MODULI SPACES OF SHEAVES ON K3 SURFACES AND GALOIS REPRESENTATIONS

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SARAH FREI

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Student: Sarah Frei

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Nicolas Addington	Chair
Ellen Eischen	Member
Alexander Polishchuk	Member
Nicholas Proudfoot	Member
Roy Chan	Institutional Representative

and

Janet Woodruff-Borden	Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Sarah Frei

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We consider two K3 surfaces defined over an arbitrary field, together with a smooth proper moduli space of stable sheaves on each. When the moduli spaces have the same dimension, we prove that if the étale cohomology groups with $\mathbb{Q}_\ell$ coefficients of the two surfaces are isomorphic as Galois representations, then the same is true of the two moduli spaces. In particular, if the field of definition is finite and the K3 surfaces have equal zeta functions, then so do the moduli spaces, even when the moduli spaces are not birational. This generalizes works of Mukai, O'Grady, and Markman, who have studied these moduli spaces of sheaves defined over the complex numbers.

CURRICULUM VITAE

NAME OF AUTHOR: Sarah Frei

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Colorado State University, Fort Collins, CO
University of Portland, Portland, OR

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2019, University of Oregon
Master of Science, Mathematics, 2014, Colorado State University
Bachelor of Science, Mathematics and Chemistry, 2010, University of Portland

AREAS OF SPECIAL INTEREST:

Algebraic Geometry and Arithmetic Geometry

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2014-2019

Graduate Teaching Assistant, Colorado State University, 2012-2014

GRANTS, AWARDS AND HONORS:

Anderson Distinguished Graduate Teaching Award, Department of Mathematics, University of Oregon, 2018

Civin Graduate Award, Department of Mathematics, University of Oregon, 2017

Heidi J. Arnold Early Teaching Award, Department of Mathematics, University of Oregon, 2015

PUBLICATIONS:

Moduli spaces of sheaves on K3 surfaces and Galois representations. Submitted. *arXiv preprint, arxiv.1810.06735.*

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For my parents, Michael and Vickie

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CHAPTER I

INTRODUCTION

1.1. Overview

Given a K3 surface S defined over an arbitrary field k, we can study moduli spaces M of stable sheaves on S with fixed Chern classes. Under mild conditions on the Chern classes, each such moduli space is a smooth, projective, geometrically irreducible variety with a natural symplectic structure. The best-studied example of such a moduli space is the Hilbert scheme of points, $S^{[n]}$, parameterizing zero-dimensional subschemes of length n in S. These spaces have been well-studied over $\mathbb{C}$ because they are one of the few known families of compact hyperkähler manifolds. It is a well-known result due to Huybrechts [26], O'Grady [47] and Yoshioka [60], recently summarized in [51], that when $k = \mathbb{C}$ such a moduli space M is actually deformation equivalent to $S^{[n]}$ for $n = \frac{1}{2}\dim M$. This result was recently generalized to arbitrary fields by Charles in his proof of the Tate conjecture for K3 surfaces over finite fields [5]. However, these moduli spaces are typically not birational to the Hilbert scheme.

For a projective variety X defined over a finite field, let $Z(X,t)$ denote the zeta function of X. We prove here that the zeta function of a moduli space of sheaves M is determined by the zeta function of S.

Theorem 1. *Let S_1 and S_2 be K3 surfaces defined over a finite field such that $Z(S_1,t) = Z(S_2,t)$. Let M_1 and M_2 be smooth proper moduli spaces of stable sheaves on S_1 and S_2, respectively, with $\dim M_1 = \dim M_2$. Then $Z(M_1,t) = Z(M_2,t)$.*

Since any two such moduli spaces need not be birational, the equality in Theorem 1 is surprising. In particular, there need not be a geometric map between the moduli spaces that realizes this equality in point-counts over finite fields.

Consider the case where $S_1 = S_2$. When the moduli space M is fine and two-dimensional, M is a K3 surface derived equivalent to the original K3 surface. In this case, our result about zeta functions for two moduli spaces on a fixed K3 surface was already proved by Lieblich and Olsson [36, Thm. 1.2] and independently by Huybrechts [28, Prop. 16.4.6]. We extend their result to also hold when M is not a fine moduli space. Their work was also generalized by Honigs [22] to hold for any derived equivalent surfaces. In higher dimensions, it is an open question whether any two moduli spaces corresponding to a given K3 surface, under possible conditions on Chern classes, are derived equivalent once their dimensions coincide. If we speculate for a moment that they are [28, Ch. 10 Questions and open problems], then our result is consistent with Orlov's conjecture that derived equivalent smooth, projective varieties have isomorphic motives with rational coefficients [50, Conj. 1]. In particular, this conjecture would imply that derived equivalent smooth, projective varieties over a finite field have equal zeta functions. On the other hand, if we suppose instead that there are two such moduli spaces of the same dimension which are not derived equivalent, our result suggests that for this family of varieties, the zeta function is a very coarse invariant.

By the Lefschetz trace formula, the zeta function is determined by the action of the Frobenius endomorphism on the cohomology ring. Thus we will deduce Theorem 1 from the following more general statement. Let ℓ be a prime different from the characteristic of k, and for any of the varieties X below, let $\overline{X} = X \times_k \overline{k}$ where $\overline{k}$ is the algebraic closure of k.

Theorem 2. *Let S_1 and S_2 be K3 surfaces defined over an arbitrary field k such that $H^2_{ét}(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2_{ét}(\overline{S}_2, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations. Additionally, let M_1 and M_2 be smooth proper moduli spaces of stable sheaves on S_1 and S_2, respectively, with $\dim M_1 = \dim M_2$. Then for all $i \geq 0$, $H^i_{ét}(\overline{M}_1, \mathbb{Q}_\ell) \cong H^i_{ét}(\overline{M}_2, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations.*

We remark that when the moduli spaces are fine, the isomorphism $H^2_{ét}(\overline{M}_1, \mathbb{Q}_\ell) \cong H^2_{ét}(\overline{M}_2, \mathbb{Q}_\ell)$ follows almost immediately from the work of Charles [5], who built off of work done by O'Grady [47] over the complex numbers. We extend their result to non-fine moduli spaces, and then the bulk of the work required to prove Theorem 2 is to construct the Galois-equivariant isomorphisms for the higher cohomology groups. This new work comprises the majority of this dissertation.

1.2. Future Work

The study of moduli spaces of sheaves on K3 surfaces fits into a broader framework, which is the study of irreducible symplectic varieties over arbitrary fields: smooth projective varieties with trivial étale fundamental group for which there is a non-degenerate 2-form spanning $H^0(X, \Omega^2_{X/k})$. Over the complex numbers, these varieties are compact hyperkähler manifolds and have been studied extensively. They became objects of interest when the Beauville-Bogomolov Decomposition Theorem was proved in 1983, establishing that every compact Kähler variety with trivial first Chern class is, up to a finite cover, a product of abelian varieties, Calabi-Yau varieties, and hyperkähler varieties [2]. They were recently used in a profound way by Charles in his proof of the Tate conjecture for

K3 surfaces over finite fields [5], and have only recently begun to be studied more generally in positive characteristic [11].

Irreducible symplectic varieties are higher-dimensional generalizations of K3 surfaces. These surfaces have many properties analogous to elliptic curves and have been a popular object of research since the 1950s [28]. In dimension two, all irreducible symplectic varieties are K3 surfaces. In higher dimensions, despite being well-studied there are only a few known examples, every one of which is deformation equivalent to one of the following: a moduli space of stable sheaves on a K3 surface, a generalized Kummer variety, or one of two sporadic examples in dimensions six and ten ([48], [49]). For the higher-dimensional examples, much is still unknown about their arithmetic properties.

Here we discuss some projects and conjectures about irreducible symplectic varieties which are natural extensions of the main results above.

1.2.1. Generalized Kummer varieties

It is natural to ask whether or not the Theorem 2 also holds for the other known family of irreducible symplectic varieties: generalized Kummer varieties. For the Hilbert scheme $A^{[n+1]}$ where A is an abelian surface, we can consider the map

$$\begin{aligned} s_{n+1} \colon A^{[n+1]} &\to A \\ Z &\mapsto \sum_{p \in A} \ell(\mathcal{O}_{Z,p}) p. \end{aligned}$$

The generalized Kummer variety is $K^{[n]}(A) := s_{n+1}^{-1}(p)$ for any rational point p in A. It is a $2n$-dimensional irreducible symplectic variety that has been well-studied over the complex numbers, and was only recently considered in positive characteristic in [11].

Conjecture 1.2.2. *Let A_1 and A_2 be abelian surfaces defined over an arbitrary field k such that $H^2_{ét}(\overline{A}_1, \mathbb{Q}_\ell) \cong H^2_{ét}(\overline{A}_2, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations. Additionally, let $K^{[n]}(A_1)$ and $K^{[n]}(A_2)$ be smooth generalized Kummer varieties on A_1 and A_2, respectively. Show that for all $i \geq 0$, $H^i_{ét}(\overline{K^{[n]}(A_1)}, \mathbb{Q}_\ell) \cong H^i_{ét}(\overline{K^{[n]}(A_2)}, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations.*

Yoshioka shows in [60] that for a moduli space of sheaves on an abelian surface, $M(v)$ with $\dim M(v) \geq 6$, a fiber $K(v)$ of the albanese map $\mathfrak{a}_v \colon M(v) \to A \times \hat{A}$, where $\hat{A}$ is the dual abelian surface, is deformation equivalent to $K^{[v^2/2-1]}(A)$ and is also an irreducible symplectic variety. Thus by studying moduli spaces of sheaves on abelian surfaces, we hope to gain insight into the arithmetic properties of generalized Kummer varieties.

In particular, Theorem 2 should hold for moduli spaces of sheaves on abelian surfaces under mild constrants on the Mukai vector. Since de Cataldo and Migliorini's work [6] holds for any smooth algebraic surface, we can again reduce to the case of a single abelian surface A and a geometrically primitive Mukai vector v. Work of Honigs, Lombardi, and Tirabassi can be easily modified to identify when the moduli space $M = M(v)$ is a smooth projective variety [24, Thm. 2.10]. Additionally, it would be interesting to generalize Markman's most recent work [41, Sec. 8], which uses an isometry of the Mukai lattice to construct a ring isomorphism between the cohomologies of two moduli spaces of sheaves on a complex abelian surface.

The cohomology of $K^{[n]}(A)$ is much richer than the cohomology of the moduli space of sheaves on A. It not only depends on the cohomology of A, but also on

$A[n]$, the $n+1$-torsion points of A. For example, Hassett and Tschinkel show in [19, Prop. 4.1] that for X a smooth projective complex variety deformation equivalent to $K^{[2]}(A)$, the Lie algebra $\mathfrak{so}(4,5)$ acts on $H^*(X)$, giving the decomposition

$$H^*(X) = \mathrm{Sym}((H^2(X)) \oplus \mathbf{1}_X^{80} \oplus (H^3(X) \oplus H^5(X)).$$

In the case where $X = K^{[2]}(A)$, they construct 81 distinguished rational surfaces in X whose classes in $H^*(X)$ span an 81-dimensional subspace containing the summand $\mathbf{1}_X^{80}$. These surfaces correspond bijectively to the 81 points in $A[3]$. We expect that over non-algebraically closed fields (of any characteristic), the Galois group does not act trivially on this 80-dimensional subspace of $H^*(X)$ but rather permutes the classes of the surfaces according to the Galois action on $A[3]$.

1.2.3. Chow motives of moduli spaces

The theory of motives was first introduced by Grothendieck in the 1960's in an attempt to unify various cohomology theories for smooth projective varieties. It would be interesting, especially in light of Orlov's conjecture [50, Conj. 1], to better understand the motives of moduli spaces of sheaves on K3 surfaces. There is a growing collection of closely related work in this direction. In 2017, Huybrechts showed in [29] and [30] that isogeneous and derived equivalent K3 surfaces have isomorphic Chow motives. Recently, Bülles showed that the Chow motive of a moduli space of sheaves on a complex projective K3 or abelian surface is a direct summand of motives of various powers of the surface [4]. The Chow rings of irreducible symplectic varieties have similarly been widely studied in the last decade (see, for example, [59], [52], [58]) and continue to be a subject of interest to many algebraic geometers.

The objects of the category of Chow motives $\mathrm{Mot}(k)$ are smooth projective varieties over the field k along with some extra data. Roughly speaking, morphisms between objects X and Y are given by elements of the Chow ring $\mathrm{CH}^*(X \times Y)_{\mathbb{Q}}$. The Chow motive of a smooth projective variety X is denoted $\mathfrak{h}(X)$. Isomorphic Chow motives immediately implies isomorphic rational Chow groups, but not necessarily rational Chow rings. If an isomorphism of Chow rings is given by an invertible class in $\mathrm{CH}^*(X \times Y)$, then such a class also induces an isomorphism of Chow motives.

For each prime $\ell \neq \operatorname{char} k$, there is a functor from the category $\mathrm{Mot}(k)$ to the category of $\mathrm{Gal}(\overline{k}/k)$-representations over $\mathbb{Q}_\ell$ which sends $\mathfrak{h}(X)$ to $H^*_{\acute{e}t}(X, \mathbb{Q}_\ell)$, and from this it follows that two varieties with isomorphic Chow motives automatically have isomorphic étale cohomology groups as Galois representations. It would be a notable strengthening of Theorem 2 to show that the isomorphism of Galois representations actually comes from an isomorphism of motives.

Conjecture 1.2.4. *Let S_1 and S_2 be K3 surfaces defined over an arbitrary field k such that $\mathfrak{h}(S_1) \cong \mathfrak{h}(S_2)$, and let M_1 and M_2 be smooth proper moduli spaces of stable sheaves on S_1 and S_2, respectively, with $\dim M_1 = \dim M_2$. Then $\mathfrak{h}(M_1) \cong \mathfrak{h}(M_2)$.*

One way to approach this question is to apply the strategies used in the proof of Theorem 2 described in Chapter III. Immediately, [6, Thm. 6.2.4] implies $\mathfrak{h}(S_1^{[n]}) \cong \mathfrak{h}(S_2^{[n]})$, so we can again consider the case of a single K3 surface S and compare $\mathfrak{h}(S^{[n]})$ to $\mathfrak{h}(M)$ for M a smooth projective moduli space of sheaves on S of dimension $2n$.

In Section 3.7 we will make use of a cohomology class constructed by Markman [40, Sec. 3.4] which induces a ring isomorphism on cohomology rings. This class is the middle Chern class of a class from K-theory, and can be considered as an element of $\mathrm{CH}^{2n}(M \times S^{[n]})$. An approach to proving Conjecture 1.2.4 is to show that this class induces an isomorphism of Chow rings. Since the isomorphism is given by a correspondence, it would imply that the motives are isomorphic. Even over the complex numbers, this would be an interesting new result. Over $\mathbb{C}$, $\mathrm{CH}^*(M_1) \cong \mathrm{CH}^*(M_2)$ when M_1 and M_2 are birational [52, Thm. 3.2], so solving Conjecture 1.2.4 in this way would be a strengthening of that result.

