213,321\}}(x,q,z) = \frac{1-(1+q)z+2qz^2-xqz^3}{(1-z)(1-xz)(1-qz)}$$

i)

$$F_{\{123,132,321\}}(x,q,z) = F_{\{123,213,321\}}(x,q,z) = 1+xz+(x^2+q)z^2+(xq+q^2+q)z^3+q^2z^4$$

j)

$$F_{\{123,231,321\}}(x,q,z) = 1+xz+(x^2+q)z^2+(2xq+q)z^3+q^2z^4$$

j')

$$F_{\{123,312,321\}}(x,q,z) = 1+xz+(x^2+q)z^2+(2xq+q^2)z^3+q^2z^4$$

Proof. Throughout this proof we will use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$ described in Section 2.2.

a) As in the proof of Proposition 5.11, we have that $\pi \in \mathcal{S}_n(132)$ avoids 123 if and only if the Dyck path $\varphi(\pi)$ has height at most 2. Similarly, from the proof of Proposition 5.16, π avoids 213 if and only if $\varphi(\pi)$ is a sequence of pyramids. Thus, φ induces a bijection between $\mathcal{S}_n(123,132,213)$ and $\mathcal{P}yr_n^{\leq 2} := \mathcal{P}yr^{\leq 2} \cap \mathcal{D}_n$, where $\mathcal{P}yr^{\leq 2}$ denotes the set of sequences of pyramids of height at most 2. By Lemma 5.1,

$$F_{\{123,132,213\}}(x,q,z) = \sum_{D\in\mathcal{P}yr^{\leq 2}} x^{\mathrm{ct}(D)}q^{\mathrm{rt}(D)}z^{|D|}.$$

To count centered and right tunnels, we distinguish cases according to which steps are the middle steps of D. A path in $\mathcal{P}yr^{\leq 2}$ of height 0 at the middle can be split in two elements of $\mathcal{P}yr^{\leq 2}$ of equal length, only the right one producing right tunnels. Since the number of $D \in \mathcal{P}yr_n^{\leq 2}$ is F_{n+1}, the GF for paths of height 0 at the middle is

$$\sum_{n\geq 0} F_{m+1}^2 q^m z^{2m} = \frac{1-qz^2}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

Multiplying this expression by xz (resp. by x^2z^2) we obtain the GF for paths in $\mathcal{P}yr^{\leq 2}$ having in the middle a centered pyramid of height 1 (resp. of height 2).

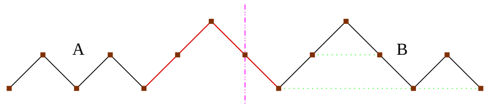


Figure 5-14: A sequence of pyramids of height at most 2.

Paths $D \in \mathcal{P}yr^{\leq 2}$ whose two middle steps are **dd** can be written as $D = A\mathbf{uudd}B$, where $A, B \in \mathcal{P}yr^{\leq 2}$ and $|B| = |A| + 1$ (see Figure 5-14). Thus, the corresponding GF is

$$\sum_{n\geq 1} F_m F_{m+1} q^m z^{2m+1} = \frac{qz^3}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

By symmetry, multiplying this expression by q we get the GF for paths whose two middle steps are **uu**.

Adding up all the cases, we get the desired GF

$$F_{\{123,132,213\}}(x,q,z) = \frac{(1+xz+x^2z^2)(1-qz^2)}{(1+qz^2)(1-3qz^2+q^2z^4)} + \frac{(q+q^2)z^3}{(1+qz^2)(1-3qz^2+q^2z^4)}.$$

b) Using the same reasoning as in **a)**, we have that $\pi \mapsto \varphi(\bar{\pi})$ induces a bijection between $\mathcal{S}_n(231,312,321)$ and $\mathcal{P}yr_n^{\leq 2}$. Now, Lemma 5.1 implies that

$$F_{\{231,312,321\}}(x,q,z) = \sum_{D\in\mathcal{P}yr^{\leq 2}} x^{\mathrm{td}_0(D)} q^{\mathrm{td}_{<0}(D)} z^{|D|}.$$

Each pyramid of height 1 produces a tunnel of depth 0, and each pyramid of height 2 creates a tunnel of negative depth. Therefore,

$$F_{\{231,312,321\}}(x,q,z) = \frac{1}{1-xz-qz^2}.$$

c) We saw in the proof of Proposition 5.18 that $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if the Dyck path $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step. Therefore, φ induces a bijection between $\mathcal{S}_n(123,132,312)$, and paths in $\mathcal{D}_n$ with the above condition and height at most 2. Such paths (except the empty one) can be expressed uniquely as $D = \mathbf{u}A\mathbf{d}B$, where A and B are sequences of hills (i.e, they have the form $(\mathbf{ud})^k$ for some $k \geq 0$). Lemma 5.1 reduces the problem to enumerating centered tunnels and right tunnels on these paths.

If B is empty, $D = \mathbf{u}A\mathbf{d}$ has a centered tunnel at height 0. The contribution of paths of this kind to our GF is $\frac{xz}{1-qz^2}$ for $|A|$ even, and $\frac{x^2z^2}{1-qz^2}$ for $|A|$ odd.

Assume now that $|A| < |B|$, so that A is within the left half of $D = \mathbf{u}A\mathbf{d}B$. If the middle of D is at height 0, then D is determined by the length of A and the number of hills in B to the left of the middle. Thus, the contribution of this subset to the GF is

$$\frac{qz^2}{(1-qz^2)^2}.$$

Multiplying this expression by xz gives the GF for paths whose midpoint is on top of a hill of B.

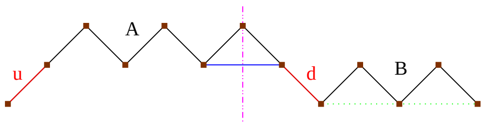


Figure 5-15: An example with $|A| = 3$ and $|B| = 2$.

It remains the case in which $|A| \geq |B| > 0$. If $|A| - |B|$ is even, the contribution of these paths to the GF is

$$z \cdot \frac{qz^2}{1-qz^2} \cdot \frac{1}{1-qz^2},$$

where the last factor counts how larger A is than B. If $|A| - |B|$ is odd, the corresponding GF is

$$z \cdot \frac{qz^2}{1-qz^2} \cdot \frac{xz}{1-qz^2},$$

since in this case there is a centered tunnel of height 1 inside A (see Figure 5-15).

Summing up all the cases, we get

$$F_{\{123,132,231\}}(x,q,z) = 1 + \frac{xz + x^2z^2}{1-qz^2} + \frac{(1+xz)qz^2}{(1-qz^2)^2} + \frac{qz^3(1+xz)}{(1-qz^2)^2}.$$

c') By Lemma 1.2, we have that $F_{\{123,132,312\}}(x,q,z) = F_{\{123,132,231\}}(x/q, 1/q, qz)$, so the formula follows from part **c)**.

d) As in the proof of Proposition 5.18, we use that $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if the Dyck path $\varphi(\pi)$ does not have any two consecutive up-steps after the first down-step. Besides, as in Proposition 5.20, $\pi \in \mathcal{S}_n(132)$ avoids 321 if and only if $\varphi(\pi)$ has at most two peaks. Thus, $\pi \in \mathcal{S}_n(132, 231, 321)$ if and only if $\varphi(\pi) \in \mathcal{D}_n$ has the form $\mathbf{u}^k B \mathbf{d}^k$, where B is either empty or is a pair of pyramids, the second of height 1. Fixed points and excedances of π are mapped to centered tunnels and right tunnels of $\varphi(\pi)$ respectively, by Lemma 5.1. Thus, $F_{\{123,132,312\}}(x,q,z)$ equals the GF enumerating centered and right tunnels in these paths.

If B is not empty, the contribution of the first pyramid is $\frac{z}{1-z}$, and the second pyramid contributes qz. Centered tunnels come from the steps outside B. Hence,

$$F_{\{123,132,312\}}(x,q,z) = \frac{1}{1-xz}\left(1 + \frac{z}{1-z} \cdot qz\right).$$

d') It follows from part **d)** and Lemma 1.2.

e) Let $\pi \in \mathcal{S}_n(132)$. We have seen that the condition that π avoids 213 translates into $\varphi(\pi)$ being a sequence of pyramids. The additional restriction of π avoiding 231 implies that all but the first pyramid of the sequence $\varphi(\pi)$ must have height 1. Thus, by Lemma 5.1, $F_{\{132,213,231\}}(x,q,z)$ can be obtained enumerating centered and right tunnels in paths of the form $D = AB$, where A is any pyramid and B is a sequence of hills.

The contribution of such paths when B is empty is just $\frac{1}{1-xz}$. Assume now that B is not empty. If $|A| > |B|$, the corresponding contribution is

$$\frac{qz^2}{1-qz^2} \cdot \frac{z}{1-z},$$

where the second factor counts how larger A is than B. It remains the case $|A| \leq |B|$, in which A is within the left half of D. If the middle of D is at height 0, then D is determined by the length of A and the number of hills in B to the left of the middle. Thus, the contribution of this subset to the GF is

$$\frac{qz^2}{(1-qz^2)^2}.$$

Multiplying this expression by xz gives the GF for paths whose midpoint is on top of a hill of B.

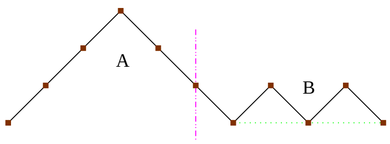


Figure 5-16: A pyramid followed by a sequence of hills.

Summing all this up, we get

$$F_{\{132,213,231\}}(x,q,z) = \frac{1}{1-xz} + \frac{qz^3}{(1-z)(1-qz^2)} + \frac{(1+xz)qz^2}{(1-qz^2)^2}.$$

e') It follows from part **e)** and Lemma 1.2.

f) Reasoning as in the proof of **e)**, we see that $\pi \mapsto \varphi(\bar{\pi})$ induces a bijection between $\mathcal{S}_n(123,231,312)$ and the subset of paths in $\mathcal{D}_n$ consisting of a pyramid followed by a sequence of hills. By Lemma 5.1, it is enough to enumerate these paths according to the statistics td_0 and $\mathrm{td}_{<0}$. If the path is nonempty, the first pyramid contributes $\frac{xz}{1-qz^2}$ if it has odd size (since then it contains a tunnel of depth 0) and $\frac{qz^2}{1-qz^2}$ if it has even size. The sequence of hills contributes $\frac{1}{1-xz}$. Therefore,

$$F_{\{132,231,312\}}(x,q,z) = 1 + \frac{xz+qz^2}{1-qz^2} \cdot \frac{1}{1-xz}.$$

g) Let $\pi \in \mathcal{S}_n(132)$. We have seen that π avoids 213 if and only if $\varphi(\pi)$ is a sequence of pyramids, and that π avoids 321 if and only if $\varphi(\pi)$ has at most two peaks. In other words, φ induces a bijection between $\mathcal{S}_n(132,213,321)$ and the subset of paths in $\mathcal{D}_n$ that are a sequence of at most two pyramids. Composing with the complementation operation, we have that $\pi \in \mathcal{S}_n(123,231,312)$ if and only if $\varphi(\bar{\pi})$ is in that subset. Now, Lemma 5.1 implies that $F_{\{123,231,312\}}$ can be obtained enumerating sequences of at most 2 pyramids according to td_0 and $\mathrm{td}_{<0}$. Each pyramid contributes $\frac{xz}{1-qz^2}$ if it has odd size and $\frac{qz^2}{1-qz^2}$ if it has even size. Thus,

$$F_{\{123,231,312\}}(x,q,z) = 1 + \frac{xz+qz^2}{1-qz^2} + \left(\frac{xz+qz^2}{1-qz^2}\right)^2.$$

h) We have shown in the proof of **g)** that $\pi \in \mathcal{S}_n(132,213,321)$ if and only if $\varphi(\pi)$ is a sequence of at most two pyramids. Using Lemma 5.1, it is enough to enumerate centered tunnels and right tunnels in such paths. The contribution of paths with exactly two pyramids is

$$\frac{z}{1-z} \cdot \frac{qz}{1-qz},$$

since only the one on the right gives right tunnels. Centered tunnels appear when there is only one pyramid. Thus we obtain

$$F_{\{132,213,321\}}(x,q,z) = \frac{1}{1-xz} + \frac{qz^2}{(1-z)(1-qz)}.$$

i, j, j') These cases are trivial because only permutations of length at most 4 can avoid 123 and 321 simultaneously. □

After having studied all the cases of double and triple restrictions, the next step is to consider restrictions of higher multiplicity. However, for $\Sigma \subseteq \mathcal{S}_3$ with $|\Sigma| \geq 4$, the sets $\mathcal{S}_n(\Sigma)$ are very easy to describe (see for example [79]), and the distribution of fixed points and excedances is trivial. In particular, in these cases we have that $|\mathcal{S}_n(\Sigma)| \in \{0, 1, 2\}$ for all n.

5.5 Pattern-avoiding involutions

Recall that $\mathcal{I}_n$ denotes the set of involutions of length n, i.e., permutations $\pi \in \mathcal{S}_n$ such that $\pi = \pi^{-1}$. In terms of the array representation of π, this condition is equivalent to $\operatorname{arr}(\pi)$ being symmetric with respect to the main diagonal. In this section we consider the distribution of the statistics fp and exc in involutions avoiding any subset of patterns of length 3.

For any $\pi \in \mathcal{S}_n$, it is clear that $\operatorname{fp}(\pi) + \operatorname{exc}(\pi) + \operatorname{exc}(\pi^{-1}) = n$ (each cross in the array of π is either on, to the right of, or to the left of the main diagonal). Thus, if $\pi \in \mathcal{I}_n$, then $\operatorname{exc}(\pi) = \frac{1}{2}(n - \operatorname{fp}(\pi))$, so the number of excedances is determined by the number of fixed points. Therefore, it is enough here to consider only the statistic 'number of fixed points' in pattern-avoiding involutions.

For any set of patterns Σ, let $\mathcal{I}_n(\Sigma) := \mathcal{I}_n \cap \mathcal{S}_n(\Sigma)$, and let $i_n^k(\Sigma) := |\{\pi \in \mathcal{I}_n(\Sigma) : \operatorname{fp}(\pi) = k\}|$. Define

$$G_\Sigma(x,z) := \sum_{n\geq 0} \sum_{\pi\in\mathcal{I}_n(\Sigma)} x^{\operatorname{fp}(\pi)} z^n.$$

By the reasoning above, $\sum_{n\geq 0} \sum_{\pi\in\mathcal{I}_n(\Sigma)} x^{\operatorname{fp}(\pi)} q^{\operatorname{exc}(\pi)} z^n = G_\Sigma(xq^{-1/2}, zq^{1/2})$.

Clearly, $\widehat{\pi}$ is an involution if and only if π is an involution. Therefore, from Lemma 1.1 we get the following.

Lemma 5.26 *Let Σ be any set of patterns. We have*

(1) $G_{\widehat{\Sigma}}(x,z) = G_\Sigma(x,z)$,

(2) $G_{\Sigma^{-1}}(x,z) = G_\Sigma(x,z)$.

The property stated in the following lemma is what allows us to apply our techniques for studying statistics on pattern-avoiding permutations to the case of involutions.

Lemma 5.27 *Let $\pi \in \mathcal{S}_n(132)$ and let $D = \varphi(\pi) \in \mathcal{D}_n$. Then,*

$$\pi \text{ is an involution} \Longleftrightarrow \varphi(\pi) \text{ is symmetric.}$$

Proof. The array of crosses representing π^{-1} is obtained from the one of π by reflection over the main diagonal. Therefore, from the description of the bijection φ given in Section 2.2, we have that $\varphi(\pi^{-1}) = D^*$. It follows that π is an involution if and only if $D = D^*$, which is equivalent to D being a symmetric Dyck path. □

5.5.1 Single restrictions

It is known [79] that for $\sigma \in \{123, 132, 213, 321\}$, $|\mathcal{I}_n(\sigma)| = \binom{n}{\lfloor n/2 \rfloor}$, and that for $\sigma \in \{231, 312\}$, $|\mathcal{I}_n(\sigma)| = 2^{n-1}$. From Lemma 5.26 it follows that for all $k \geq 0$, $i_n^k(132) = i_n^k(213)$ and $i_n^k(231) = i_n^k(312)$. By Theorem 4.6 we have that in fact $i_n^k(132) = i_n^k(321)$. Thus, for single restrictions there are three cases to consider.