1.2.5. The Beauville-Bogomolov form

Let X be a compact complex hyperkähler manifold of dimension n and let $\sigma \in H^0(X, \Omega_X^2)$ be such that $\int_X (\sigma\bar{\sigma})^n = 1$. Using the Hodge decomposition, any $\alpha \in H^2(X, \mathbb{C})$ can be written $\alpha = \lambda\sigma + \beta + \mu\bar{\sigma}$ with $\beta \in H^{1,1}(X)$, and then the Beauville-Bogomolov form $q_X \colon H^2(X, \mathbb{C}) \to \mathbb{C}$ is defined by

$$q_X(\alpha) = \lambda\mu + \frac{n}{2} \int_X \beta^2 (\sigma\bar{\sigma})^{n-1}.$$

We will discuss this form in further detail in Section 2.7, and it will arise a number of times as a tool for studying the moduli spaces of sheaves on K3 surfaces. Beauville [2] and Fujiki [12] prove that there is a positive constant $c_X \in \mathbb{R}$ such that $c_X q_X$ is a primitive integral quadratic form on $H^2(X, \mathbb{Z})$.

This construction depends heavily on working over the complex numbers. However, in [5, Thm. 2.4] and [11, Prop. 4.5, Prop. 7.1], it is shown that there is a canonical quadratic form on ℓ-adic and crystalline cohomology satisfying the same defining property as the original form. Thus for arbitrary irreducible symplectic varieties, we expect that the associated quadratic form has similar properties to that in the complex setting. For example, we would like to be able to use it to tell when these varieties are birational.

Conjecture 1.2.6. *The Beauville-Bogomolov form for irreducible symplectic varieties defined over arbitrary fields is a birational invariant.*

In particular, we give an example in Section 4.1 where we have found two moduli spaces of sheaves on a K3 surface which, assuming this conjecture is true, are not birational.

This fact is well known over the complex numbers [26, Lem. 2.6], but the tools used to prove it do not easily generalize to the arbitrary setting. First, the original definition of the Beauville-Bogomolov form relies on the Hodge decomposition and no longer makes sense over an arbitrary field. Additionally, given X and X' two birational compact hyperkähler varieties, if we let $Z \subset X \times X'$ be the closure of the graph of the birational morphism, then Huybrechts studies the quadratic forms induced by q_X and $q_{X'}$ on $\tilde{Z}$, where $\tilde{Z} \to Z$ is a resolution of singularities. The question of how to resolve singularities in positive characteristic is still open, so Huybrechts' methods cannot be applied directly.

1.3. Outline

In Chapter II, we review the key objects and tools used in this dissertation. This includes K3 surfaces, moduli spaces of sheaves, zeta functions of schemes, and Galois representations. A number of examples are given and results are stated that will be used in later chapters. In Chapter III, we carry out a careful study of the moduli spaces of sheaves on K3 surfaces and prove the main results stated in the introduction. In Chapter IV, we give more examples and computations to give additional context to the results.

CHAPTER II

BACKGROUND

In this chapter, we survey the main objects and tools used throughout this dissertation. In Section 2.1, we define and give examples of K3 surfaces and discuss a number of their invariants and properties. In Section 2.2, we introduce the notion of stability of sheaves. In Section 2.3 we discuss how to construct the moduli space of stable sheaves and provide some results about such moduli spaces on K3 surfaces and over non-algebraically closed fields. In Section 2.4, the zeta function is defined and the Weil conjectures are given. In Section 2.5, we show how to generalize questions about the zeta function to questions about the induced action of the Galois group on cohomology. In Section 2.6, we introduce the notion of a Fourier-Mukai transform and the map it induces on cohomology. In Section 2.7, we give another definition of the Beauville-Bogomolov form. Finally, in Section 2.8, a discussion on the Borel Density Theorem is provided in the context in which it will be used to prove Theorem 2.

2.1. K3 surfaces

The reference for this section is [28]. An algebraic K3 surface is a complete non-singular variety S of dimension two over a field k such that $\omega_S \cong \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. By a variety over k we mean a separated, geometrically integral scheme of finite type over k.

The name for these surfaces was coined by André Weil, who named them in honor of geometers Kummer, Kähler, and Kodaira, as well as the mountain K2. K3 surfaces can be thought of as a generalization of elliptic curves to dimension two, since an elliptic curve also has a trivial canonical bundle. This class of surfaces has a number of interesting properties, so while K3 surfaces have been studied since the 1950s, they continue to be actively researched today. For example, the Kodaira dimension of the variety is one invariant which dictates the complexity of the geometry of that variety, where varieties with low (but non-negative) Kodaira dimension are special and often have interesting arithmetic properties. On the other hand, in some sense most varieties have maximal Kodaira dimension and are too general, and don't have enough defining characteristics, to be studied. For example, genus 0 curves, which have Kodaira dimension $-\infty$, are all rational and are well-understood. Curves of genus greater than 1, on the other hand, have Kodaira dimension 1, and are known to always have finitely many rational points. Curves of genus 1, which have Kodaira dimension 0, have a number of interesting behaviors, the best example being that the set of rational points forms a finitely-generated abelian group.

For surfaces, the Kodaira dimension is either $-\infty$, 0, 1 or 2. Surfaces of Kodaira dimension $-\infty$ are either rational or ruled. K3 surfaces, along with abelian surfaces, bi-elliptic surfaces, and Enriques surfaces, have Kodaira dimension 0. This means K3 surfaces are accessible but still challenging to understand. Arithmetically, there are open questions about the distribution of rational points and the structure of the Brauer group, which in many ways is similar to the torsion subgroup of the group of rational points on an elliptic curve [54].

Another reason K3 surfaces are of interest to algebraic and complex geometers is because when defined over $\mathbb{C}$ and considered as complex manifolds, they are the first example of a hyperkähler manifold, or an irreducible holomorphic sympectic manifold. These are simply-connected compact Kähler manifolds X such that

$H^0(X, \Omega^2) = \mathbb{C}\omega$ for some nondegenerate 2-form ω. An integral, nondegenerate quadratic form exists on $H^2(X, \mathbb{Z})$ which agrees with the intersection form in the case of K3 surfaces. As previously mentioned, by the Beauville-Bogomolov Decomposition Theorem every compact Kähler manifold with trivial first Chern class is, up to a finite cover, a product of complex tori, Calabi-Yau manifolds, and hyperkähler manifolds [2]. A summary of what is understood and what is still unknown in the study of hyperkähler manifolds can be found in [7].

Example 2.1.1. Let S be a smooth degree 4 hypersurface in $\mathbb{P}^3$, so that S is cut out by a section of $\mathcal{O}_{\mathbb{P}^3}(4)$. This is a smooth complete 2-dimensional variety over k. Let $i\colon S \hookrightarrow \mathbb{P}^3$ be the inclusion. We use the adjunction formula to see that

$$\omega_S \cong i^*(\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4)) = i^*(\mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_S\,.$$

Then we use the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0,$$

which induces the long exact sequence

$$\cdots \to H^1(\mathcal{O}_{\mathbb{P}^3}(-4)) \to H^1(\mathcal{O}_{\mathbb{P}^3}) \to H^1(\mathcal{O}_S) \to H^2(\mathcal{O}_{\mathbb{P}^3}(-4)) \to \cdots$$

We know that $H^1(\mathcal{O}_{\mathbb{P}^3}) = H^2(\mathcal{O}_{\mathbb{P}^3}(-4)) = 0$, which implies $H^1(\mathcal{O}_S) = 0$.

Example 2.1.2. Another example of a construction of a K3 surface is as a double covering $\pi\colon S \to \mathbb{P}^2$ branched over a smooth curve $C \subset \mathbb{P}^2$ of degree six. Let C be cut out by a section $s \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. The canonical bundles are related by

$$\omega_S = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(3)) = \pi^*(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(3)),$$

which shows that $\omega_S = \mathcal{O}_S$. We claim that $\pi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$, which shows $H^1(S, \mathcal{O}_S) = H^1(\mathbb{P}^2, \pi_* \mathcal{O}_S) = 0$. For the claim, consider $L := \operatorname{Tot} \mathcal{O}_{\mathbb{P}^2}(3)$ and $\pi\colon L \to \mathbb{P}^2$. The bundle $\pi^* \mathcal{O}_{\mathbb{P}^2}(3)$ has a tautological section $y \in \Gamma(L, \pi^* \mathcal{O}_{\mathbb{P}^2}(3))$, and $S \subset L$ is cut out by $y^2 - s \in \Gamma(L, \pi^* \mathcal{O}_{\mathbb{P}^2}(6))$. Applying π_* to the short exact sequence

$$0 \to \pi^* \mathcal{O}_{\mathbb{P}^2}(-6) \to \mathcal{O}_L \to \mathcal{O}_S \to 0$$

gives $\pi_* \mathcal{O}_S = \operatorname{coker}(\pi^* \mathcal{O}_{\mathbb{P}^2}(-6) \to \mathcal{O}_L) = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$, since

$$\pi_* \pi^* \mathcal{O}_{\mathbb{P}^2}(-6) = \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-9) \oplus \mathcal{O}_{\mathbb{P}^2}(-12) \oplus \cdots,$$

and

$$\pi_* \mathcal{O}_L = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-6) \oplus \mathcal{O}_{\mathbb{P}^2}(-9) \oplus \cdots.$$

We will give an explicit example of a K3 surface arising in this way in Example 2.2.7.

We will often consider a K3 surface S along with a fixed isomorphism class of ample line bundles on S, called a polarization $H \in \operatorname{Pic}(S)$. The degree of a polarized K3 surface is equal to H^2. K3 surfaces of the form given in Example 2.1.1 are of degree four, and those of the form given in Example 2.1.2 are of degree two. The degree is always even because the intersection form on a K3 surface is even. The recent survey article [7, Sec. 2.3] by Debarre gives a full list of descriptions of polarized K3 surfaces of degree up to 24, plus degrees 30, 34, and 38.

Let us compute the Hodge diamond of a K3 surface S defined over $\mathbb{C}$, which will allow us along the way to compute a number of invariants for K3 surfaces. Since S is a complete surface, it is automatically projective, which makes it a Kähler manifold. Since S is complex and compact, we know $h^{0,0} = 1$. By definition, we have $H^1(S, \mathcal{O}_S) = 0$, so $h^{0,1} = h^{1,0} = 0$. We also know by definition and Serre duality that

$$H^2(S, \mathcal{O}_S) \cong H^0(S, \omega_S)^* = H^0(S, \mathcal{O}_S)^*.$$

Thus $h^{0,2} = h^{2,0} = 1$. This allows us to see that

$$\chi(\mathcal{O}_S) = \sum_{i=0}^{2} (-1)^i h^i(S, \mathcal{O}_S) = 1 - 0 + 1 = 2.$$

Now by symmetry of the Hodge diamond, it remains to determine $h^{1,1}$. We have $c_1(S) = c_1(T_S) = -c_1(\Omega_S) = -c_1(\omega_S) = 0$, and we claim that $c_2(S) = 24$. The Hirzebruch-Riemann-Roch Theorem gives

$$2 = \chi(\mathcal{O}_S) = \int_S \mathrm{ch}(\mathcal{O}_S)\mathrm{td}(S) = \mathrm{td}_2(S).$$

Thus $2 = \frac{1}{12}(c_1(S)^2 + c_2(S))$ and $c_2(S) = 24$. Now we use the Hirzebruch-Riemann-Roch Theorem again and compute

$$\chi(\Omega_S) = \int_S \mathrm{ch}(\Omega_S)\mathrm{td}(S) = \mathrm{rk}\Omega_S \cdot \mathrm{td}_2(S) + \mathrm{ch}_2\Omega_S = 4 - 24 = -20.$$

The second-to-last equality comes from the fact that $c_1(\Omega_S) = c_1(\omega_S) = 0$ and $c_2(\Omega_S) = c_2(T_S) = 24$, and $\mathrm{ch}_2(\Omega_S) = \frac{1}{2}(c_1(\Omega_S)^2 - 2c_2(\Omega_S)) = -24$. This means

$$-20 = \chi(\Omega_S) = h^{1,0} - h^{1,1} + h^{1,2} = -h^{1,1}.$$

Therefore, $h^{1,1} = 20$ and the Hodge diamond is:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

More generally, this allows us to compute the singular cohomology of the underlying topological space. Since every (complex) K3 surface is simply connected [28, Cor. 7.1.4], we get that $H^1(S, \mathbb{Z}) = 0$ and $H^2(S, \mathbb{Z})$ is torsion-free. By Poincare duality, we know $H^3(S, \mathbb{Z}) = 0$. Thus,

$$H^i(S, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 4 \\ 0 & i = 3 \\ \mathbb{Z}^{22} & i = 2 \\ 0 & i = 1 \\ \mathbb{Z} & i = 0 \end{cases}.$$

2.2. Stable sheaves

The material in this section can be found in more detail in [31]. Let $\mathcal{F}$ be a torsion-free coherent sheaf on a projective scheme X, $\dim X = n$, with an ample line bundle H.

Definition 2.2.1. The *slope* of the sheaf $\mathcal{F}$ is

$$\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F}},$$

where the degree of the sheaf $\mathcal{F}$ is $\deg \mathcal{F} = c_1(\mathcal{F}).H^{n-1}$. The sheaf $\mathcal{F}$ is *slope stable* or *μ-stable* if for all subsheaves $\mathcal{G} \subset \mathcal{F}$ with $0 < \operatorname{rk} \mathcal{G} < \operatorname{rk} \mathcal{F}$ one has

$$\mu(\mathcal{G}) < \mu(\mathcal{F}).$$

We say $\mathcal{F}$ is *μ-semistable* if $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$.

Let X be an arbitrary projective scheme with an ample line bundle H. Let $\mathcal{F}$ be a sheaf on X, and note that we no longer require it to be torsion-free. Then the Hilbert polynomial of $\mathcal{F}$ is

$$P(\mathcal{F}, t) = \chi(\mathcal{F}(tH)) = \sum_{i=1}^{d} \alpha_d(\mathcal{F}) \frac{m^i}{i!},$$

where d is the dimension of the support of $\mathcal{F}$.

Definition 2.2.2. The *reduced Hilbert polynomial* of a sheaf $\mathcal{F}$ is

$$p(\mathcal{F}, t) = \frac{P(\mathcal{F}, t)}{\alpha_d(\mathcal{F})}.$$

A coherent sheaf $\mathcal{F}$ of dimension d is called *pure* if $\dim(\mathcal{F}) = dim(\mathcal{G})$ for every non-trivial subsheaf $\mathcal{G} \subset \mathcal{F}$.

Definition 2.2.3. A coherent sheaf $\mathcal{F}$ is called *stable* if $\mathcal{F}$ is pure and

$$p(\mathcal{G}, t) < p(\mathcal{F}, t), \quad t \gg 0$$

for every proper non-trivial subsheaf $\mathcal{G} \subset \mathcal{F}$. A sheaf is called *semistable* if the strict inequality is replaced with $\leq$.