Theorem 5.28 ([43, 24]) *Let $n \geq 1$, $k \geq 0$. We have*

(1) $i_n^0(123) = i_n^2(123) = \begin{cases} \binom{n-1}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$

$i_n^1(123) = \begin{cases} \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$

$i_n^k(123) = 0$ *if* $k \geq 3$.

(2) $i_n^k(132) = i_n^k(213) = i_n^k(321) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \text{if } n-k \text{ is even,} \\ 0 & \text{if } n-k \text{ is odd.} \end{cases}$

(3) $G_{231}(x,z) = G_{312}(x,z) = \dfrac{1-z^2}{1-xz-2z^2}.$

Proof. (1) Clearly a 123-avoiding permutation cannot have more than two fixed points. On the other hand, if $\pi \in \mathcal{I}_n$, we have $\mathrm{fp}(\pi) = n - 2\,\mathrm{exc}(\pi)$, which explains that $i_n^k(123) = 0$ if $n-k$ is odd. This implies that for odd n, $\mathrm{fp}(\pi) = 1$ for all $\pi \in \mathcal{I}_n$, so $i_n^1(123) = |\mathcal{I}_n(123)| = \binom{n}{\frac{n-1}{2}}$. For even n, all we have to show is that $i_n^0(123) = i_n^2(123)$.

The bijection $\psi_\lrcorner : \mathcal{S}_n(123) \longrightarrow \mathcal{D}_n$ described in Section 5.2.1 has the property that $\pi \in \mathcal{I}_n(123)$ if and only if $\psi_\lrcorner(\pi)$ is a symmetric Dyck path. If n is even, involutions $\pi \in \mathcal{I}_n$ with $\mathrm{fp}(\pi) = 2$ are mapped to symmetric Dyck paths with a peak in the middle, and those with $\mathrm{fp}(\pi) = 0$ are mapped to symmetric Dyck paths with a valley in the middle. We can establish a bijection between these two sets of Dyck paths just by changing the middle peak **ud** into a middle valley **du** (this can always be done because the height at the middle of a Dyck path of even semilength is always even, so it cannot be 1). This proves that $i_n^0(123) = i_n^2(123)$, and in particular it equals $\frac{1}{2}|\mathcal{I}_n(123)| = \binom{n-1}{\frac{n}{2}}$.

(2) We use the bijection $\varphi : \mathcal{S}_n(132) \longrightarrow \mathcal{D}_n$, which by Lemma 5.27 restricts to a bijection between $\mathcal{I}_n(132)$ and $\mathcal{D}s$. Thus, by Lemma 5.1, $G_{132}(x,z)$ can be expressed as $\sum_{D \in \mathcal{D}s} x^{\mathrm{ct}(D)} z^{|D|}$, where the sum is over all symmetric Dyck paths. But the number of centered tunnels of a symmetric Dyck path is just its height at the middle. Therefore, taking only the first half of the path, $i_n^k(132)$ counts the number of paths from $(0,0)$ to (n,k) never going below the x-axis, which equals the ballot number given in the theorem.

(3) Consider the bijection $\begin{array}{ccc} \mathcal{S}_n(312) & \longleftrightarrow & \mathcal{D}_n \\ \pi & \mapsto & \varphi(\bar{\pi}) \end{array}$. Then $\pi \in \mathcal{I}_n(312)$ if and only if $\varphi(\bar{\pi})$ is a sequence of pyramids. Together with the proof of Proposition 5.17, this implies (see also [79]) that $\mathcal{I}_n(312) = \mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$. Recall that fixed points of π are mapped to tunnels of depth 0 of $\varphi(\bar{\pi})$, which are produced by pyramids of odd size. Thus, as in Proposition 5.17,

$$G_{312}(x,z) = \frac{1}{1 - \frac{xz+z^2}{1-z^2}}.$$

□

5.5.2 Multiple restrictions

Theorem 5.29 a)

$$G_{\{123,132\}}(x,z) = G_{\{123,213\}}(x,z) = \frac{1 + xz + (x^2-1)z^2}{1-2z^2}$$

b)

$$G_{\{231,321\}}(x,z) = G_{\{312,321\}}(x,z) = \frac{1}{1-xz-z^2}$$

c)

$$G_{\{132,213\}}(x,z) = \frac{1-z^2}{(1-xz)(1-2z^2)}$$

d)

$$G_{\{231,312\}}(x,z) = \frac{1-z^2}{1-xz-2z^2}$$

e)

$$G_{\{132,231\}}(x,z) = G_{\{213,231\}}(x,z) = G_{\{132,312\}}(x,z) = G_{\{213,312\}}(x,z) = \frac{1+xz^3}{(1-xz)(1-z^2)}$$

f)

$$G_{\{132,321\}}(x,z) = G_{\{213,321\}}(x,z) = \frac{1}{(1-xz)(1-z^2)}$$

g)

$$G_{\{123,231\}}(x,z) = G_{\{123,312\}}(x,z) = \frac{1 + xz + (x^2-1)z^2 + xz^3 + z^4}{(1-z^2)^2}$$

h)

$$G_{\{123,321\}}(x,z) = 1 + xz + (x^2+1)z^2 + 2xz^3 + 2z^4$$

Proof. All the equalities between G_Σ for different Σ follow trivially from Lemma 5.26. To find expressions for these GFs, the idea is to use again the same bijections as in Section 5.3, between permutations avoiding two patterns of length 3 and certain subclasses of Dyck paths. The main difference is that here we will have to deal only with symmetric Dyck paths, as a consequence of Lemma 5.27.

a) From the proof of Proposition 5.11 and Lemma 5.27, we have that φ restricts to a bijection between $\mathcal{I}_n(123, 132)$ and symmetric Dyck paths $D \in \mathcal{D}_n$ of height at most 2. By Lemma 5.1, φ maps fixed points to centered tunnels, so all we have to do is count elements

$D \in \mathcal{D}s$ of height at most 2 according to the number of centered tunnels. Such a D can be uniquely written as $D = ABC$, where $A = C^* \in \mathcal{D}^{\leq 2}$ and B is either empty or has the form $B = \mathbf{u}B_1\mathbf{d}$, where B_1 is a sequence of hills. If $|B_1|$ is even (resp. odd), then D has one (resp. two) centered tunnels, so the contribution of B is $1+\frac{(1+xz)xz}{1-z^2}$. The contribution of A and C is $\frac{1-z^2}{1-2z^2}$. The product of these two quantities gives the expression for $G_{\{123,132\}}(x,z)$.

b) As shown above and also in [79], we have that $\mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$. Therefore, $\mathcal{I}_n(231, 321) = \mathcal{S}_n(231, 312, 321)$. This case was treated in Theorem 5.25 **b)**.

c) From the proof of Proposition 5.16 and Lemma 5.27, we have that φ gives a bijection between $\mathcal{I}_n(132, 213)$ and symmetric sequences of pyramids $D \in \mathcal{P}yr_n$, and that it maps fixed points of the permutation to centered tunnels of the Dyck path. Such a D can be written uniquely as $D = ABC$, where $A = C^* \in \mathcal{P}yr$, and B is either empty or a pyramid. The contribution of B is $\frac{1}{1-xz}$, whereas A and C contribute $\frac{1-z^2}{1-2z^2}$. Multiplying these two expressions we get a formula for $G_{\{132,213\}}(x,z)$.

d) Again, $\mathcal{I}_n(231) = \mathcal{S}_n(231, 312)$ implies that $\mathcal{I}_n(231, 312) = \mathcal{S}_n(231, 312)$, which has been considered in Proposition 5.17.

e) We have that $\mathcal{I}_n(132, 231) = \mathcal{S}_n(132, 231, 312)$, so the formula follows from Theorem 5.25 **f)**.

f) From the proof of Proposition 5.20 and Lemma 5.27, we have that φ gives a bijection between $\mathcal{I}_n(132, 321)$ and symmetric paths $D \in \mathcal{D}_n$ with at most two peaks. Counting centered tunnels in such paths is very easy, since they have the form $D = \mathbf{u}^k B\mathbf{d}^k$, where $k \geq 0$ and B is either empty or a pair of identical pyramids. The contribution of B is $\frac{1}{1-z^2}$, whereas the rest contributes $\frac{1}{1-xz}$.

g) We have that $\mathcal{I}_n(123, 231) = \mathcal{S}_n(123, 231, 312)$, so the formula follows from Theorem 5.25 **g)**.

h) It is trivial since $\mathcal{S}_n(123, 321) = \emptyset$ for $n \geq 5$. □

The case of involutions avoiding simultaneously three or more patterns of length 3 is very easy and does not involve any new idea, so we omit it here.

5.6 Expected number of fixed points

It is well known that the expected number of fixed points of a permutation $\pi \in \mathcal{S}_n$ chosen uniformly at random is 1. One can ask whether this is true for a random permutation in $\mathcal{S}_n(\sigma)$, for a given pattern σ. In the case of patterns of length 3 we can answer this question easily using the results from this chapter.

For any pattern σ, let X_n^σ the random variable on the probability space $\mathcal{S}_n(\sigma)$ that gives the number of fixed points. Let $E[X_n^\sigma]$ denote its expectation. If we have an expression for $F_\sigma(x,1,z) = \sum_{n,k\geq 0} |\{\pi \in \mathcal{S}_n(\sigma) : \mathrm{fp}(\pi) = k\}| x^k z^n$, then the derivative with respect to x gives

$$\frac{\partial F_\sigma(x,1,z)}{\partial x} = \sum_{n,k} k \cdot |\{\pi \in \mathcal{S}_n(\sigma) : \mathrm{fp}(\pi) = k\}|\, x^{k-1} z^n.$$

Evaluating at $x = 1$, we get

$$\left.\frac{\partial F_\sigma(x,1,z)}{\partial x}\right|_{x=1} = \sum_{n\geq 0}\sum_{k\geq 0} k \cdot |\{\pi \in \mathcal{S}_n(\sigma) : \mathrm{fp}(\pi) = k\}|\, z^n = \sum_{n\geq 0} E[X_n^\sigma]\, |\mathcal{S}_n(\sigma)|\, z^n. \quad (5.20)$$

5.6.1 a) 123

For the expectation of X_n^{123} we have the formula

$$E[X_n^{123}] = \frac{s_n^1(123) + 2s_n^2(123)}{\mathbf{C}_n},$$

since 123-avoiding permutations cannot have more than two fixed points. Now, applying Corollary 5.3 we get the following result.

Proposition 5.30 *For all $n \geq 1$,*

$$E[X_n^{123}] = \begin{cases} \dfrac{2\,\mathbf{C}_{n-1}}{\mathbf{C}_n} = \dfrac{n+1}{2n-1} & \text{if } n \text{ is even,} \\ \dfrac{2\,\mathbf{C}_{n-1} - \mathbf{C}^2_{\frac{n-1}{2}}}{\mathbf{C}_n} & \text{if } n \text{ is odd.} \end{cases}$$

In both cases, the expectation is asymptotically $\frac{1}{2}$, which agrees with the intuitive fact that the condition of avoiding 123 restrains the number of fixed points of a permutation.

5.6.2 b) $132 \approx 213 \approx 321$

We clearly have that $E[X_n^{132}] = E[X_n^{213}] = E[X_n^{321}]$, since the distribution of fixed points is the same in permutations avoiding each one of the three patterns. Theorem 5.6 gives the GF $F_{132}(x, 1, z) = \frac{2}{1+2(1-x)z+\sqrt{1-4z}}$, from where we get

$$\left.\frac{\partial F_{132}(x,1,z)}{\partial x}\right|_{x=1} = \frac{1-2z-\sqrt{1-4z}}{2z} = \mathbf{C}(z) - 1 = \sum_{n\geq 1} \mathbf{C}_n z^n.$$

So, by (5.20), we get the following proposition.

Proposition 5.31 *For all $n \geq 1$,*

$$E[X_n^{132}] = E[X_n^{213}] = E[X_n^{321}] = 1.$$

This shows the a priori surprising phenomenon that the expected number of fixed points of a random permutation is not affected by the condition that it avoids one of the patterns 132, 213 or 321. This contrasts with the fact that, given $1 \leq i \leq n$, whereas for a random $\pi \in \mathcal{S}_n$ the probability that i is a fixed point of π is $\frac{1}{n}$ and does not depend on i, for a random $\pi \in \mathcal{S}_n(321)$ the probability that i is a fixed point of π is $\frac{\mathbf{C}_{i-1}\mathbf{C}_{n-i}}{\mathbf{C}_n}$, since $\pi_i = i$ if and only if $\pi_1\pi_2\cdots\pi_{i-1} \in \mathcal{S}_{i-1}(321)$ and $(\pi_{i+1}-i)(\pi_{i+2}-i)\cdots(\pi_n-i) \in \mathcal{S}_{n-i}(321)$. Note that in particular this gives a more direct way to compute $E[X_n^{321}]$: for $1 \leq i \leq n$, let $X_{n,i}^{321}$ be the indicator random variable that equals 1 if i is a fixed point of $\pi \in \mathcal{S}_n(321)$ and 0 otherwise. Clearly, $X_n^{321} = \sum_i X_{n,i}^{321}$, and

$$E[X_n^{321}] = \sum_{i=1}^{n} E[X_{n,i}^{321}] = \sum_{i=1}^{n} \frac{\mathbf{C}_{i-1}\mathbf{C}_{n-i}}{\mathbf{C}_n} = 1. \tag{5.21}$$

One can also ask what proportion of 132-avoiding permutations do not have fixed points. For permutations in $\mathcal{S}_n$, it is known (see for example [85]) that the proportion of derange-

ments is
$$\frac{\#\{\pi \in \mathcal{S}_n : \mathrm{fp}(\pi)=0\}}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!},$$
which tends to e^{-1} as n goes to infinity. We showed that the number of 132-avoiding derangements is $s_n^0(132) = s_n^0(213) = s_n^0(321) = \mathbf{F}_n$, the n-th Fine number. Therefore, the proportion of them in $\mathcal{S}_n(132)$ is $\frac{\mathbf{F}_n}{\mathbf{C}_n}$, which tends to the constant $\frac{4}{9}$ as n goes to infinity.