Proposition 2.2.4. *We have that μ-stable implies stable implies semistable implies μ-semistable.*

Proof. We consider the reduced Hilbert polynomial of a sheaf on a smooth integral scheme X of dimension n. We know

$$\chi(\mathcal{F}) = \int_X \operatorname{ch}(\mathcal{F})\operatorname{td}(X),$$

with $\operatorname{ch}(\mathcal{F}) = \operatorname{rk}\mathcal{F} + c_1(\mathcal{F}) + \frac{1}{2}(c_1(\mathcal{F})^2 - 2c_2(\mathcal{F})) + ...$ and $\operatorname{ch}(\mathcal{O}(t)) = 1 + tH + \frac{1}{2}t^2H^2 + ...$ Since we also know that $\operatorname{td}(X) = 1 + \frac{1}{2}c_1(X) + \frac{1}{12}(c_1(X)^2 + c_2(X)) + ...$, we can put all of this together to see that

$$\chi(\mathcal{F}(t)) = \operatorname{rk}\mathcal{F} \cdot \deg X \frac{t^n}{n!} + \left(\operatorname{rk}\mathcal{F} \cdot \frac{H^{n-1}.c_1(X)}{2} + H^{n-1}.c_1(\mathcal{F})\right)\frac{t^{n-1}}{(n-1)!} + ... + \chi(\mathcal{F}).$$

Note that $\deg X = H^n$ and $\deg\mathcal{F} = H^{n-1}.c_1(\mathcal{F})$. Thus, for the reduced Hilbert polynomial, we divide by $\operatorname{rk}\mathcal{F}\deg X$ to get

$$p_{\mathcal{F}}(t) = \frac{t^n}{n!} + \frac{1}{\deg X}\left(\frac{H^{n-1}.c_1(X)}{2} + \frac{\deg\mathcal{F}}{\operatorname{rk}\mathcal{F}}\right)\frac{t^{n-1}}{(n-1)!} + ...$$

Observe that the coefficient on t^{n-1} in $p_{\mathcal{F}}(t)$ is $\mu(\mathcal{F})$ plus some additional topological data about X. This means the function sending $\mu(\mathcal{F})$ to $\frac{H^{n-1}.c_1(X)}{2} + \mu(\mathcal{F})$ is an increasing linear function. Thus being μ-stable immediately implies being stable, which also immediately implies being semistable. Again using the fact that $\mu(\mathcal{F}) \mapsto \frac{H^{n-1}.c_1(X)}{2} + \mu(\mathcal{F})$ is a linear function, it follows that being semistable implies μ-semistability. $\square$

Example 2.2.5. Rank 1 torsion-free sheaves are stable because they have no saturated subsheaves. That is, it is enough to check stability on saturated subsheaves, and there aren't any, so the sheaf is vacuously stable. In particular, any line bundle is stable.

Proposition 2.2.6. *Let $\mathcal{F}$ be a semistable sheaf of positive rank on a K3 surface X with polarization H. Suppose* $\gcd(rk\mathcal{F}, c_1(\mathcal{F}).H) = 1$*. Then $\mathcal{F}$ is μ-stable.*

Proof. Since $\mathcal{F}$ is semistable, it is μ-semistable, and to prove stability, it is enough to prove μ-stability. Let $\mathcal{G}$ be any torsion-free subsheaf $\mathcal{G} \subsetneq \mathcal{F}$. We can assume that $\mathcal{G}$ is saturated, because if it weren't, we would have

$$0 \to \mathcal{G} \to \mathcal{F} \xrightarrow{\pi} \mathcal{F} \,/\, \mathcal{G} \to 0,$$

and we can consider $\mathcal{G} \subset \mathcal{G}' := \pi^{-1}(\mathcal{T}) \subset \mathcal{F}$ where $\mathcal{T}$ is the torsion subsheaf of $\mathcal{F} \,/\, \mathcal{G}$. We note that $\mathcal{G}$ and $\mathcal{G}'$ are sheaves of the same rank, since $\mathcal{F} \,/\, \mathcal{G}$ and $\mathcal{F} \,/\, \mathcal{G}'$ differ only by torsion and are hence the same rank. We claim that $\deg \mathcal{G} \leq \deg \mathcal{G}'$, which means $\mu(\mathcal{G}) \leq \mu(\mathcal{G}')$. To see this, we take the top wedge power of the line bundles, so we have $\det \mathcal{G} \hookrightarrow \det \mathcal{G}'$, and then we take the double dual to get a map of line bundles. If we are on a curve, this map gives a section of $\det \mathcal{G}' \otimes (\det \mathcal{G})^*$, which means the degree of this line bundle is non-negative. Then in general, we know that the degree of $\det \mathcal{G}' \otimes (\det \mathcal{G})^*$ is equal to its degree when restricted to a hyperplane section, giving the result. So we may suppose $\mathcal{G}$ is saturated, and we will show that $\mu(\mathcal{G}) \neq \mu(\mathcal{F})$. Since $\mathcal{G}$ is saturated, $\mathcal{F} \,/\, \mathcal{G}$ is torsion free, and since $\mathcal{G} \neq \mathcal{F}$, this means $\operatorname{rk} \mathcal{G} < \operatorname{rk} \mathcal{F}$. Suppose for the sake of a contradiction that $\mu(\mathcal{G}) = \mu(\mathcal{F})$, so

$$\frac{c_1(\mathcal{G}).H}{\operatorname{rk} \mathcal{G}} = \frac{c_1(\mathcal{F}).H}{\operatorname{rk} \mathcal{F}},$$

and hence

$$c_1(\mathcal{G}).H\mathrm{rk}\,\mathcal{F} = c_1(\mathcal{F}).H\mathrm{rk}\,\mathcal{G}.$$

By assumption, $\mathrm{rk}\,\mathcal{F}$ does not divide $c_1(\mathcal{F}).H$, so $\mathrm{rk}\mathcal{F}$ must divide $\mathrm{rk}\mathcal{G}$. But this is also impossible since $\mathrm{rk}\mathcal{G} < \mathrm{rk}\,\mathcal{F}$. Thus we have reached a contradiction and can conclude that $\mathcal{F}$ is stable. $\square$

Example 2.2.7. Here we give an explicit example of a geometrically stable sheaf on a K3 surface (where the definition of geometrically stable is given in Definition 2.3.14). Moreover, we will give a family of polarizations for which the stability of the given sheaf changes throughout the family.

The K3 surface: Let X be the K3 surface over $\mathbb{F}_3$ cut out by

$$\begin{aligned} w^2 =& 2y^2(x^2+2xy+2y^2)^2 + (2x+z)(x^5+x^4y+x^3yz+x^2y^3+x^2y^2z+2x^2z^3 \\ &+ xy^4+2xy^3z+xy^2z^2+y^5+2y^4z+2y^3z^2+2z^5) \end{aligned}$$

in $\mathbb{P}(3,1,1,1)$ which is the reduction modulo 3 of a K3 surface defined over $\mathbb{Q}$ in [20, Section 5]. This K3 surface is a double cover of $\mathbb{P}^2$, as described in Example 2.1.2. The branch curve in $\mathbb{P}^2$ has a tritangent line (a line which is tangent to the sextic above in three points) given by $2x + z = 0$. Let C be the preimage of this line in X, so C is defined over $\mathbb{F}_3$. We see that over $\mathbb{F}_9$, C splits as two copies of $\mathbb{P}^1$, we'll call them C_1 and C_2, intersecting in 3 points. Let H be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ on X, so that $H = C_1 + C_2$. We will describe a sheaf $\mathcal{L}$ on X for which the stability of $\mathcal{L}$ with respect to $H' = H + \epsilon C_1$ changes as we vary ϵ.

The sheaf: Let $\mathcal{L}$ be a degree 3 line bundle on C such that $\deg \mathcal{L}|_{C_1} = 0$ and $\deg \mathcal{L}|_{C_2} = 3$. By abuse of notation, we will also use $\mathcal{L}$ to denote the pushforward of $\mathcal{L}$ to X. We will see later that there is a $\mathbb{P}^2$'s worth of such line bundles. Since

$\mathcal{L}$ is degree 3 and C has arithmetic genus 2, $\chi(\mathcal{L}) = d + 1 - g = 2$. To see that the genus of C is 2, we compute the genus of a smooth fiber $Y \in |H|$ using the Riemann-Hurwitz formula: a generic $\mathbb{P}^1$ in $\mathbb{P}^2$ intersects the branch locus in 6 points, and so the fiber Y in X satisfies

$$2g(Y) - 2 = 2(-2) + 6.$$

Now, we have $\mathcal{L}|_{C_1} = \mathcal{O}_{C_1}$ and $\mathcal{L}|_{C_2} = \mathcal{O}_{C_2}(3)$. We will frequently make use of the sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{L} \to \mathcal{O}_{C_1} \to 0.$$

This sequence comes from the sequence

$$0 \to I_{C_1/C_1 \cup C_2} \to \mathcal{O}_C \to \mathcal{O}_{C_1} \to 0,$$

and $I_{C_1/C_1 \cup C_2} \cong I_{C_1 \cap C_2/C_2} \cong \mathcal{O}_{C_2}(-3)$, tensored with $\mathcal{L}$.

Computing reduced Hilbert polynomials: Let us first compute the Hilbert polynomials of these sheaves with respect to $H' = H + \epsilon C_1$. We will do this by computing the Mukai vectors (where Mukai vectors are defined in 2.3.6), since the third coordinate of the vector will be the Euler characteristic. We have $v(\mathcal{L}) = (\operatorname{rk} \mathcal{L}, c_1(\mathcal{L}), \chi(\mathcal{L})) = (0, H, 2)$ and $v(\mathcal{O}_{C_1}) = v(\mathcal{O}_{C_2}) = (0, C_1, 1)$. Then:

$$v(\mathcal{L}(tH')) = (0, H, 2)(1, tH', \frac{1}{2}t^2 H'^2) = (0, H, tH.H' + 2) = (0, H, (2 + \epsilon)t + 2),$$

since $H.H' = H.(H + \epsilon C_1) = 2 + \epsilon H.C_1 = 2 + \epsilon$. Similarly,

$$v(\mathcal{O}_{C_1}(tH')) = (0, C_1, 1)(1, tH', \frac{1}{2}t^2 H'^2) = (0, C_1, tC_1.H' + 1) = (0, C_1, (1 - 2\epsilon)t + 1),$$

since $C_1.H' = C_1.(H + \epsilon C_1) = 1 + C_1^2 = 1 - 2\epsilon$. Recall that $C_1^2 = C_2^2 = -2$ for the following reason: each is a $\mathbb{P}^1 \subset X$ a K3 surface, and by the adjunction formula,

$$\omega_{C_i} \cong (\omega_X \otimes \mathcal{O}(C_i))|_{C_i} = \mathcal{O}(C_i)|_{C_i},$$

so taking degree on both sides gives $2g_{C_i} - 2 = C_i^2$.

Lastly,

$$v(\mathcal{O}_{C_2}(tH')) = (0, C_2, 1)(1, tH', \frac{1}{2}t^2H'^2) = (0, C_2, tC_2.H' + 1) = (0, C_2, (1 + 3\epsilon)t + 1),$$

since $C_2.H' = C_2.(H + \epsilon C_1) = 1 + \epsilon C_1.C_2 = 1 + 3\epsilon$.

Therefore, we see that

$$P(\mathcal{L}, t) = (2 + \epsilon)t + 2,$$
$$P(\mathcal{O}_{C_1}, t) = (1 - 2\epsilon)t + 1,$$
$$P(\mathcal{O}_{C_2}, t) = (1 + 3\epsilon)t + 1.$$

Then we get the following for reduced Hilbert polynomials:

$$p(\mathcal{L}) = t + \frac{2}{2 + \epsilon},$$
$$p(\mathcal{O}_{C_1}) = t + \frac{1}{1 - 2\epsilon},$$
$$p(\mathcal{O}_{C_2}) = t + \frac{1}{1 + 3\epsilon}.$$

The cases for stability: We can see that for $-\frac{1}{3} < \epsilon < 0$, $p(\mathcal{O}_{C_2}) > p(\mathcal{L}) > p(\mathcal{O}_{C_1})$. Thus, via the sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{L} \to \mathcal{O}_{C_1} \to 0,$$

we have found a destabilizing subsheaf of $\mathcal{L}$ (equivalently, a destabilizing quotient sheaf).

When $\epsilon = 0$, we have $p(\mathcal{O}_{C_2}) = p(\mathcal{L}) = p(\mathcal{O}_{C_1})$. Since $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_2}$ are both semistable and have the same reduced Hilbert polynomial, it follows that $\mathcal{L}$ is also semistable.

Lastly, for $0 < \epsilon < \frac{1}{2}$, the inequalities are $p(\mathcal{O}_{C_2}) < p(\mathcal{L}) < p(\mathcal{O}_{C_1})$. We claim that in this case, $\mathcal{L}$ is geometrically stable with respect to H'.

Showing that $\mathcal{L}$ is stable: Now suppose $F \subset \mathcal{L}$ is a proper saturated subsheaf. Then we get a subsequence of sheaves

$$0 \to \mathcal{O}_{C_2} \cap F \to F \to F/(\mathcal{O}_{C_2} \cap F) \to 0.$$

The intersection $\mathcal{O}_{C_2} \cap F$ could be 0, or otherwise it is a subsheaf of $\mathcal{O}_{C_2}$ which means it is of the form $\mathcal{O}_{C_2}(m)$ for $m \leq 0$. Similarly, the quotient $F/(\mathcal{O}_{C_2} \cap F)$ could be 0, or otherwise it is a subsheaf of $\mathcal{O}_{C_1}$ which means it is of the form $\mathcal{O}_{C_1}(n)$ for $n \leq 0$. We see that for most combinations of the options above, F has smaller reduced Hilbert polynomial than $\mathcal{L}$. The possible issues are if we have one of the following:

$$0 \to 0 \to F \xrightarrow{\sim} \mathcal{O}_{C_1} \to 0,$$

$$0 \to \mathcal{O}_{C_2} \to F \to \mathcal{O}_{C_1} \to 0.$$

The second sequence would mean $F = \mathcal{L}$, but we are assuming F is a proper subsheaf. So we only need to worry about the first sequence, which gives an inclusion $\mathcal{O}_{C_1} \hookrightarrow \mathcal{L}$. Note that this gives a splitting of the sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{L} \to \mathcal{O}_{C_1} \to 0,$$

which is a contradiction since we're assuming $\mathcal{L}$ is a non-trivial extension. We could alternatively argue in the following way: first, use the sequence

$$0 \to \mathcal{L} \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}(3) \to \mathcal{O}_{3pts} \to 0,$$

which comes from tensoring the following with $\mathcal{L}$:

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0.$$

The inclusion $\mathcal{O}_{C_1} \hookrightarrow \mathcal{L}$ would give an inclusion $\mathcal{O}_{C_1} \hookrightarrow \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}$, and since there are no non-trivial maps between $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_2}$, we must have $\mathcal{O}_{C_1} \hookrightarrow \mathcal{O}_{C_1}$. Such a map corresponds to a section of $\mathcal{O}_{C_1}$, and by the short exact sequence above, this section must vanish on $C_1 \cap C_2$. But a section of $\mathcal{O}_{C_1}$ vanishing at 3 points must be 0. Thus, no such inclusion exists, and we conclude that for every proper saturated subsheaf $F \subset \mathcal{L}$, $p_{H'}(F) < p_{H'}(L)$. Thus we can finally conclude that L is geometrically stable.