5.6.3 c, c') $231 \sim 312$

Clearly $E[X_n^{231}] = E[X_n^{312}]$. The generating function $F_{312}(x,1,z)$ was computed in Theorem 5.8. An alternative way of expressing it is as $F_{312}(x,1,z) = A_0(x,z)$ where A_i is defined by the recurrence
$$A_h(x,z) = \frac{1}{1 - z[(x-1)\mathbf{C}_h z^h + A_{h+1}(x,z)]}$$
for all $h \geq 0$. Taking partial derivatives, we get
$$\frac{\partial A_h(x,z)}{\partial x} = \frac{z\left[\mathbf{C}_h z^h + \dfrac{\partial A_{h+1}(x,z)}{\partial x}\right]}{(1 - z[(x-1)\mathbf{C}_h z^h + A_{h+1}(x,z)])^2}.$$
Let $B_i(z) = \left.\frac{\partial A_i(x,z)}{\partial x}\right|_{x=1}$. Evaluating at $x=1$, we obtain the following recurrence for the B_i:
$$B_h(z) = \frac{z(\mathbf{C}_h z^h + B_{h+1}(z))}{(1 - zA_{h+1}(1,z))^2} = z\mathbf{C}(z)^2(\mathbf{C}_h z^h + B_{h+1}(z)),$$
where the last equality follows from the fact that $A_h(1,z) = \frac{1}{1-zA_{h+1}(1,z)}$ for all h, and therefore $A_h(1,z) = \mathbf{C}(z) = \frac{1}{1-z\mathbf{C}(z)}$. Denoting $K := z\mathbf{C}(z)^2 = \mathbf{C}(z) - 1$, we can write the recurrence as $B_h(z) = K(\mathbf{C}_h z^h + B_{h+1}(z))$. Expanding it for $B_0(z) = \left.\frac{\partial F_{312}(x,1,z)}{\partial x}\right|_{x=1}$ we get
$$\begin{aligned} B_0(z) &= K(1 + K(z + K(2z^2 + K(5z^3 + K(\cdots(\mathbf{C}_h z^h + K(\ \cdots\ ))\cdots))))) \\ &= K + K^2 z + K^3 2z^2 + K^4 5z^3 + \cdots + K^{h+1}\mathbf{C}_h z^h + \cdots \\ &= K\sum_{h\geq 0}\mathbf{C}_h (zK)^h = K\mathbf{C}(zK) = (\mathbf{C}(z)-1)\,\mathbf{C}(z(\mathbf{C}(z)-1)). \end{aligned} \tag{5.22}$$

This proves the following proposition.

Proposition 5.32
$$\sum_{n\geq 1} E[X_n^{312}]\mathbf{C}_n z^n = \frac{1 - \sqrt{-1 + 4z + 2\sqrt{1-4z}}}{2z}.$$

In particular, it follows from expression (5.22) that $E[X_n^{312}] > 1$ for $n \geq 3$.

5.6.4 Other cases

From the GFs in Sections 5.3 and 5.4, one can easily compute in the same way the expectation of the random variables X_n^{Σ} for the number of fixed points in a random permutation in $\mathcal{S}_n(\Sigma)$.

It is interesting to observe that whereas $E[X_n^{132}] = E[X_n^{321}] = 1$, we have that

$$E[X_n^{\{132,321\}}] = \frac{\binom{n}{3}+n}{\binom{n}{2}+1} \neq 1.$$

In fact, the only subsets $\Sigma \subset \mathcal{S}_3$ of two or three patterns for which $E[X_n^{\Sigma}] = 1$ for all $n \geq 1$ are $\Sigma = \{123, 231, 312\}$ and $\Sigma = \{132, 213, 321\}$.

A natural question to ask is whether there exist patterns $\sigma \in \mathcal{S}_k$ with $k \geq 4$ for which $E[X_n^{\sigma}] = 1$ for all $n \geq 1$. We have checked that no such patterns exist for $k = 4, 5$. Note that the condition $E[X_k^{\sigma}] = 1$, together with the fact that $\mathcal{S}_k(\sigma) = \mathcal{S}_k \setminus \{\sigma\}$, forces σ to have exactly one fixed point.

5.7 Final remarks and possible extensions

Looking at the results of this chapter, one can observe that the GFs $F_{\Sigma}(x, q, z)$ that we have obtained for $\Sigma \subseteq \mathcal{S}_3$ are all rational functions when $|\Sigma| \geq 2$. This is in contrast with the fact that they are not rational when $|\Sigma| = 1$, since in that case $F_{\Sigma}(1, 1, z) = \frac{1-\sqrt{1-4z}}{2z} = \mathbf{C}(z)$. For the case of involutions, all the GFs $G_{\Sigma}(x, z)$ for $\Sigma \subseteq \mathcal{S}_3$ are rational except when $\Sigma \in \{\{123\}, \{132\}, \{213\}, \{321\}\}$.

Regarding possible extensions of this work, one could try to find a generating function for fixed points and excedances in 123-avoiding permutations, the only case of patterns of length 3 that remains unsolved. Even for the enumeration of fixed points in these permutations, we expect that a simpler expression than the one in Theorem 5.4 can be given.

Another interesting extension would be to study the distribution of statistics in permutations avoiding longer patterns. The enumeration of such permutations is itself a very difficult problem, and not even the case of length 4 is completely solved (see [11, 38, 40, 64] for single patterns, [13, 53, 91, 92] for pairs of patterns of length 4, and [19, 52, 59, 60, 61] for other pairs). For the case of patterns of length 4, we have checked by computer that the only cases in which the number of derangements in $\mathcal{S}_n(\sigma)$ is the same for different patterns $\sigma \in \mathcal{S}_4$ are those in which there exists a trivial bijection (such as $\pi \mapsto \pi^{-1}$ or $\pi \mapsto \widehat{\pi}$) proving this fact. Therefore, Theorems 1.4 and 2.1 do not seem to have an analogue for patterns of length 4. Still, there would be some interest in finding generating functions to enumerate permutations avoiding patterns of length 4 or more with respect to statistics such as the number of fixed points and the number of excedances. For permutations avoiding a pattern of length 4 there are 13 different equivalence classes with respect to the distribution of the statistic fp.

It is possible that Theorem 2.1 admits generalizations to other permutation statistics, or some variations. We have not succeeded in finding any other case of equidistribution of a statistic for different patterns having such an interesting and nontrivial proof. An example of a much simpler result is that the statistic number of descents has the same distribution in $\mathcal{S}_n(132)$, $\mathcal{S}_n(213)$, $\mathcal{S}_n(231)$ and $\mathcal{S}_n(312)$.

5.7.1 Cycle structure

Another further direction of research would consist in describing the cycle structure of pattern-avoiding permutations. Using the same bijective techniques as in Section 5.3, we can easily derive generating functions for the augmented cycle index of permutations in $\mathcal{S}_n(231,312)$, in $\mathcal{S}_n(231,321)$ and in $\mathcal{S}_n(132,321)$. However, it is not clear whether for permutations avoiding other subsets of patterns of length 3, the distribution of the cycle type has a simple description.

For $\pi \in \mathcal{S}_n$, define $Z(\pi) = t_1^{c_1} t_2^{c_2} \cdots t_n^{c_n}$, where c_i is the number of cycles of length i of π (in particular, $\sum ic_i = n$). For any subset $A \in \mathcal{S}_n$, its *augmented cycle index* is defined as

$$\widetilde{Z}(A) = \sum_{\pi \in A} Z(\pi).$$

In some cases we can give GFs for the cycle index.

Proposition 5.33

$$\sum_{n\geq 0} \widetilde{Z}(\mathcal{S}_n(231,312))z^n = \frac{1-t_2z^2}{1-t_1z-2t_2z^2},$$

$$\sum_{n\geq 0} \widetilde{Z}(\mathcal{S}_n(231,321))z^n = \frac{1}{1-\sum_{i\geq 1} t_iz^i},$$

$$\sum_{n\geq 0} \widetilde{Z}(\mathcal{S}_n(132,321))z^n = \frac{1}{1-t_1z}\left[1+\sum_{i,j\geq 1} t_{\frac{i+j}{\gcd(i,j)}}^{\gcd(i,j)} z^{i+j}\right].$$

Proof. In the proof of the third part of Theorem 5.28 we saw that all the permutations in $\mathcal{S}_n(231,312)$ are involutions. In Proposition 5.17 we gave a bijection between $\mathcal{S}_n(231,312)$ and sequences of pyramids. Each pyramid contributes $\frac{t_1z+t_2z^2}{1-t_2z^2}$ to the GF for the augmented cycle index, so the first part follows.

In the case of $\mathcal{S}_n(231,321)$ and $\mathcal{S}_n(312,321)$, we saw in Proposition 5.13 that these sets are in bijection with Dyck paths of height at most 2. Such paths can be written as sequences of paths of the form $\mathbf{u}A\mathbf{d}$, where A is a (possibly empty) sequence of hills. Each part $\mathbf{u}A\mathbf{d}$ contributes to the cycle index of the permutation as t_iz^i, where $i-1$ is the semilength of A. This gives the second formula.

For the last part of the proposition, recall the bijection given in Proposition 5.20 between $\mathcal{S}_n(132,321)$ and Dyck paths D with at most two peaks. These can be uniquely written as $D = \mathbf{u}^k D' \mathbf{d}^k$, where $k \geq 0$ and D' is either empty or a pair of adjacent pyramids. The part outside D' corresponds to k fixed points of the permutation, hence the term $\frac{1}{1-t_1z}$ in the GF. Let us assume now that $k = 0$ and that D' is not empty, and let i and j be the semilenghts of the two pyramids in D'. Then, the permutation π such that $\varphi(\pi) = D$ can be easily described as

$$\pi_m = \begin{cases} m+i & \text{if } m \leq j, \\ m-j & \text{if } m > j. \end{cases}$$

It follows that all the cycles of π have length $\frac{i+j}{\gcd(i,j)}$, so we get the contribution $t_{\frac{i+j}{\gcd(i,j)}}^{\gcd(i,j)} z^{i+j}$ to the GF. □

One might wonder if the fact that the number of fixed points has the same distribution in both 321- and in 132-avoiding permutations admits a generalization concerning the cycle structure in $\mathcal{S}_n(321)$ and $\mathcal{S}_n(132)$. We have for example that $\widetilde{Z}(\mathcal{I}_n(321)) = \widetilde{Z}(\mathcal{I}_n(132))$ for all n. However, it is not true that $\widetilde{Z}(\mathcal{S}_6(321)) = \widetilde{Z}(\mathcal{S}_6(132))$, as shown in [24].

Chapter 6

Motzkin permutations

In this chapter we study a very special case of pattern-avoiding permutations, which we call *Motzkin permutations*. This name is motivated by the fact that they are enumerated by the Motzkin numbers. A permutation π is a Motzkin permutation if it avoids 132 and there do not exist indices $a < b$ such that $\pi_a < \pi_b < \pi_{b+1}$. This definition falls inside a more general notion of pattern avoidance, which is described in Section 6.1.1. We study the distribution of several statistics on Motzkin permutations, including the length of the longest increasing and decreasing subsequences and the number of ascents. Much of the work in this chapter was suggested by Toufik Mansour.

We start by introducing some preliminaries. In Section 6.2 we exhibit a bijection between the set of Motzkin permutations and the set of Motzkin paths. Then we use it to obtain generating functions of Motzkin permutations with respect to the length of the longest decreasing and increasing subsequences together with the number of ascents. The chapter ends with another application of the bijection, to the enumeration of fixed points in permutations avoiding simultaneously 231 and 32-1.

6.1 Preliminaries

6.1.1 Generalized patterns

In [4], Babson and Steingrímsson introduced the notion of *generalized patterns*, which allows the requirement that two adjacent letters in a pattern must be adjacent in the permutation. A generalized pattern is written as a sequence where two adjacent elements may or may not be separated by a dash. In this context, we write a classical pattern with dashes between any two adjacent letters of the pattern (for example, 1423 as 1-4-2-3). If we omit the dash between two letters, we mean that for it to be an occurrence in a permutation π, the corresponding elements of π have to be adjacent. For example, in an occurrence of the pattern 12-3-4 in a permutation π, the entries in π that correspond to 1 and 2 are adjacent. The permutation $\pi = 3542617$ has only one occurrence of the pattern 12-3-4, namely the subsequence 3567, whereas π has two occurrences of the pattern 1-2-3-4, namely the subsequences 3567 and 3467.

If σ is a generalized pattern, $\mathcal{S}_n(\sigma)$ denotes the set of permutations in $\mathcal{S}_n$ that have no occurrences of σ in the sense described above. Here is a remark about notation. Throughout this chapter, a pattern represented with no dashes will always denote a classical pattern (i.e., with no requirement about elements being consecutive) unless otherwise stated, following the same notation used in the previous chapters. All the generalized patterns that we will consider here will have at least one dash.

6.1.2 Motzkin paths

A *Motzkin path* of length n is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(n,0)$ consisting of up-steps $(1,1)$, down-steps $(1,-1)$, and horizontal steps $(1,0)$ which never goes below the x-axis. Denote by $\mathcal{M}_n$ the set of Motzkin paths with n steps, and let $\mathcal{M} = \bigcup_{n\geq 0} \mathcal{M}_n$. We will write $|M| = n$ if $M \in \mathcal{M}_n$. Sometimes it will be convenient to encode each up-step by a letter $\mathbf{u}$, each down-step by $\mathbf{d}$, and each horizontal step by $\mathbf{h}$. Denote by $\mathbf{M}_n = |\mathcal{M}_n|$ the n-th Motzkin number. The generating function for these numbers is $\mathbf{M}(z) = \sum_{n\geq 0} \mathbf{M}_n z^n = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$.

The definition of tunnel of a Dyck path can be generalized naturally to Motzkin paths. A *tunnel* of $M \in \mathcal{M}$ is a horizontal segment between two lattice points of the path that intersects M only in these two points, and stays always below M. Tunnels are in one-to-one correspondence with decompositions of the path as $M = X\mathbf{u}Y\mathbf{d}Z$, where $Y \in \mathcal{M}$. In the decomposition, the tunnel is the segment that goes from the beginning of the $\mathbf{u}$ to the end of the $\mathbf{d}$.

Define a *Motzkin permutation* π to be a 132-avoiding permutation in which there do not exist indices $a < b$ such that $\pi_a < \pi_b < \pi_{b+1}$. In the context of generalized patterns, the second condition is just saying that π avoids 1-23. Denote by $\mathfrak{M}_n = \mathcal{S}_n(132, 1\text{-}23)$ the set of all Motzkin permutations in $\mathcal{S}_n$. For example, there are exactly 4 Motzkin permutations of length 3, namely, $\mathfrak{M}_3 = \{213, 231, 312, 321\}$. The main reason for the term "Motzkin permutation" is that $|\mathfrak{M}_n| = \mathbf{M}_n$, as we will see in Section 6.2. For any pattern σ, denote by $\mathfrak{M}_n(\sigma)$ the set of Motzkin permutations of length n that avoid σ.

It follows from the definition that the set $\mathfrak{M}_n$ is the same as the set of 132-avoiding permutations $\pi \in \mathcal{S}_n$ where there is no a such that $\pi_a < \pi_{a+1} < \pi_{a+2}$. Indeed, assume that $\pi \in \mathcal{S}_n(132)$ has an occurrence of 1-23, say $\pi_a < \pi_b < \pi_{b+1}$ with $a < b$. Now, if $\pi_{b-1} > \pi_b$, then π would have an occurrence of 132, namely $\pi_a\pi_{b-1}\pi_{b+1}$. Therefore, $\pi_{b-1} < \pi_b < \pi_{b+1}$, so π has three consecutive increasing elements.

6.2 The bijection Υ

In this section we establish a bijection between Motzkin permutations and Motzkin paths. This allows us to study the distribution of certain statistics on the set of Motzkin permutations.

6.2.1 Definition of Υ

The construction of the bijection $\Upsilon : \mathfrak{M}_n \longrightarrow \mathcal{M}_n$ is done in two parts. The first step is again the bijection φ from $\mathcal{S}_n(132)$ to $\mathcal{D}_n$ given in Section 2.2. One can see that $\pi \in \mathcal{S}_n(132)$ avoids 1-23 if and only if the Dyck path $\varphi(\pi)$ does not contain three consecutive up-steps (a *triple rise*). Indeed, assume that $\varphi(\pi)$ has three consecutive up-steps. Then, the path from the lower-left corner to the upper-right corner of $\mathrm{arr}(\pi)$ used to define $\varphi(\pi)$ has three consecutive north steps. The crosses in the corresponding three rows give three consecutive increasing elements in π (this follows from the description of φ^{-1}), and hence an occurrence of 1-23.

Reciprocally, assume now that π has an occurrence of 1-23. The path from the lower-left to the upper-right corner of $\mathrm{arr}(\pi)$ must have two consecutive north steps in the rows of the crosses corresponding to '2' and '3'. But if $\varphi(\pi)$ has no triple rise, the next step of this path must be an east step, and the cross corresponding to '2' must be right below it. But then all the crosses above this cross are to the right of it, which contradicts the fact that this was an occurrence of 1-23.