How to explicitly find such a sheaf: Finally, we need to see that such an $\mathcal{L}$ actually exists. Using the sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{L} \to \mathcal{O}_{C_1} \to 0.$$

one last time, we note that $\mathcal{L}$ corresponds to a class in $\mathrm{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2})$. We claim that $\dim \mathrm{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = 3$, and so there is a $\mathbb{P}^2$'s worth of choices for this sheaf $\mathcal{L}$. For the claim, we observe that $\mathrm{Hom}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = 0$ since C_1 and C_2 intersect in only three points. By Serre duality,

$$\mathrm{Ext}^2(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) \cong \mathrm{Hom}(\mathcal{O}_{C_2}, \mathcal{O}_{C_1} \otimes \omega_X)^\vee = \mathrm{Hom}(\mathcal{O}_{C_2}, \mathcal{O}_{C_1})^\vee = 0.$$

Thus, $\chi(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) = -\dim \mathrm{Ext}^1(\mathcal{O}_{C_1}, \mathcal{O}_{C_2})$. Using the Hirzebruch-Riemann-Roch Theorem,

$$\begin{aligned}
\chi(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}) &= \chi(\mathcal{O}_{C_1}^* \otimes \mathcal{O}_{C_2}) \\
&= \int_X \mathrm{ch}(\mathcal{O}_{C_1}^* \otimes \mathcal{O}_{C_2}) \mathrm{td} X \\
&= \int_X \mathrm{ch}(\mathcal{O}_{C_1})^\vee \mathrm{ch}(\mathcal{O}_{C_2}) \mathrm{td} X \\
&= \int_X (0, -C_1, 1)(0, C_2, 1)(1, 0, 2) \\
&= -C_1.C_2 \\
&= -3.
\end{aligned}$$

This completes the claim.

Lastly, we observe that a similar example can be constructed by picking $\mathcal{L}$ such that $\deg \mathcal{L}|_{C_1} = 1$ and $\deg \mathcal{L}|_{C_2} = 2$, or $\deg \mathcal{L}|_{C_1} = 2$ and $\deg \mathcal{L}|_{C_2} = 1$, or $\deg \mathcal{L}|_{C_1} = 3$ and $\deg \mathcal{L}|_{C_2} = 0$. We will revisit this example in Section 4.2.

Remark 2.2.8. We observe that the example above could have been done by considering rank two sheaves on X instead of rank 0 sheaves. These rank two sheaves correspond to the rank 0 sheaves above in the following way. Since $\mathcal{L}$ is a degree 3 line bundle on a curve C of arithmetic genus 2, we know by the Riemann-

Roch Theorem for curves that

$$h^0(\mathcal{L}) - h^1(\mathcal{L}) = d + 1 - g = 2,$$

and we claim that $h^1(\mathcal{L}) = 0$ so that $\mathcal{L}$ has exactly two global sections. To verify this, we consider the long exact sequence induced by the sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{L} \to \mathcal{O}_{C_1} \to 0,$$

which gives

$$\cdots \to H^0(\mathcal{O}_{C_1}) \to H^1(\mathcal{O}_{C_2}) \to H^1(\mathcal{L}) \to H^1(\mathcal{O}_{C_1}) \to \cdots .$$

Since $C_1 \cong C_2 \cong \mathbb{P}^1$, we know that $H^1(\mathcal{O}_{C_2}) = H^1(\mathcal{O}_{C_1}) = 0$, and so $H^1(\mathcal{L}) = 0$.

These two global sections give rise to the short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^2 \to \mathcal{L} \to 0.$$

Then $\mathcal{K}$ is a rank 2 sheaf on X with

$$v(\mathcal{K}) = 2v(\mathcal{O}_X) - v(\mathcal{L}) = (2, 0, 2) - (0, H, 2) = (2, -H, 0).$$

The analysis carried out above can be repeated to find that $\mathcal{K}$ is either stable, semistable, or unstable with respect to H', given appropriate choices of H'.

2.3. Moduli spaces of sheaves

2.3.1. General theory

The reference for this section is [31]. Let us momentarily assume k is an algebraically closed field. We will discuss what happens when we remove this assumption in Section 2.3.13.

For a smooth projective variety X over a field k, we would like to construct a moduli space which parameterizes sheaves on X. However, this is certainly too much information to study at once, and maybe we would hope that by restricting to just sheaves of a fixed rank, the moduli space would be a reasonable scheme. As a first example, we can consider the moduli space of line bundles (not even all rank-one sheaves) on X, which is called the Picard scheme Pic_X. This scheme is disconnected and is not projective, nor is it of finite type over k. In this case, we can fix this by looking at line bundles with a fixed Hilbert polynomial, but more generally, fixing a Hilbert polynomial does not ensure that a moduli space is of finite type. Thus we see that even by fixing some invariants about the sheaves, it is possible to end up with a moduli space which is too big. Worse yet, even after fixing some invariants and asking for bundles of that type, the resulting moduli space need not be separated.

It turns out that, by adding enough extra conditions on the sheaves, we can get a well-behaved moduli space. The appropriate condition (which in particular results in a separated moduli space) turns out to exactly be stability (using the definition of stability given in 2.2.3). Before constructing the moduli space, we must introduce one more concept. Fix a polarization H on X (which is necessary in order to discuss stability).

Proposition 2.3.2. *[31, Prop. 1.5.2] Let $\mathcal{F}$ be a semistable sheaf on X. Then there exists a filtration (called the Jordan-Hölder filtration) of $\mathcal{F}$*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_\ell = \mathcal{F},$$

such that the factors $\mathcal{F}_i \,/\, \mathcal{F}_{i-1}$ are stable with reduced Hilbert polynomial $p(\mathcal{F}, t)$. Moreover, up to isomorphism, the sheaf $gr(\mathcal{F}) := \oplus_i \mathcal{F}_i \,/\, \mathcal{F}_{i+1}$ does not depend on the choice of the filtration.

Definition 2.3.3. Two semistable sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ with the same reduced Hilbert polynomial are called *S-equivalent* if $gr(\mathcal{F}_1) \cong gr(\mathcal{F}_2)$.

Now, fix a polynomial $P \in \mathbb{Q}[t]$. Then there is a functor

$$\mathcal{M}_H(P)\colon (\mathrm{Sch}/k)^{op} \to (\mathrm{Sets})$$

which sends a scheme S to the set of isomorphism classes of S-flat families of H-semistable sheaves on X with Hilbert polynomial P, up to an equivalence which identifies $E \in \mathrm{Coh}(X \times S)$ with $E \otimes p^*L$ for any line bundle L on S.

Theorem 2.3.4. *The functor is corepresented by a projective k-scheme $M_H(P)$, i.e. there exists a natural transformation $\mathcal{M}_H(P) \to h_{M_H(P)}$ such that for any other $\mathcal{M}_H(P) \to h_N$ there exists a unique morphism $M_H(P) \to N$ such that*

$$\begin{tikzcd}
\mathcal{M}_H(P) \arrow[r] \arrow[dr] & h_{M_H(P)} \arrow[d, dotted, "\exists!"] \\
& h_N.
\end{tikzcd}$$

We call $M_H(P)$ the moduli space for $\mathcal{M}_H(P)$. Moreover, the closed points of $M_H(P)$ parameterize S-equivalence classes of semistable sheaves on X with Hilbert polynomial P.

We briefly summarize the construction of $M(P) = M_H(P)$. It can be shown that the family of semistable sheaves on X with Hilbert polynomial P is bounded, which implies there is some integer m such that $\mathcal{F}(m)$ is globally generated for any such sheaf $\mathcal{F}$. Then $h^0(\mathcal{F}(m)) = P(m)$, and we can consider the sheaf $G := \mathcal{O}_X(-m)^{P(m)}$. We have a natural surjection $\mathcal{O}_X^{P(m)} \twoheadrightarrow \mathcal{F}(m)$ which equivalently gives a surjection $G \twoheadrightarrow \mathcal{F}$. This defines a closed point in the Quot scheme $\mathrm{Quot}(G, P)$. All of the semistable sheaves are contained in an open subset $R \subset \mathrm{Quot}(G, P)$, but a choice was made for these points based on a choice of basis for $H^0(F(m))$. Changing the basis gives an action of $GL_{P(m)}(k)$ on R, and taking the quotient of R by this action, using geometric invariant theory, gives the moduli space.

2.3.5. On K3 surfaces

In many cases, including for K3 surfaces, it turns out to be more convenient to fix additional topological data than just the Hilbert polynomial of the sheaves. In this section, we work over the complex numbers (although most statements hold more generally). This leads us to the definition of the Mukai vector:

Definition 2.3.6. For a coherent sheaf $\mathcal{F}$ on a K3 surface S, the *Mukai vector* of $\mathcal{F}$ is

$$v(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td(S)}} \in H^*(S, \mathbb{Z}).$$

Note that the Mukai vector of a sheaf on a smooth projective variety X has the same definition but need not be an integral class, i.e. it is an element of $H^*(X, \mathbb{Q})$. We also remark that in positive characteristic $v(\mathcal{F})$ is considered in either $H^*_{ét}(X, \mathbb{Z}_\ell)$ for ℓ different from the characteristic, or the numerical Grothendieck group. We will comment on this further in the following section.

At first glance, the definition of the Mukai vector may seem a little odd since it involves $\sqrt{\mathrm{td}(S)}$. This is computed completely formally, using the power series expansion for $\sqrt{1+y}$. This is because $\mathrm{td}(S) = 1 + \frac{1}{2}c_1(S) + \frac{1}{12}(c_1(S)^2 + c_2(S))....$ Then the expansion is

$$\sqrt{1+y} = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + ...$$

Since for a K3 surface we have $c_1(S) = 0$, it follows that $y = \frac{1}{12}c_2(S)$, which means (since y is in $H^4(S, \mathbb{Z})$) we don't need to go past the 2nd term in the expansion. Recalling that $c_2(S) = 24$, we get that

$$\sqrt{\mathrm{td}(S)} = 1 + \frac{1}{24}c_2(S) = (1, 0, 1) \in H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}).$$

Thus, for a K3 surface,

$$\begin{aligned} v(\mathcal{F}) &= (\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \mathrm{ch}_2(\mathcal{F}))(1, 0, 1) \\ &= (\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \mathrm{rk}\,\mathcal{F} + \mathrm{ch}_2(\mathcal{F})) \\ &= (\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \chi(\mathcal{F}) - \mathrm{rk}\,\mathcal{F}), \end{aligned}$$

where the last equality follows by the Hirzebruch-Riemann-Roch Theorem:

$$\begin{aligned}\chi(\mathcal{F}) &= \int_S \mathrm{ch}(\mathcal{F})\mathrm{td}(S) \\ &= \int_S (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), \mathrm{ch}_2(\mathcal{F})(1,0,2) \\ &= \int_S (\mathrm{rk}(\mathcal{F}), c_1(\mathcal{F}), 2\mathrm{rk}(\mathcal{F}) + \mathrm{ch}_2(\mathcal{F})) \\ &= 2\mathrm{rk}(\mathcal{F}) + \mathrm{ch}_2(\mathcal{F}).\end{aligned}$$

Example 2.3.7. We give a few examples of Mukai vectors. First, $v(\mathcal{O}_X) = (1,0,1)$ because it is a line bundle on S, and $\chi(\mathcal{O}_X) = 2$. For any line bundle $\mathcal{L}$, we have $v(L) = (1, c_1(L), c_1(L)^2/2 + 1)$. For a skyscraper sheaf, $v(\mathcal{O}_{pt}) = (0,0,1)$.

There is a pairing on these vectors, called the Mukai pairing, given by

$$\langle \alpha, \beta \rangle := -\alpha_0.\beta_4 + \alpha_2.\beta_2 - \alpha_4.\beta_0,$$

for $\alpha = (\alpha_0, \alpha_2, \alpha_4)$ and $\beta = (\beta_0, \beta_2, \beta_4)$. As the following proposition shows, this pairing only differs by a sign from the pairing given by the Euler characteristic.

Proposition 2.3.8. *For two sheaves $\mathcal{E}$ and $\mathcal{F}$, $\chi(\mathcal{E}, \mathcal{F}) = -\langle v(\mathcal{F}), v(\mathcal{F}) \rangle$.*

Proof.

$$\begin{aligned}
\chi(\mathcal{E},\mathcal{F}) &= \sum(-1)^i \dim \operatorname{Ext}^i(\mathcal{E},\mathcal{F}) \\
&= \chi(\mathcal{E}^* \otimes \mathcal{F}) \\
&= \int_S \mathrm{ch}(\mathcal{E}^*)\mathrm{ch}(\mathcal{F})\mathrm{td}(S) \\
&= \int_S \mathrm{ch}(\mathcal{E}^*)\sqrt{\mathrm{td}(S)}\mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}(S)} \\
&= v(\mathcal{E}^*).v(\mathcal{F}) \\
&= -\langle v(\mathcal{E}), v(\mathcal{F})\rangle. \qquad \square
\end{aligned}$$

Thus, we get that

$$P(\mathcal{E},t) = \chi(\mathcal{E}(t)) = \chi(\mathcal{E},\mathcal{O}(-t)) = -\langle v(\mathcal{E}), v(\mathcal{O}(-t))\rangle,$$

and the Mukai vector determines the Hilbert polynomial. This means that fixing a Mukai vector v fixes a Hilbert polynomial, and we get that the functor $\mathcal{M}_H(v)$, where the fixed Hilbert Polynomial P is replaced by the fixed Mukai vector v, is corepresented by a projective scheme $M_H(v)$ as in 2.3.4. Note that two sheaves with a fixed Hilbert polynomial can have different Mukai vectors, so $M_H(v)$ is a union of connected components of $M_H(P)$ (in fact $M_H(v)$ is often connected, but this is non-trivial to prove).

We hope that, under appropriate conditions, we are able to get a nicely-behaved moduli space. We might ask that in addition to being projective, $M_H(v)$ be a smooth variety. For smoothness, we have the following:

Proposition 2.3.9. *At a point $t \in M_H(v)$ corresponding to a stable sheaf $\mathcal{F}$ on S, $M_H(v)$ is smooth.*

Proof. By [28, Prop. 10.1.11], if the trace map $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \to H^2(S,\mathcal{O}_S)$ is injective and Pic_S is smooth at the point corresponding to $\det \mathcal{F}$, then M is smooth at $t \in M$. We know that stable sheaves are simple, and so by Serre duality, $\mathrm{Ext}^2(\mathcal{F},\mathcal{F}) \cong \mathrm{Hom}(\mathcal{F},\mathcal{F})^\vee \cong k$, giving injectivity of the trace map. Furthermore, all points in the Picard scheme of a K3 surface are smooth, so the result follows. □

Thus the locus of strictly stable sheaves $M_H(v)^s \subset M_H(v)$ is smooth if it is non-empty. We claim that $T_tM_H(v) \cong \mathrm{Ext}^1(\mathcal{F},\mathcal{F})$. To see this, recall that $M_H(v)$ is constructed as a $\mathrm{GL}_{P(m)}(k)$-quotient of $R \subset Q := \mathrm{Quot}(G,P)$ where $G := \mathcal{O}_S(-m)^{P(m)}$ for $m \gg 0$. A point in Q corresponds to a surjection $G \twoheadrightarrow \mathcal{F}$, and the tangent space at this point is naturally identified with $\mathrm{Hom}(\mathcal{K},\mathcal{F})$, where $\mathcal{K}$ is the kernel of $G \twoheadrightarrow \mathcal{F}$. This is just the sheaf-version of the fact that the tangent space of the Grassmannian at a point corresponding to the subspace $W \subset V$ is $\mathrm{Hom}(W, V/W)$. Recall that $M_H(v)$ is obtained by quotienting out by different choices of basis for G, which at a point is the same as quotienting out by the possible choices of surjections $G \twoheadrightarrow \mathcal{F}$ which are not just different due to an endomorphism of $\mathcal{F}$. We can compute this by applying $\mathrm{Hom}(-,\mathcal{F})$ to the short exact sequence

$$0 \to \mathcal{K} \to G \to \mathcal{F} \to 0,$$

which gives rise to the exact sequence

$$0 \to \mathrm{End}(\mathcal{F}) \to \mathrm{Hom}(G,\mathcal{F}) \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{K},\mathcal{F}) \to \mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \to 0.$$

Note that the last term in the sequence is 0 because m was chosen so that $H^i(\mathcal{F}(m)) = 0$ for all $i > 0$ and so

$$\mathrm{Ext}^1(G, \mathcal{F}) = \mathrm{Ext}^1(\mathcal{O}_S(-m)^{P(m)}, \mathcal{F}) = H^1(\mathcal{F}(m))^{P(m)} = 0.$$

We see by the description above that $T_t M_H(v)$ is exactly $\mathrm{Hom}(G, \mathcal{F})/\mathrm{im}(\alpha)$, which by exactness is isomorphic to $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$.

Thus it follows that $\dim M_H(v)^s = \dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{F})$. We observe that

$$v(\mathcal{F})^2 = -\chi(\mathcal{F}, \mathcal{F}) = -\sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) = \dim M_H(v)^s - 2.$$

So we see that the dimension of $M_H(v)^s$ is $v^2 + 2$, if it is non-empty. However, this subspace $M_H(v)^s$ is open, so we get only a quasi-projective scheme. By restricting to a specific class of Mukai vectors, we are able to eliminate the occurrence of properly semistable sheaves.

Definition 2.3.10. A Mukai vector v is called *primitive* if it cannot be written as a scalar multiple of some other class in $H^*(S, \mathbb{Z})$.

Proposition 2.3.11. *[28, Prop. 10.2.5] Let v be a primitive Mukai vector. Then with respect to a generic polarization H, any semistable sheaf $\mathcal{F}$ with $v(\mathcal{F}) = v$ is stable.*

In this case, we see that $M_H(v)^s = M_H(v)$, and the moduli space of stable sheaves on S is a smooth projective variety. We compare this result with Proposition 2.2.6, which gives another criterion for eliminating the existence of properly semistable sheaves.

Example 2.3.12. Moduli spaces of sheaves can be thought of as generalizations of Hilbert schemes of points. We demonstrate here that Hilbert schemes of points on a K3 surface parameterize rank one sheaves on the surface. Let $v = (1, 0, 1-n)$ for some integer $n \geq 1$, so that

$$v^2 = -(1-n) - (1-n) = 2n - 2 \geq 0$$

for $n \geq 1$. We observe that v is primitive, so there is some polarization H on S for which there are no properly semistable sheaves in $M_H(v)$. For $\mathcal{F} \in M_H(v)$, $\mathcal{F}$ is torsion-free and of rank 1, so we have an injection $\mathcal{F} \hookrightarrow \mathcal{F}^{**}$, and $\mathcal{F}^{**} = L$ is a line bundle since $\mathcal{F}^{**}$ is a rank one reflextive sheaf on a regular scheme [17, Prop. 1.9]. Also, a torsion-free sheaf of rank 1 is free in codimension 1, so this map is an ismorphism in codimension 1. Tensoring this map with L^*, we get an injection $\mathcal{F} \otimes L^* \to \mathcal{O}_S$ which is also an isomorphism in codimension 1. Then $\mathcal{F} \otimes L^*$ is the ideal sheaf of a subscheme $Z \subset S$. If we write $\mathcal{F} \otimes L^* = \mathcal{I}$, we have $\mathcal{I}_x \cong \mathcal{O}_{S,x}$ for all $x \in S$ of codimension 1. Then $\mathcal{O}_S / \mathcal{I}$ is supported in codimension 2, which on a surface is a zero-dimensional subscheme. So $\mathcal{F} = \mathcal{I}_Z \otimes L$ where Z is a zero-dimensional subscheme of S and L is a line bundle on S. We have

$$0 \to \mathcal{I}_Z \otimes L \to L \to \mathcal{O}_Z \to 0,$$

which means $v(\mathcal{F}) = v(L) - v(\mathcal{O}_Z) = (1, c_1(L), \chi(L)-1) - (0, 0, \chi(\mathcal{O}_Z)) = (1, 0, 1-n)$. This immediately tells us that $c_1(L) = 0$ and so $L \cong \mathcal{O}_S$. Thus, $\mathcal{F} = \mathcal{I}_Z$, and $v(\mathcal{O}_S) - v(\mathcal{O}_Z) = (1, 0, 1-n)$, and we conclude that $h^0(\mathcal{O}_Z) = n$. This allows us to define a map

$$M_H(1, 0, 1-n) \to \mathrm{Hilb}^n(S),$$

which is in fact an isomorphism. The map in the other direction is given by sending a subscheme $Z \subset S$ to its ideal sheaf $\mathcal{I}_Z$.

2.3.13. Over non-algebraically closed fields

The goal in this dissertation is to study moduli spaces of sheaves on a K3 surface defined over an arbitrary field. Thus we must introduce the tools and results necessary to conduct such a study. For a coherent sheaf $\mathcal{F}$ on a smooth projective variety X defined over a non-algebraically closed field k, there may be destabilizing subsheaves of $\mathcal{F}$ which are only defined over some field extension of k. This concern naturally leads to the following definition:

Definition 2.3.14. A coherent sheaf $\mathcal{F}$ is *geometrically stable* if for any field extension K of k, the pull-back $\mathcal{F} \otimes_k K$ is a stable sheaf on $X \times_k \operatorname{Spec} K$.

A careful study of the stability of sheaves in positive characteristic was carried out by Langer in [35]. In particular, he proves the following result which is a generalization of Theorem 2.3.4. Let R be a universally Japanese ring, for example a field, a Noetherian complete local ring, $\mathbb{Z}$, a Dedekind domain with characteristic zero fraction field, or a finite type extension ring of any of the above (the actual definition of a universally Japanese ring is unenlightening, see [53, Tag 032E]). Let $X \to S$ be a projective morphism of R-schemes of finite type with geometrically connected fibers, and let $\mathcal{O}_X(1)$ be a relatively ample line bundle. For a fixed polynomial $P \in \mathbb{Q}[t]$, we can consider the functor

$$\mathcal{M}_{X/S}(P) \colon (\mathrm{Sch}/S)^{op} \to (\mathrm{Sets})$$

which sends a scheme T to the set of S-equivalence classes of T-flat families of semistable sheaves on the fibers of $T \times_S X \to T$ with Hilbert polynomial P. Then the following is true:

Theorem 2.3.15. *[35, Thm. 0.2] There exists a projective S-scheme $M_{X/S}(P)$ of finite type over S which uniformly corepresents the functor $\mathcal{M}_{X/S}(P)$, and there is an open scheme $M^s_{X/S}(P) \subset M_{X/S}(P)$ that universally corepresents the subfunctor of families of geometrically stable sheaves.*

In particular, this result allows us to study moduli spaces of sheaves in families. For example, suppose $S \to \operatorname{Spec} \mathbb{Z}_p$ is a relative K3 surface for some prime p with a relatively ample line bundle. Again for K3 surfaces, we can consider Mukai vectors instead of Hilbert polynomials, and for a vector v on S, we get the moduli space $M_{S/\mathbb{Z}_p}(v)$ which is projective over $\mathbb{Z}_p$. This morphism has two fibers:

$$\begin{tikzcd} M_{S/\mathbb{Q}_p}(v) \arrow[r] \arrow[d] & M_{S/\mathbb{Z}_p}(v) \arrow[d] & M_{S/\mathbb{F}_p}(v) \arrow[l] \arrow[d] \\ \operatorname{Spec} \mathbb{Q}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}_p & \operatorname{Spec} \mathbb{F}_p. \arrow[l] \end{tikzcd}$$

This idea will allow us to move between studying moduli spaces of sheaves in positive characteristic and studying them in characteristic zero, a technique that will be used frequently throughout the proof of Theorem 2. In particular, it is a classic result due to Deligne [10, Thm. 1.6] that for a pair (S, L) with S a K3 surface and L an ample line bundle on S, both defined over a perfect field k, there exists a DVR W' which is finite over the ring of Witt vectors $W(k)$ and a smooth proper scheme $\mathcal{S} \to \operatorname{Spec} W'$ together with a line bundle $\mathcal{L}$ on $\mathcal{S}$ such that $\mathcal{S} \times_{W'} k \cong S$ and $\mathcal{L} \otimes_{W'} k \cong L$. That is, S and L can both be lifted to characteristic zero.

In order to lift the moduli space as discussed above, we need to lift the K3 surface S and the polarization H, but we in addition need to lift the Mukai vector v. This requires being able to lift c_1, where $v = (r, c_1, s)$. The following result allows us to lift up to nine line bundles on the K3 surface.

Proposition 2.3.16. *[5, Prop. 1.5] Let S be a K3 surface over an algebraically closed field k of positive characteristic, and let $L_1, ..., L_r$ be line bundles on S with L_1 ample. If $r \leq 10$, there exists a DVR W' which is finite and flat over $W(k)$ and a smooth projective relative K3 surface $\mathcal{S} \to \operatorname{Spec} W'$ such that $\mathcal{S} \times_{W'} k \cong S$ and the image of $\operatorname{Pic}(\mathcal{S}) \to \operatorname{Pic}(S)$ contains $L_1, ..., L_r$.*

As mentioned above, when working over an arbitrary field, Mukai vectors are considered as elements of the Mukai lattice but we can no longer use the singular cohomology for the lattice. The notion of a Mukai lattice makes sense in any Weil cohomology theory and is discussed in great generality in [36] as well as [22]. We will make use of Mukai's original construction over $\mathbb{C}$ as well as the construction for étale cohomology, which we recall here. We will omit the subscript ét which usually denotes étale cohomology, and will rather assume from here on that the cohomology is étale unless stated otherwise. The standard reference for the study of étale cohomology is [43].

Definition 2.3.17. Let ℓ be a prime different from the characteristic of k. The *ℓ-adic Mukai lattice* of S is the $\operatorname{Gal}(\overline{k}/k)$-module

$$\widetilde{H}(\overline{S}, \mathbb{Z}_\ell) := H^0(\overline{S}, \mathbb{Z}_\ell) \oplus H^2(\overline{S}, \mathbb{Z}_\ell(1)) \oplus H^4(\overline{S}, \mathbb{Z}_\ell(2))$$

endowed with the *Mukai pairing*

$$(\alpha, \beta) = -\alpha_0.\beta_4 + \alpha_2.\beta_2 - \alpha_4.\beta_0.$$

It is worth pointing out a couple of things about this definition. First, we have made use of Tate twists. For $m \geq 1$, the Tate twist $\mathbb{Z}/\ell^n\mathbb{Z}(m)$ is the sheaf $\mu_{\ell^n}^{\otimes m}$ on $\overline{S}_{ét}$. For $m < 0$, $\mathbb{Z}/\ell^n\mathbb{Z}(m)$ is defined to be the dual of $\mathbb{Z}/\ell^n\mathbb{Z}(-m)$. Then $H^i(\overline{S}, \mathbb{Z}_\ell(m)) := \varprojlim H^i(\overline{S}, \mathbb{Z}/\ell^n\mathbb{Z}(m))$. Over a finite field $\mathbb{F}_q$, this twisting has the effect of scaling the eigenvalues of the induced action of the Frobenius endomorphim f^* on the cohomology. As an example, if f^* acts on $H^2(\overline{S}, \mathbb{Z}_\ell)$ with an eigenvalue of $\lambda \in \mathbb{C}$, then f^* acts on $H^2(\overline{S}, \mathbb{Z}_\ell(1))$ with an eigenvalue of $\dfrac{\lambda}{q}$.

Secondly, note that we have defined the Mukai lattice in weight zero but will continue to use the usual sign on the Mukai pairing. That is, it is standard over the complex numbers to twist the cohomology into weight two for the Mukai lattice:

$$H^0(\overline{S}, \mathbb{Z}_\ell(-1)) \oplus H^2(\overline{S}, \mathbb{Z}_\ell) \oplus H^4(\overline{S}, \mathbb{Z}_\ell(1)).$$

The twists on H^0 and H^4 explain the negative signs present in the Mukai pairing. When the Mukai lattice is instead placed in weight zero, the signs on the terms in the pairing should be changed so that H^0 and H^4 terms remain positive and the pairing in H^2 is negated. However, the pairing given in Definition 2.3.17 is so standard that we will continue to use it as is so as not to confuse the reader.

We define the Mukai vector of a coherent sheaf $\mathcal{F}$ on S as above, but now it is an element of $\widetilde{H}(\overline{S}, \mathbb{Z}_\ell)$.

Definition 2.3.18. Let $\omega \in H^4(\overline{S}, \mathbb{Z}_\ell(2))$ be the numerical equivalence class of a point on $\overline{S}$. A *Mukai vector on S* is an element of

$$N(S) := \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}\omega,$$

and $N(S)$ is considered as a subgroup of $\widetilde{H}(\overline{S}, \mathbb{Z}_\ell)$ under the natural inclusion. A Mukai vector is often denoted by $v = (r, c_1, s)$.

2.4. Zeta functions of schemes and the Weil Conjectures

Let X be a smooth projective variety defined over $\mathbb{F}_q$ with $q = p^m$ for some m and some prime number p. Let $N_r(X)$ be the number of points of X defined over $\mathbb{F}_{q^r}$.

Definition 2.4.1. The *zeta function* of X is

$$Z(X, t) := \exp\left(\sum_{r=1}^{\infty} N_r(X)\frac{t^r}{r}\right).$$

We use this as the definition for the zeta function because it gives a generating function for the values $N_r(X)$. However, it can equivalently be defined using the following proposition, in which case the function more directly resembles its namesake, the Riemann zeta function.