Denote by $\mathcal{T}_n$ the set of Dyck paths of length $2n$ with no triple rise. We have given a bijection between $\mathfrak{M}_n$ and $\mathcal{T}_n$. The second step is to exhibit a bijection between $\mathcal{T}_n$ and $\mathcal{M}_n$, so that Υ will be defined as the composition of the two bijections. Given $D \in \mathcal{T}_n$, divide it in n blocks, splitting after each down-step. Since D has no triple rises, each block is of one of these three forms: **uud**, **ud**, **d**. From left to right, transform the blocks according to the rule

$$\begin{array}{rcl} \mathbf{uud} & \to & \mathbf{u}, \\ \mathbf{ud} & \to & \mathbf{h}, \\ \mathbf{d} & \to & \mathbf{d}. \end{array} \tag{6.1}$$

We obtain a Motzkin path of length n. This step is clearly a bijection.

Up to reflection of the Motzkin path over a vertical line, Υ is essentially the same bijection that was given by Claesson [20] between $\mathfrak{M}_n$ and $\mathcal{M}_n$, using a recursive definition.

6.2.2 Statistics on $\mathfrak{M}_n$

Here we show how the bijection Υ can be applied to give generating functions for some statistics on Motzkin permutations. The following lemma follows from the definition of Υ and from Proposition 2.4. Recall from Chapter 1 that lds, asc and dr denote the length of the longest decreasing subsequence, the number of ascents, and the number of double rises respectively.

Lemma 6.1 *Let $\pi \in \mathfrak{M}_n$, let $D = \varphi(\pi) \in \mathcal{D}_n$, and let $M = \Upsilon(\pi) \in \mathcal{M}_n$. We have*

(1) $\mathrm{lds}(\pi) = \#\{\textit{peaks of } D\} = \#\{\textit{steps } \mathbf{u} \textit{ in } M\} + \#\{\textit{steps } \mathbf{h} \textit{ in } M\}$,

(2) $\mathrm{lis}(\pi) = \textit{height of } D = \textit{height of } M + 1$,

(3) $\mathrm{asc}(\pi) = \mathrm{dr}(D) = \#\{\textit{steps } \mathbf{u} \textit{ in } M\}$.

Theorem 6.2 *The generating function for Motzkin permutations with respect to the length of the longest decreasing subsequence and to the number of ascents is*

$$A(v,y,z) := \sum_{n\geq 0} \sum_{\pi \in \mathfrak{M}_n} v^{\mathrm{lds}(\pi)} y^{\mathrm{asc}(\pi)} z^n = \frac{1 - vz - \sqrt{1 - 2vz + (v^2 - 4vy)z^2}}{2vyz^2}.$$

Moreover,

$$A(v,y,z) = \sum_{n\geq 0} \sum_{m\geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{m+2n}{2n} z^{m+2n} v^{m+n} y^n.$$

Proof. By Lemma 6.1, we can express A as

$$A(v,y,z) = \sum_{M \in \mathcal{M}} v^{\#\{\text{steps } \mathbf{u} \text{ in } M\} + \#\{\text{steps } \mathbf{h} \text{ in } M\}} y^{\#\{\text{steps } \mathbf{u} \text{ in } M\}} z^{|M|}.$$

Using the standard decomposition of Motzkin paths, we obtain the following equation for the generating function A.

$$A(v,y,z) = 1 + vzA(v,y,z) + vyz^2A^2(v,y,z). \tag{6.2}$$

Indeed, any nonempty $M \in \mathcal{M}$ can be written uniquely as either $M = \mathbf{h}M_1$ or $M = \mathbf{u}M_2\mathbf{d}M_3$, where M_1, M_2, M_3 are arbitrary Motzkin paths. In the first case, the number of horizontal steps of $\mathbf{h}M_1$ is one more than in M_1, the number of up-steps is the same, and $|\mathbf{h}M_1| = |M_1|+1$, so we get the term $vzA(v,y,z)$. Similarly, the second case gives the term $vyz^2A^2(v,y,z)$. Solving equation (6.2) we get the desired expression. $\square$

Theorem 6.3 *For $k > 0$, let*

$$B_k(v,y,z) = \sum_{n\geq 0} \sum_{\pi\in\mathfrak{M}_n(12\ldots(k+1))} v^{\mathrm{lds}(\pi)} y^{\mathrm{asc}(\pi)} z^n$$

be the generating function for Motzkin permutations avoiding $12\ldots(k+1)$ with respect to the length of the longest decreasing subsequence and to the number of ascents. Then we have the recurrence

$$B_k(v,y,z) = \frac{1}{1 - vz - vyz^2B_{k-1}(v,y,z)},$$

with $B_1(v,y,z) = \frac{1}{1-vz}$. Thus, B_k can be expressed as

$$B_k(v,y,z) = \cfrac{1}{1 - vz - \cfrac{vyz^2}{\cfrac{\ddots}{1 - vz - \cfrac{vyz^2}{1-vz}}}},$$

where the fraction has k levels, or in terms of Chebyshev polynomials of the second kind, as

$$B_k(v,y,z) = \frac{U_{k-1}\left(\frac{1-vz}{2z\sqrt{vy}}\right)}{z\sqrt{vy}U_k\left(\frac{1-vz}{2z\sqrt{vy}}\right)}.$$

Proof. The condition that π avoids $12\ldots(k+1)$ is equivalent to the condition $\mathrm{lis}(\pi) \leq k$. By Lemma 6.1, permutations in $\mathfrak{M}_n$ whose longest increasing subsequence has length at most k are mapped by Υ to Motzkin paths of height strictly less than k. Thus, we can express B_k as

$$B_k(v,y,z) = \sum_{M\in\mathcal{M} \text{ of height}<k} v^{\#\{\text{steps } \mathbf{u} \text{ in } M\}+\#\{\text{steps } \mathbf{h} \text{ in } M\}} y^{\#\{\text{steps } \mathbf{u} \text{ in } M\}} z^{|M|}.$$

For $k > 1$, we use again the standard decomposition of Motzkin paths. In the first of the above cases, the height of $\mathbf{h}M_1$ is the same as the height of M_1. However, in the second case, in order for the height of $\mathbf{u}M_2\mathbf{d}M_3$ to be less than k, the height of M_2 has to be less than $k-1$. So we obtain the equation

$$B_k(v,y,z) = 1 + vzB_k(v,y,z) + vyz^2B_{k-1}(v,y,z)B_k(v,y,z).$$

For $k = 1$, the path can have only horizontal steps, so we get $B_1(v, y, z) = \frac{1}{1-vz}$. Now, using the above recurrence and equation (1.2) we get the desired result. $\square$

6.3 Fixed points in the reversal of Motzkin permutations

Here we show another application of Υ. A slight modification of it will allow us to enumerate fixed points in another class of pattern-avoiding permutations closely related to Motzkin permutations. Recall that $\pi^R = \pi_n \ldots \pi_2\pi_1$ denotes the reversal of $\pi \in \mathcal{S}_n$. Let $\mathfrak{M}_n^R := \{\pi \in \mathcal{S}_n : \pi^R \in \mathfrak{M}_n\}$. In terms of pattern avoidance, $\mathfrak{M}_n^R = \mathcal{S}_n(231, 32\text{-}1)$, i.e., it is the set of 231-avoiding permutations $\pi \in \mathcal{S}_n$ where there do not exist $a < b$ such that $\pi_{a-1} > \pi_a > \pi_b$.

Theorem 6.4 *The generating function* $\sum_{n\geq 0}\sum_{\pi\in\mathfrak{M}_n^R} x^{\mathrm{fp}(\pi)}z^n$ *for permutations avoiding simultaneously* 231 *and* 32-1 *with respect to the number of fixed points is*

$$\cfrac{1}{1 - xz - \cfrac{z^2}{1 - z - \mathbf{M}_0(x-1)z^2 - \cfrac{z^2}{1 - z - \mathbf{M}_1(x-1)z^3 - \cfrac{z^2}{1-z-\mathbf{M}_2(x-1)z^4 - \cfrac{z^2}{\ddots}}}}}, \qquad (6.3)$$

where after the second level, the coefficient of $(x-1)z^{n+2}$ *is the Motzkin number* $\mathbf{M}_n$.