Proposition 2.4.2. *Let* $\zeta_X(s) = \prod_{x \in |X| \text{ closed}} \frac{1}{1 - N(x)^{-s}}$, *where* $N(x) = |\kappa(x)|$, *the residue field at* x. *Then*

$$\zeta_X(s) = Z(X, q^{-s}).$$

Proof. First, we note that because the product in ζ_X is taken over only closed points in X, the residue field $\kappa(x)$ is a finite extension of the base field $\mathbb{F}_q$, and so

$$N(x) = |k(x)| = q^{\deg x},$$

where $\deg x = [\kappa(x) : \mathbb{F}_q]$. Let us use $|X|^{cl}$ to denote the set of closed points in X. If we begin by setting $t = q^{-s}$, we get

$$\zeta_X(s) = \prod_{x \in |X|^{cl}} \frac{1}{1 - N(x)^{-s}} = \prod_{x \in |X|^{cl}} \frac{1}{1 - t^{\deg x}}.$$

Taking the natural log on both sides gives

$$\ln \zeta_X(s) = \sum_{x \in |X|^{cl}} \ln\left(\frac{1}{1 - t^{\deg x}}\right) = \sum_{x \in |X|^{cl}} \sum_{r=1}^{\infty} \frac{t^{r \deg x}}{r} = \sum_{n=1}^{\infty} T_n t^n,$$

for some coefficients T_n. By analyzing the terms in the double sum, we see that

$$\begin{aligned} T_n &= \#\{x \in |X|^{cl} : \deg x = n\} + \frac{1}{2}\#\left\{x \in |X|^{cl} : \deg x = \frac{n}{2}\right\} + ... \\ &= \sum_{j=1}^{\infty} \frac{1}{j}\#\left\{x \in |X|^{cl} : \deg x = \frac{n}{j}\right\}. \end{aligned}$$

Observe that the set $\left\{x \in |X|^{cl} : \deg x = \frac{n}{j}\right\}$ is empty if j does not divide n, and so a number of the terms drop out of the sum. We can rearrange the sum by summing over $k = \frac{n}{j}$, in which case $\frac{1}{j} = \frac{k}{n}$. This gives

$$T_n = \frac{1}{n} \sum_{k|n} k\#\{x \in |X|^{cl} : \deg x = k\}.$$

We would like to relate this coefficient T_n to $N_n(X)$, so let us consider an $\mathbb{F}_{q^n}$-point of X. Such a point is determined by a morphism of schemes $\operatorname{Spec} \mathbb{F}_{q^n} \to X$, which equivalently corresponds to a closed point of X along with an $\mathbb{F}_q$-linear map of residue fields $\mathbb{F}_{q^{\deg x}} = \kappa(x) \to \mathbb{F}_{q^n}$. But such maps only exist when $\deg x$ divides n, in which case the number of maps which are $\mathbb{F}_q$-linear is $|\mathrm{Gal}(\mathbb{F}_{q^{\deg x}}/\mathbb{F}_q)| = \deg x$. Thus,

$$\begin{aligned} N_n(X) &= \sum_{k|n} \# \operatorname{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^k}, \mathbb{F}_{q^n}) \cdot \#\{x \in |X|^{cl} : \deg x = k\} \\ &= \sum_{k|n} k\#\{x \in |X|^{cl} : \deg x = k\}, \end{aligned}$$

and we see that $T_n = \dfrac{N_n(X)}{n}$. Finally, we have

$$\ln \zeta_X(s) = \sum_{n=1}^{\infty} \frac{N_n(X)}{n} t^n,$$

and hence, recalling that $t = q^{-s}$, we conclude that

$$\zeta_X(s) = \exp\left(\sum_{n=1}^{\infty} \frac{N_n(X)}{n} t^n \right) = Z(X, q^{-s}).$$

$\square$

Remark 2.4.3. The function $\zeta_X(s)$ makes sense more generally, when X is a scheme of finite type over $\mathbb{Z}$. If $X = \operatorname{Spec} \mathbb{Z}$, then the closed points correspond to the prime ideals (p) of $\mathbb{Z}$, in which case the residue field is $\mathbb{F}_p$. Then

$$\zeta_{\operatorname{Spec} \mathbb{Z}}(s) = \prod_{\text{primes } p} \frac{1}{1 - p^{-s}} = \zeta(s),$$

the usual Riemann zeta function. More generally for $X = \operatorname{Spec} \mathcal{O}_K$ where $\mathcal{O}_K$ is the ring of integers in a number field K, $\zeta_X(s)$ is the Dedekind zeta function for K. Historically this generalization to the level of schemes was made in an attempt to prove the Riemann hypothesis.

Remark 2.4.4. It is worth pointing out that there is also an equality

$$Z(X,t) = \sum_{r=1}^{\infty} \#\mathrm{Sym}^r(X)(\mathbb{F}_q)t^r,$$

which is what you get when you apply the counting measure to the motivic zeta function of X.

In 1949, André Weil made conjectures about these numbers $N_m(X)$ which suggested a beautiful connection between the arithmetic properties of varieties defined over finite fields and the geometric properties of varieties defined over the complex numbers. The proof of his conjectures was not completed until 1973, and it was in attempting to prove the conjectures that the framework of modern algebraic geometry was developed. Although the results are due to Dwork, Grothendieck, and Deligne (among others who played a part in developing the theory of ℓ-adic cohomology), they continue to be called the Weil Conjectures.

Theorem 2.4.5. *[8, 9] (The Weil Conjectures) Let X be a smooth projective variety of dimension n over a finite field.*

1. *(Rationality of the zeta function) The zeta function $Z(X,t)$ is a rational function. More specifically, it is of the form*

$$Z(X,t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)},$$

where each $P_i(t) \in \mathbb{Z}[t]$. Furthermore, $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$, and the other $P_i(t)$ factor as $\prod_j (1 - \alpha_{i,j} t)$ for some $\alpha_{i,j} \in \mathbb{C}$.

2. *(Functional equation and Poincaré Duality) The zeta function satisfies*

$$Z\left(X, \frac{1}{q^n t}\right) = \pm q^{\frac{nE}{2}} t^E Z(X,t),$$

where E is the Euler characteristic of X. In particular, this means for each i, $\{\alpha_{2n-i,j}\}_j = \left\{\frac{q^n}{\alpha_{i,j}}\right\}_j$.

3. *(Riemann hypothesis) $|\alpha_{i,j}| = q^{i/2}$ for all $1 \le i \le 2n-1$ and all j. This means all of the zeros of $P_i(t)$ lie on the line of complex numbers s with $Re(s) = \frac{i}{2}$.*

4. *(Betti numbers) If there is a smooth variety Y defined over a number field such that X is the reduction of Y modulo a prime ideal, then the degree of P_i is equal to the i^{th} Betti number of Y (considered as a variety over $\mathbb{C}$ via an inclusion of the number field into $\mathbb{C}$).*

Example 2.4.6. We show that the Weil conjectures hold for $X = \mathbb{P}^2$ over $\mathbb{F}_q$, which is a smooth projective variety of dimension 2. We can consider $\mathbb{P}^2$ as $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^1 \cup \{*\}$, so $N_r(X) = q^{2r} + q^r + 1$. Then

$$\begin{aligned} Z(X,t) &= \exp\left(\sum_{r=1}^{\infty} (q^{2r} + q^r + 1)\frac{t^r}{r}\right) \\ &= \exp\left(\sum_{r=1}^{\infty} \frac{(q^2t)^r}{r}\right) \exp\left(\sum_{r=1}^{\infty} \frac{(qt)^r}{r}\right) \exp\left(\sum_{r=1}^{\infty} \frac{t^r}{r}\right) \\ &= \frac{1}{1-q^2t}\frac{1}{1-qt}\frac{1}{1-t} \\ &= \frac{1}{(1-t)(1-qt)(1-q^2t)}. \end{aligned}$$

We immediately observe the rationality of the zeta function, where $P_0(t) = 1 - t$, $P_2(t) = 1 - qt$ and $P_2(t) = 1 - q^2t$. Next, we check the functional equation. We know $\chi(\mathbb{P}^2) = 3$, so we will multiply through the top and bottom by q^3t^3:

$$\begin{aligned} Z\left(X, \frac{1}{q^2t}\right) &= \frac{1}{(1-\frac{1}{q^2t})(1-\frac{1}{qt})(1-\frac{1}{t})} \cdot \frac{q^3t^3}{q^3t^3} \\ &= \frac{q^3t^3}{(q^2t-1)(qt-1)(t-1)} \\ &= -q^3t^3 Z(X,t). \end{aligned}$$

Now, we observe that $|\alpha_0| = 1 = q^{0/2}$, $|\alpha_2| = q = q^{2/2}$, and $|\alpha_4| = q^2 = q^{4/2}$, which agrees with the Riemann hypothesis. Lastly, the degrees $\deg P_0 = \deg P_2 = \deg P_4 = 1$ and $\deg P_1 = \deg P_3 = 0$ match the Betti numbers of $\mathbb{P}^2_{\mathbb{C}}$.

Example 2.4.7. The cohomology of a K3 surface over $\mathbb{C}$ was computed in Section 2.1. Let S be a K3 surface defined over a finite field $\mathbb{F}_q$. The Weil Conjectures for K3 surfaces says that there are 22 algebraic numbers $\alpha_i \in \overline{\mathbb{Q}}$ for $1 \leq i \leq 22$ such that

$$Z(S,t) = \frac{1}{(1-t)\prod_{1\leq i\leq 22}(1-\alpha_i t)(1-q^2t)}.$$

Moreover, $|\alpha_i| = q$ and we can assume $\alpha_i = \pm q$ for $i = 1, ..., 2k$, for some $k \leq 11$, and for $i > 2k$, we have $\alpha_i \neq \pm q$ and $\alpha_{2j-1} \cdot \alpha_{2j} = q^2$ for $j > k$.

Here is an explicit example for the K3 surface introduced in Example 2.2.7. Recall this is the surface X over $\mathbb{F}_3$ cut out by

$$\begin{aligned} w^2 =& 2y^2(x^2+2xy+2y^2)^2 + (2x+z)(x^5+x^4y+x^3yz+x^2y^3+x^2y^2z+2x^2z^3 \\ &+ xy^4 + 2xy^3z + xy^2z^2 + y^5 + 2y^4z + 2y^3z^2 + 2z^5) \end{aligned}$$

in $\mathbb{P}(3,1,1,1)$. Then Hassett and Várilly-Alvarado [20, Section 5] compute that

$$\begin{aligned}P_2(t) = (&1 + 3t + 15t^2 + 45t^3 + 162t^4 + 162t^5 + 486t^6 - 2187t^7 - 8748t^8 \\ &- 52488t^9 - 118098t^{10} - 472392t^{11} - 708588t^{12} - 1594323t^{13} \\ &+ 3188646t^{14} + 9565938t^{15} + 86093442t^{16} + 215233605t^{17} \\ &+ 645700815t^{18} + 1162261467t^{19} + 3486784401t^{20})(1 + 3t)(1 - 3t)\end{aligned}$$

where

$$Z(S,t) = \frac{1}{(1-t)P_2(t)(1-9t)}.$$

2.5. Galois representations

This dissertation was motivated by wanting to understand the zeta functions of various moduli spaces of sheaves on a fixed K3 surface over a finite field $\mathbb{F}_q$. However, the study of zeta functions can be generalized to a study of Galois representations, because the zeta function is determined by the action of the Frobenius endomorphism, an element of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Suppose X is a smooth projective variety of dimension d defined over $k = \mathbb{F}_q$ with $q = p^n$. Let $F_X \colon X \to X$ be the absolute Frobenius map, which is the identity on points of X and is the p^{th} power map on the structure sheaf $\mathcal{O}_X$. Note that this map is not a morphism of $\mathbb{F}_q$-schemes if $q \neq p$. Then let $\overline{X} = X \times_k \overline{k}$ and define $f := F_X^n \times \mathrm{id} \colon \overline{X} \to \overline{X}$, which is the n^{th} power of the relative Frobenius morphism.

Example 2.5.1. Suppose $X = \mathbb{A}^n_{\mathbb{F}_p} = \operatorname{Spec} \mathbb{F}_p[x_1, ..., x_n]$. Then F_X is induced by the p^{th} power map on $\mathbb{F}_p[x_1, ..., x_n]$, under which it can be checked that primes ideals are fixed (so the induced map is the identity on points of X).

The map $f = F_X \times \mathrm{id}\colon \overline{X} \to \overline{X}$ is induced by the map $f^{\#}\colon \overline{\mathbb{F}}_p[x_1, ..., x_n] \to \overline{\mathbb{F}}_p[x_1, ..., x_n]$ which sends x_i to x_i^p for each i and fixes the $\overline{\mathbb{F}}_p$-coefficients. As an example, for the point $\mathfrak{p} = (x_1 - a_1, ..., x_n - a_n) \in \overline{X}$ we have

$$f(\mathfrak{p}) = (f^{\#})^{-1}(\mathfrak{p}) = (x_1 - a_1^p, ..., x_n - a_n^p),$$

since $x_i - a_i^p \mapsto x_i^p - a_i^p = (x_i - a_i)^p$.

If we let $x_1, ..., x_n$ be coordinates for $\overline{X} = \mathbb{A}^n_{\overline{\mathbb{F}}_p}$, this means $f(a_1, ..., a_n) = (a_1^p, ..., a_n^p)$. This means f fixes the point $(a_1, ..., a_n)$ if and only if $a_i \in \mathbb{F}_p$ for all $1 \leq i \leq n$.

By the example above, we see that for an arbitrary (smooth, projective) variety X and for all closed points $x \in \overline{X}$, $f^r(x) = x$ if and only if x has coordinates in $\mathbb{F}_{q^r}$. Thus, $N_r(X)$ is equal to the number of fixed points of f^r. This number can be computed using the Lefschetz fixed point theorem, which tells us that

$$N_r(X) = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(f^{r*}\colon H^i_{ét}(\overline{X}, \mathbb{Q}_\ell) \to H^i_{ét}(\overline{X}, \mathbb{Q}_\ell)).$$

Then we find that

$$\begin{aligned} Z(X,t) &= \exp\left(\sum_{r=1}^{\infty} \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(f^{r*}|_{H^i_{ét}(\overline{X},\mathbb{Q}_\ell)}) \frac{t^r}{r} \right) \\ &= \prod_{i=0}^{2d} \exp\left(\sum_{r=1}^{\infty} \mathrm{tr}(f^{r*}|_{H^i_{ét}(\overline{X},\mathbb{Q}_\ell)}) \frac{t^r}{r} \right)^{(-1)^i} \\ &= \prod_{i=0}^{2d} \det(1 - f^* t|_{H^i_{ét}(\overline{X},\mathbb{Q}_\ell)})^{(-1)^{i+1}}, \end{aligned}$$

where the last equality follows from a linear algebra identity (see [43, V Lem. 2.7] for a proof). Moreover, this new equality shows that the numbers $\{\alpha_{i,j}\}_j$ are the eigenvalues of f^* acting on $H^i_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)$. Thus the zeta function is determined by the action of f^* on $H^i_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)$ for each $0 \leq i \leq 2d$.

We would like to relate this to considering the induced action of the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ instead of f^*, but f is not itself an element of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Rather, if we consider the absolute Frobenius on $\overline{X}$, $F_{\overline{X}}\colon \overline{X} \to \overline{X}$, then by [43, VI Lem. 13.2], $F_{\overline{X}}$ acts as the identity on the cohomology of $\overline{X}$. We also have that $F_{\overline{X}} = F_X \times F_{\overline{\mathbb{F}}_q}$, where $F_{\overline{\mathbb{F}}_q}$ is the usual q^{th} power map on $\overline{\mathbb{F}}_q$. Thus, on cohomology, f^* and $F^*_{\overline{\mathbb{F}}_q}$ are inverses to each other, and $F_{\overline{\mathbb{F}}_q} \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Thus if we can determine the induced action of $F_{\overline{\mathbb{F}}_q}$ on the cohomology of X, then we have determined $Z(X,t)$. This naturally leads us to consider the more general situation.

Let X be a smooth projective variety defined over an arbitrary field k. Then any $\sigma \in \mathrm{Gal}(\overline{k}/k)$ acts on $\overline{X}$ and induces an action σ^* on $H^i_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell)$. Now, rather than just studying zeta functions of these varieties, i.e. the action of a specific element of the Galois group, we are led to study the cohomology groups as Galois representations.

2.6. Derived categories and a conjecture of Orlov

The main reference for this section is [27]. Another perspective from which to study a variety X is through its bounded derived category of coherent sheaves, $D(X) := D^b(\mathrm{Coh}(X))$. The objects of this category are complexes

$$\mathcal{F}^\bullet = \cdots \to \mathcal{F}^{i-1} \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \cdots$$

with $\mathcal{F}^i \in \mathrm{Coh}(X)$ and $\mathcal{F}^i = 0$ for $|i| \gg 0$. A morphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is given by an equivalence class of roofs $\mathcal{E}^\bullet \xleftarrow{\psi} \mathcal{G}^\bullet \xrightarrow{\varphi} \mathcal{F}^\bullet$ where ψ is a quasi-isomorphism.

Another description of this category is as the localization of the homotopy category of $\mathrm{Coh}(X)$ by quasi-isomorphisms.

Two smooth projective varieties over a field k are said to be derived equivalent if there exists a k-linear exact equivalence $F\colon D(X) \xrightarrow{\sim} D(Y)$. Orlov shows that, in fact, the following is true:

Theorem 2.6.1. *[27, Cor. 5.17] The equivalence F is isomorphic to a Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to an object $\mathcal{P} \in D(X \times Y)$ that is unique up to isomorphism.*

The Fourier-Mukai transform $\Phi_{\mathcal{P}}$ associated to $\mathcal{P} \in D(X \times Y)$ is defined as follows. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the two projection morphisms. Then $\Phi_{\mathcal{P}}\colon D(X) \to D(Y)$ is defined by $\mathcal{F}^\bullet \mapsto q_*(p^*\mathcal{F}^\bullet \otimes \mathcal{P})$. The fact that Fourier-Mukai transforms are determined by a single object in the derived category makes them particularly nice to work with, and so we see the strength of Orlov's result.

In order to study the Galois representations which arise from the cohomology groups of moduli spaces of sheaves on K3 surfaces, we are interested in maps between cohomology groups and rings, and Fourier-Mukai transforms induce such maps on cohomology. To get from the derived category $D(X)$ to cohomology, we must pass through the Grothendieck group $K(X)$. The map $D(X) \to K(X)$ is given by

$$\mathcal{F}^\bullet \mapsto \sum_i (-1)^i [\mathcal{F}^i],$$

where $[\mathcal{F}^i]$ denotes the equivalence class of $\mathcal{F}^i$. The Mukai vector is then used to get from the Grothendieck group to cohomology. That is, a sheaf $\mathcal{F}$ can be sent to $v(\mathcal{F}) \in H^*(X, \mathbb{Q})$, and this can be extended additively to a map $v\colon K(X) \to$

$H^*(X, \mathbb{Q})$. Then the following diagram commutes:

$$\begin{tikzcd} D(X) \arrow[r, "\Phi_{\mathcal{P}}"] \arrow[d, "{[\,]}"'] & D(Y) \arrow[d, "{[\,]}"] \\ K(X) \arrow[d, "v"'] & K(Y) \arrow[d, "v"] \\ H^*(X, \mathbb{Q}) \arrow[r, "\Phi^H_{v(\mathcal{P})}"'] & H^*(Y, \mathbb{Q}), \end{tikzcd}$$

where $\Phi^H_{v(\mathcal{P})}$ is given by $\beta \mapsto q_*(p^*(\beta).v(\mathcal{P}))$. We note that we could more generally define a cohomological Fourier-Mukai transform $\Phi^H_\alpha \colon H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z})$ associated to any class $\alpha \in H^*(X \times Y, \mathbb{Z})$, given by $\beta \mapsto q_*(p^*(\beta).\alpha)$. This type of map will make an appearance numerous times throughout Chapter III.

2.6.2. A conjecture of Orlov

It is natural to ask how well the derived category captures the geometry of the variety. By a theorem of Bondal and Orlov, if there exists an equivalence $D(X) \cong D(Y)$ for two smooth projective varieties X and Y, and the canonical bundle of X is either ample or anti-ample, then X and Y are isomorphic. Thus it remains to fully understand what happens for varieties with trivial canonical bundle, such as K3 surfaces and moduli spaces of sheaves on K3 surfaces (among others). Orlov has made the following conjecture:

Conjecture 2.6.3. *[50, Conj. 1] If there exists an equivalence $D(X) \cong D(Y)$ for X and Y smooth, projective varieties, then X and Y have isomorphic rational Chow motives.*

We see immediately that this conjecture is verified when ω_X is either ample or anti-ample, since being isomorphic is stronger than having isomorphic rational

Chow motives. It was shown in [30, Thm. 0.1] that the conjecture holds for K3 surfaces. However, it is unclear what to expect in higher dimensions when ω_X is trivial.

As mentioned in Section 1.2.3, the motive of a variety captures its rational Chow ring as well as its cohomology for any Weil cohomology theory. In particular, this conjecture predicts that two derived equivalent smooth, projective varieties over a field k will have isomorphic étale cohomology groups with $\mathbb{Q}_\ell$-coefficients, for ℓ a prime different from the characteristic of k, and this isomorphism will be as $\mathrm{Gal}(\overline{k}/k)$-representations.

2.7. The Beauville-Bogomolov form

As moduli spaces of sheaves on K3 surfaces over $\mathbb{C}$ are hyperkähler varieties, their second cohomology group comes endowed with a quadratic form. This extra structure is a useful tool in better understanding the cohomology of these moduli spaces. Recall that the Beauville-Bogomolov form was introduced in Section 1.2.5, and it is clear that the definition depends on the Hodge decomposition. As remarked in the introduction, there is a canonical quadratic form on ℓ-adic and crystalline cohomology satisfying the same defining property as the original form. Alternatively, in [16, Def. 26.19], Huybrechts defines an unnormalized version of the Beauville-Bogomolov form on a hyperkähler variety X as

$$\tilde{q}_X(\alpha) = \int_X \alpha^2 \sqrt{\operatorname{td} X},$$

and he shows that this differs from the standard definition of the Beauville-Bogomolov form by a non-zero constant which depends on the topology of X.

Up to scaling, we can take $\tilde{q}_X$ as our definition for the Beauville-Bogomolov form $q\colon H^2_{sing}(X,\mathbb{Z}) \to \mathbb{Z}$, and now this definition makes sense for étale cohomology as well.

2.8. The Borel Density Theorem

A key component of the argument that will be made in Chapter III will be showing that a set of $\mathbb{Z}$-points in a variety is Zariski dense. An outline of the results used in that argument are given here.

Let G be a linear, semisimple Lie group with only finitely many connected components. That is, G is a closed subgroup of $\mathrm{SL}(\ell,\mathbb{R})$ for some number ℓ, and is isomorphic to a finite direct product of simple Lie groups (possibly modulo a finite group).

Definition 2.8.1. A subgroup Γ of G is a *lattice* in G if Γ is a discrete subgroup of G and G/Γ has finite volume.

The volume form on G is given by the Haar measure. As an example, $\mathrm{SL}(2,\mathbb{Z})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$. Given such a group G, we would like to determine whether or not the $\mathbb{Z}$-points $G(\mathbb{Z})$ are Zariski dense in G. We can consider $\mathrm{SL}(\ell,\mathbb{R})$ as a subvariety of $\mathbb{A}^{\ell\times\ell}_{\mathbb{R}}$, and so it makes sense to consider the Zariski topology on a subset $G \subset \mathrm{SL}(\ell,\mathbb{R})$. By the following result, we see that this question is related to whether or not $G(\mathbb{Z})$ is a lattice in G. Note that G has a maximal compact subgroup K, and asking that Γ project densely into the maximal compact factor of G means that the image of Γ under the projection $G \to G/K$ is dense.

Theorem 2.8.2. *[44, 4.5.6] (Borel Density Theorem) Let G be a linear, semisimple Lie group as above and Γ a lattice in G. If Γ projects densely into the maximal compact factor of G and G is connected, then Γ is Zariski dense in G.*

We can get around the condition that G needs to be connected by considering G°, the connected component of the identity element in G, and then using information about G° to make conclusions about G. Moreover, there is a straightforward criterion to verify when we have found a Lie group G such that $G(\mathbb{Z})$ is a lattice in G. Write the coordinate ring of $\mathbb{A}_{\mathbb{R}}^{\ell\times\ell}$ as $\mathbb{R}[x_{1,1}, ..., x_{\ell,\ell}]$, adjoining the variables $x_{i,j}$ for $1 \leq i, j \leq \ell$.

Definition 2.8.3. [44, 5.1.2] A closed subgroup H of $\mathrm{SL}(\ell, \mathbb{R})$ is *defined over* $\mathbb{Q}$ if there exists a subset $Q \subset \mathbb{Q}[x_{1,1}, ..., x_{\ell,\ell}]$ such that:

- $V(Q)$ is a subgroup of $\mathrm{SL}(\ell, \mathbb{R})$,
- $H^\circ = V(Q)^\circ$, and
- H has only finitely many components.

This definition may at first sound non-standard, but it turns out to be exactly the conditions necessary to ensure that the integer points of G form a lattice. The example we will be most interested in is the Lie group $\mathrm{SO}(m, n) \subset \mathrm{SL}(m + n, \mathbb{R})$. We see that $\mathrm{SO}(m, n)$ is defined over $\mathbb{Q}$ because more generally, for any $A \in \mathrm{SL}(\ell, \mathbb{Q})$, the group $\mathrm{SO}(A)$ is given by

$$Q = \left\{ \sum_{1\leq p,q\leq\ell} x_{i,p}A_{p,q}x_{j,q} - A_{i,j} \ : \ 1 \leq i, j \leq \ell \right\}.$$

Proposition 2.8.4. *[44, 5.1.11] If G is defined over $\mathbb{Q}$, then $G(\mathbb{Z})$ is a lattice in G.*

Thus if G is a linear, semisimple Lie group with finitely many components which is defined over $\mathbb{Q}$, and if $G(\mathbb{Z})$ projects densely into the maximal compact factor of G (and G is connected), then the $\mathbb{Z}$-points of G are Zariski dense in G. This gives a clear list of criteria to check in order to conclude the density of $G(\mathbb{Z})$.

We would additionally like to apply these results in the case of complex Lie groups. But this is straightforward since, by definition, the Zariski closure of a real semisimple linear Lie group G in $\mathrm{SL}(\ell, \mathbb{C})$ is the complexification $G_{\mathbb{C}}$ (see [44, Rmk. 18.1.8(3)]). In the case that G is defined over $\mathbb{Q}$, so that $G^\circ = V(Q)^\circ$ for $Q \subset \mathbb{Q}[x_{1,1}, ..., x_{\ell,\ell}]$, then $G_{\mathbb{C}} = V_{\mathbb{C}}(Q) := \{g \in \mathrm{SL}(\ell, \mathbb{C}) \ : \ f(g) = 0 \text{ for all } f \in Q\}$. Then so long as such a group G satisfies the hypotheses of Theorem 2.8.2, it will follow that the $\mathbb{Z}$-points of the complex group $G_{\mathbb{C}}$ are Zariski dense.

CHAPTER III

PROOFS AND GALOIS REPRESENTATIONS

This chapter contains new results in the study of moduli spaces of sheaves on K3 surfaces over arbitrary fields and their Galois representations. In particular, we will prove Theorem 2. In Section 3.1 we show that the moduli space of geometrically stable sheaves on a K3 surface is a smooth, projective, geometrically irreducible variety. We show in Section 3.2 that $H^2(\overline{M}, \mathbb{Z}_\ell(1))$ is isometric to a specific sublattice in $H^*(\overline{S}, \mathbb{Z}_\ell)$ and in Section 3.3 that, after tensoring with $\mathbb{Q}_\ell$, the same sublattice can be identified with a fixed sublattice of $H^*(\overline{S}, \mathbb{Q}_\ell)$, which depends only on the dimension of M. In Section 3.4, we reduce to the case of considering just one K3 surface S and comparing M to the Hilbert scheme $S^{[n]}$. In Section 3.5 we construct a ring R which surjects via a ring homomorphism h onto the cohomology of the moduli space. In Section 3.6, we prove that this surjection h is equivariant with respect to an orthogonal group which acts on both the ring R and the cohomology ring. In Section 3.7, we complete the proof of Theorem 2 by constructing a Galois equivariant ring isomorphism between the cohomologies of the two moduli spaces M and $S^{[n]}$.

3.1. The moduli space over an arbitrary field

Let S be a K3 surface defined over an arbitrary field k with algebraic closure $\overline{k}$, and let $\overline{S} = S \times_k \overline{k}$. Recall the definitions of the Mukai lattice and Mukai vectors discussed in Section 2.3.

Given a Mukai vector v on S and an ample class H in $\mathrm{NS}(S)$, we can form the moduli space $M_H(S, v)$ of Gieseker geometrically H-stable sheaves $\mathcal{F}$ on S such that $v(\mathcal{F}) = v$. These spaces were originally constructed over algebraically closed fields in [42] and [14]. When the notation is clear, we will simply write M or $M(v)$ in place of $M_H(S, v)$. By [35, Thm. 0.2] recalled in Theorem 2.3.15, M is a quasi-projective scheme of finite type over k. In order for the moduli space to be a non-empty, smooth, projective variety, we will require the Mukai vector to satisfy the following conditions.

Definition 3.1.1. A Mukai vector $v \in N(S)$ is *geometrically primitive* if its image under $N(S) \to N(\overline{S})$ is primitive.

Geometrically primitive is the same as primitive when $\mathrm{Br}(k) = 0$, or when S has a zero-cycle of degree one (for example, a k-point), in which case there is an isomorphism $\mathrm{Pic}(S) \xrightarrow{\sim} \mathrm{Pic}(\overline{S})^{\mathrm{Gal}(\overline{k}/k)}$ coming from the Hochschild-Serre spectral sequence [28, Sec. 18.1, Eq. 1.10 and 1.13].