Proof. We have the following composition of bijections:

$$\begin{array}{ccccccc} \mathfrak{M}_n^R & \longleftrightarrow & \mathfrak{M}_n & \longleftrightarrow & \mathcal{T}_n & \longleftrightarrow & \mathcal{M}_n \\ \pi & \mapsto & \pi^R & \mapsto & \varphi(\pi^R) & \mapsto & \Upsilon(\pi^R) \end{array}$$

The idea of the proof is to look at how the fixed points of π are transformed by each of these bijections.

Fixed points of π are mapped by the reversal operation to elements j such that $\pi_j^R = n+1-j$, which in the array of π^R correspond to crosses on the secondary diagonal. Each cross in this array naturally corresponds to a tunnel of the Dyck path $\varphi(\pi^R)$, namely the one determined by the north step in the same row as the cross and the east step in the same column as the cross. The reasoning given in Section 5.1 to prove Lemma 5.1 shows that crosses on the secondary diagonal correspond to tunnels of depth 0 in the Dyck path (i.e., tunnels T satisfying the condition $\mathrm{height}(T) + 1 = \frac{1}{2}\mathrm{length}(T)$).

The next step is to see how these tunnels are transformed by the bijection from $\mathcal{T}_n$ to $\mathcal{M}_n$. Tunnels of height 0 and length 2 in the Dyck path $D := \varphi(\pi^R)$ are just hills **ud** on the x-axis. By the rule (6.1) they are mapped to horizontal steps at height 0 in the Motzkin path $M := \Upsilon(\pi^R)$. Assume now that $k \geq 1$. A tunnel T of height k and length $2(k+1)$ in D corresponds to a decomposition $D = X\mathbf{u}Y\mathbf{d}Z$ where X ends at height k and $Y \in \mathcal{D}_{2k}$. Note that Y has to begin with an up-step (since it is a nonempty Dyck path) followed by a down-step, otherwise D would have a triple rise. Thus, we can write $D = X\mathbf{uud}Y'\mathbf{d}Z$ where $Y' \in \mathcal{D}_{2(k-1)}$. When we apply to D the bijection given by rule (6.1), X is mapped to an initial segment $\widetilde{X}$ of a Motzkin path ending at height k, **uud** is mapped to **u**, Y' is mapped to a Motzkin path $\widetilde{Y'} \in \mathcal{M}_{k-1}$ of length $k-1$, the **d** following Y' is mapped to **d** (since it is preceded by another **d**), and Z is mapped to a final segment $\widetilde{Z}$ of a Motzkin path going from height k to the x-axis. Thus, we have that $M = \widetilde{X}\mathbf{u}\widetilde{Y'}\mathbf{d}\widetilde{Z}$. It follows that tunnels T of D satisfying $\text{height}(T)+1 = \frac{1}{2}\text{length}(T)$ are transformed by the bijection into tunnels $\widetilde{T}$ of M satisfying $\text{height}(\widetilde{T})+1 = \text{length}(\widetilde{T})$. We will call *good* tunnels the tunnels of M satisfying this last condition. It remains to show that the generating function for Motzkin paths where x marks the number of good tunnels plus the number of horizontal steps at height 0, and z marks the length of the path, is given by (6.3).

To do this we repeat the technique used in Theorem 5.8 to enumerate fixed points in 312-avoiding permutations. We will separate good tunnels according to their height. Notice that if a good tunnel of M corresponds to a decomposition $M = X\mathbf{u}Y\mathbf{d}Z$, then M has no good tunnels inside the part given by Y. In other words, the projections on the x-axis of all the good tunnels of a given Motzkin path are disjoint. Clearly, they are also disjoint from horizontal steps at height 0. Using this we will express our generating function as a continued fraction.

For every $k \geq 1$, let $\text{gt}_k(M)$ be the number of tunnels of M of height k and length $k+1$. Let $\text{hor}(M)$ be the number of horizontal steps at height 0. We have seen that for $\pi \in \mathfrak{M}_n^R$, $\text{fp}(\pi) = \text{hor}(\Upsilon(\pi^R)) + \sum_{k\geq 1}\text{gt}_k(\Upsilon(\pi^R))$. We will show now that for every $k \geq 1$, the generating function for Motzkin paths where x marks the statistic $\text{hor}+\text{gt}_1+\cdots+\text{gt}_{k-1}$ is given by the continued fraction (6.3) truncated at level k, with the $(k+1)$-st level replaced with $\mathbf{M}(z)$.

A Motzkin path M can be written uniquely as a sequence of horizontal steps $\mathbf{h}$ and elevated Motzkin paths $\mathbf{u}M'\mathbf{d}$, where $M' \in \mathcal{M}$. In terms of the generating function $\mathbf{M}(z) = \sum_{M\in\mathcal{M}} z^{|M|}$, this translates into the equation $\mathbf{M}(z) = \frac{1}{1-z-z^2\mathbf{M}(z)}$. The generating function where x marks horizontal steps at height 0 is just

$$\sum_{M\in\mathcal{M}} x^{\text{hor}(M)} z^{|M|} = \frac{1}{1-xz-z^2\mathbf{M}(z)}.$$

If we want x to mark also good tunnels at height 1, each M' from the elevated paths above has to be decomposed as a sequence of horizontal steps and elevated Motzkin paths $\mathbf{u}M''\mathbf{d}$. In this decomposition, a tunnel of height 1 and length 2 is produced by each empty M'', so we have

$$\sum_{M\in\mathcal{M}} x^{\text{hor}(M)+\text{gt}_1(M)} z^{|M|} = \cfrac{1}{1-xz-\cfrac{z^2}{1-z-z^2[x-1+\mathbf{M}(z)]}}. \tag{6.4}$$

Indeed, the case where M'' is empty has to be counted as x, not as 1.

Let us now enumerate simultaneously horizontal steps at height 0 and good tunnels at heights 1 and 2. We can rewrite (6.4) as

$$\cfrac{1}{1-xz-\cfrac{z^2}{1-z-z^2\left[x-1+\cfrac{1}{1-z-z^2\mathbf{M}(z)}\right]}}.$$

Combinatorially, this corresponds to expressing each M'' as a sequence of horizontal steps and elevated paths $\mathbf{u}M'''\mathbf{d}$, where $M''' \in \mathcal{M}$. Notice that since $\mathbf{u}M'''\mathbf{d}$ starts at height 2, a tunnel of height 2 and length 3 is created whenever $M''' \in \mathcal{M}_1$. Thus, if we want x to mark also these tunnels, such an M''' has to be counted as xz, not z. The corresponding generating function is

$$\sum_{M\in\mathcal{M}} x^{\mathrm{hor}(M)+\mathrm{gt}_1(M)+\mathrm{gt}_2(M)} z^{|M|}$$

$$= \cfrac{1}{1-xz-\cfrac{z^2}{1-z-z^2\left[x-1+\cfrac{1}{1-z-z^2[(x-1)z+\mathbf{M}(z)]}\right]}}.$$

Now it is clear how iterating this process indefinitely we obtain the continued fraction (6.3). From the generating function where x marks $\mathrm{hor}+\mathrm{gt}_1+\cdots+\mathrm{gt}_k$, we can obtain the one where x marks $\mathrm{hor}+\mathrm{gt}_1+\cdots+\mathrm{gt}_{k+1}$ by replacing the $\mathbf{M}(z)$ at the lowest level with

$$\frac{1}{1-z-z^2[\mathbf{M}_k(x-1)z^k+\mathbf{M}(z)]},$$

to account for tunnels of height k and length $k+1$, which in the decomposition correspond to the $\mathbf{M}_k$ possible elevated Motzkin paths at height k. $\square$

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