Definition 3.1.2. A Mukai vector $v = (r, c_1, s) \in N(S)$ is *effective* if $r > 0$, or $r = 0$ and c_1 is effective, or if $r = c_1 = 0$ and $s > 0$.

These conditions are necessary to ensure that $M(v)$ is non-empty.

Definition 3.1.3. A polarization $H \in \mathrm{Pic}(S)$ is *v-generic* if it is not contained in the locally finite union of hyperplanes in $\mathrm{NS}(\overline{S})_{\mathbb{R}}$ defined in [31, Def. 4.C.1].

On $\overline{S}$, there are many choices of v-generic polarizations. However, it is possible that $\mathrm{NS}(S) \subset \mathrm{NS}(\overline{S})$ is contained entirely in one of the hyperplanes defined in [31, Def. 4.C.1], resulting in the existence of properly semistable sheaves and causing the moduli space of stable sheaves to be only quasi-projective. This is demonstrated in the following example.

Example 3.1.4. Let S be the K3 surface defined over $\mathbb{F}_3$ first introduced in Example 2.2.7, which has $\mathrm{NS}(S) = \mathbb{Z}H$ (which is demonstrated below). We claim that there is no v-generic polarization on S for $v = (0, H, 2)$.

In the proof of Proposition 5.5 in [20], Hassett and Várilly-Alvarado show that $\mathrm{rank}(\mathrm{NS}(\overline{S})) = 2$. We see explicitly that $C_1, C_2 \in \mathrm{NS}(\overline{S})$ and their intersection matrix

$$\begin{pmatrix} -2 & 3 \\ 3 & -2 \end{pmatrix}$$

has determinant -5 so these classes are independent in $\mathrm{NS}(\overline{S})$. Moreover, the determinant is square-free, so the span of C_1 and C_2 forms a primitive sublattice in $\mathrm{NS}(\overline{S})$. That is, if C_1 and C_2 generated an index-N sublattice, then the discriminant would be divisible by N^2. Thus, $\mathrm{NS}(\overline{S}) = \mathbb{Z}C_1 \oplus \mathbb{Z}C_2$. By looking at the eigenvalues of Frobenius given in the proof of Proposition 5.5 in [20], we see that the Frobenius action swaps C_1 and C_2, so $\mathrm{NS}(S) \subset \mathrm{NS}(\overline{S})^G = \mathbb{Z}(C_1 + C_2)$ for $G = \mathrm{Gal}(\overline{\mathbb{F}}_3/\mathbb{F}_3)$. Since $C_1 + C_2 = H \in \mathrm{NS}(S)$, we conclude that $\mathrm{NS}(S) = \mathbb{Z}H$, and the inclusion $\mathrm{NS}(S) \subset \mathrm{NS}(\overline{S})$ is given by $H = C_1 + C_2$.

Finally, we saw in Example 2.2.7 that there are properly H-semistable sheaves $\mathcal{L}$ on S with $v(\mathcal{L}) = (0, H, 2)$ (this is the case of $H' = H + \epsilon C_1$ with $\epsilon = 0$). Since H and multiples of H are the only choices for a polarization on S over $\mathbb{F}_3$ with which to compute stability, there will always be properly semistable sheaves, and hence the locus of geometrically stable sheaves with be a quasi-projective subvariety of the moduli space of semistable sheaves on S.

In order to avoid this behavior, we will restrict ourselves to situations in which this does not happen. This can be guaranteed, for example, if the components of v satisfy a gcd condition given by Charles in [5] (and which is similar to that given in Proposition 2.2.6), or if $\operatorname{rank}(\mathrm{NS}(S)) = \operatorname{rank}(\mathrm{NS}(\overline{S}))$ as Huybrechts assumes in [30].

Proposition 3.1.5. *Let $v \in N(S)$ be an effective and geometrically primitive Mukai vector with $v^2 \geq 0$, and let H be a v-generic polarization on S. Then M is a non-empty, smooth, projective, geometrically irreducible variety over k of dimension $v^2 + 2$.*

This was proven in [5, Thm. 2.4(i)] under the stronger assumption that v satisfy condition (C) given in [ibid., Def. 2.3], which in particular implies that M is a fine moduli space. See also [11, Prop. 4.5] for a similar result which is slightly more general than [5, Thm. 2.4(i)], but which still requires v to have positive rank.

Proof of Proposition 3.1.5. First, we show that M is projective. It is enough to show that any semistable sheaf $\mathcal{F}$ is actually geometrically stable, since by [35, Thm. 0.2] the moduli space $M(P)$ of Gieseker H-semistable sheaves on S with Hilbert polynomial P is a projective scheme of finite type over k. Hence we consider the pullback of $\mathcal{F}$ to $\overline{k}$, and note that the notions of semistable and geometrically semistable coincide [31, Thm. 1.3.7]. Since v is geometrically primitive and H is v-generic, [28, Ch. 10 Prop. 2.5] shows that the pullback of $\mathcal{F}$ is stable. Thus, $\mathcal{F}$ is geometrically stable. Lastly, fixing the Mukai vector v fixes the Hilbert polynomial P, so $M = M(v)$ is a closed subscheme of the projective scheme $M(P)$, and is hence also projective.

For smoothness, we know $\overline{M} = M_{\overline{k}}$ is smooth by [45, Cor. 0.2], and hence M is also smooth. Once we know M is non-empty, discussed below, [45, Cor. 0.2] also shows that $\dim M = v^2 + 2$.

We show next that M is geometrically irreducible. First, suppose that $\operatorname{char} k = 0$. If $k = \mathbb{C}$, this fact is well-known: see [32, Thm. 4.1] or [60, Thm. 8.1].

Otherwise, we will apply the Lefschetz principle. Since M is a projective variety, it is defined by finitely many equations determined by a finite set of coefficients $\{a_i\}_{i\in I}$ with $a_i \in k$ for each $i \in I$. Then we can consider the subfield $k' = \mathbb{Q}(a_i) \subset k$ generated by all of the a_i over $\mathbb{Q}$, and we see that M is defined over k'. There are inclusions $\overline{k'} \hookrightarrow \mathbb{C}$ and $\overline{k'} \hookrightarrow \overline{k}$ giving $\overline{M}$ and $M_{\mathbb{C}}$ as geometric fibers of $M_{\overline{k'}} \to \operatorname{Spec} \overline{k'}$. Since $M_{\mathbb{C}}$ is irreducible, it follows that $\overline{M}$ is as well.

Now suppose $\operatorname{char} k = p > 0$. To show that $\overline{M}$ is irreducible, we will show that it is connected. As discussed in Section 2.3.13, we will lift $\overline{S}$ to characteristic zero. By [5, Prop. 1.5], there is a finite flat morphism $\operatorname{Spec} W' \to \operatorname{Spec} W$, where W' is a discrete valuation ring and W is the ring of Witt vectors of $\overline{k}$, and there exists a smooth projective relative K3 surface $\mathcal{S} \to \operatorname{Spec} W'$ with special fiber isomorphic to $\overline{S}$. By the same result, there are lifts $\mathcal{H}$ of H and $\tilde{c}_1$ of c_1 to $\mathcal{S}$, so we can form the relative moduli space $f\colon \mathcal{M}_{\mathcal{H}}(\mathcal{S}, v_{W'}) \to \operatorname{Spec} W'$ parameterizing geometrically stable sheaves on the fibers of $\mathcal{S} \to \operatorname{Spec} W'$, as constructed in [35, Thm. 0.2] (see Theorem 2.3.15 for a discussion). For the sake of notation, we will denote $\mathcal{M}_{\mathcal{H}}(\mathcal{S}, v_{W'})$ by $\mathcal{M}$. We must show that f is a smooth morphism. Since smoothness is an open condition, we need only show that the morphism is smooth at closed points in the central fiber. These are the closed points of $\overline{M}$, so they correspond to geometrically stable sheaves $\mathcal{F}$ on $\overline{S}$. By [23, Lem. 3.1.5], f is smooth at such a point $[\mathcal{F}]$ if and only if $\operatorname{Pic}(\mathcal{S}/W')$ is smooth at $[\det \mathcal{F}]$. The latter holds because $\det \mathcal{F} = c_1$ lifted from $\overline{S}$ to $\mathcal{S}$. Therefore, $f\colon \mathcal{M} \to \operatorname{Spec} W'$ is smooth.

To complete the proof of irreducibility, we claim that all geometric fibers of f are connected. Since all closed points in the central fiber are geometrically stable and this property is also an open condition, we conclude that all closed points in $\mathcal{M}$ correspond to geometrically stable sheaves. Then by [35, Thm. 0.2]

f is projective, so in particular it is flat and proper with reduced geometric fibers. From [53, Tag 0E0N] it follows that the number of connected components of the geometric fibers is constant. Thus the closed fiber $\overline{M}$ is connected and smooth, hence irreducible.

Lastly, the non-emptiness of M over $k = \mathbb{C}$ is proven in [46, Thm. 5.1 & 5.4] for $v^2 = 0$, in [61, Thm. 3.16] for $r > 0$, and in [62, Cor. 3.5] otherwise (this result is summarized in [28, Thm. 10.2.7]), from which it follows that when $\operatorname{char} k = 0$, M is not empty. For $\operatorname{char} k > 0$, the fact that the number of connected components of the geometric fibers of $f \colon \mathcal{M} \to \operatorname{Spec} W'$ is constant implies that the closed fiber $\overline{M}$, and hence also M, is non-empty. □

3.2. Generalizing results of Mukai and O'Grady

In [46], Mukai showed that for a complex projective K3 surface S and a primitive Mukai vector v with $v^2 = 0$, there is an isomorphism $v^\perp/\langle v\rangle \cong H^2_{sing}(M, \mathbb{Z})$, where $v^\perp$ is the orthogonal complement of v in the Mukai lattice and $M = M(v)$. When $v^2 > 0$, O'Grady [47] proved that $v^\perp \cong H^2_{sing}(M, \mathbb{Z})$. Moreover, both of these isomorphisms were shown to be isometries, where the pairing on $H^2_{sing}(M, \mathbb{Z})$ is given by the Beauville-Bogomolov form. We will make use of the definition of the Beauville-Bogomolov form given at the beginning of Section 2.7.

We will show here that the isometries proven by Mukai and O'Grady also hold when S is defined over an arbitrary field k and v is an effective and geometrically primitive Mukai vector.

Proposition 3.2.1. *Let S be a K3 surface defined over an arbitrary field k and $v \in N(S)$ an effective and geometrically primitive Mukai vector with a v-generic polarization $H \in \mathrm{NS}(S)$.*

1. *When* $v^2 > 0$, *there is a Galois equivariant isometry*

$$v^\perp \cong H^2(\overline{M}, \mathbb{Z}_\ell(1)).$$

2. *When* $v^2 = 0$, *there is a Galois equivariant isometry*

$$v^\perp/\langle v\rangle \cong H^2(\overline{M}, \mathbb{Z}_\ell(1)).$$

Charles in [5, Thm. 2.4(v)] proved this result when $v^2 > 0$ and assuming v satisfies condition (C) in [ibid., Def. 2.3]. We follow his technique to prove the more general result, making modifications where necessary. We will prove Proposition 3.2.1 in great detail so that we can easily refer back to it in similar situations later.

Proof of Proposition 3.2.1. For both (i) and (ii), we must first show that a quasi-universal sheaf exists on $S \times M$ in the sense of [31, Def. 4.6.1]. We claim that a quasi-univeral sheaf $\mathcal{U}$ exists by using the same proof of existence given in [31, Prop. 4.6.2] but appealing to work by Langer for moduli of sheaves in arbitrary characteristic. Langer proves in [34, Thm. 4.3] that the open subset $\mathscr{R}$ of the Quot-scheme parameterizing Gieseker semistable sheaves is equal to the set of semistable points under the $\mathrm{GL}(V)$-action. The quotient is a $\mathrm{PGL}(V)$-principal bundle in the fppf topology and by [43, I.3.26], it also has local sections in the étale topology. Then the proof of [31, Prop. 4.6.2] gives that the universal sheaf on $\mathscr{R}^s$ descends to a quasi-universal sheaf on $S \times M$.

The quasi-universal sheaf is used to define the Mukai map which we will show gives the desired isomorphisms. Introducing some notation, we consider the projections from $S \times M$:

$$\begin{tikzcd} & S \times M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S & & M. \end{tikzcd}$$

The Mukai map $\theta_v \colon \widetilde{H}(\overline{S}, \mathbb{Z}_\ell) \to H^2(\overline{M}, \mathbb{Z}_\ell(1))$ is defined by

$$\alpha \mapsto \frac{1}{\rho} \left[\pi_{2*}(v(\mathcal{U}) \cdot \pi_1^*(\alpha)) \right]_2 ,$$

where $v(\mathcal{U})$ is the Mukai vector of $\mathcal{U}$ and ρ is the similitude of $\mathcal{U}$ (that is, the rank of the sheaf W in [31, Def. 4.6.1]).

We are now ready to prove that if $v^2 > 0$, $\theta_v|_{v^\perp} \colon v^\perp \xrightarrow{\sim} H^2(\overline{M}, \mathbb{Z}_\ell(1))$ is a Galois equivariant isometry. This will be done in different cases depending on the field k. If $k = \mathbb{C}$, then θ_v was proven in [47, Main Thm.] to be an isometry for singular cohomology with coefficients in $\mathbb{Z}$. This isomorphism can be tensored with $\mathbb{Z}_\ell(1)$, and then the comparison theorem for singular and étale cohomology gives the isomorphism $v^\perp \cong H^2(M, \mathbb{Z}_\ell(1))$.

Now suppose k is an arbitrary field of characteristic zero. Again there is a field k' with inclusions $\overline{k'} \hookrightarrow \mathbb{C}$ and $\overline{k'} \hookrightarrow \overline{k}$ such that S and M are defined over k'. The inclusions give the following horizontal isomorphisms by smooth base change:

$$\begin{tikzcd} H^2(\overline{M}, \mathbb{Z}_\ell(1)) & H^2(M_{\overline{k'}}, \mathbb{Z}_\ell(1)) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H^2(M_{\mathbb{C}}, \mathbb{Z}_\ell(1)) \\ v^\perp_{\overline{k}} \arrow[u, "\theta_v"] & v^\perp_{\overline{k'}} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u] & v^\perp_{\mathbb{C}}, \arrow[u, "\wr"', "\theta_v"] \end{tikzcd}$$

where $v^\perp_{\overline{k'}} \subseteq \widetilde{H}(S_{\overline{k'}}, \mathbb{Z}_\ell)$, and similarly for $v^\perp_{\overline{k}}$ and $v^\perp_{\mathbb{C}}$. The right-most vertical arrow is an isomorphism by the argument above, and by commutativity this implies the other vertical arrows are isomorphisms as well. Since $v(\mathcal{U})$ is defined over k, we see that $\theta_v\colon v^\perp \xrightarrow{\sim} H^2(\overline{M}, \mathbb{Z}_\ell(1))$ is Galois equivariant.

Next, suppose k is an arbitrary field of characteristic $p > 0$. As in the proof of Proposition 3.1.5, we form the relative moduli space $\mathcal{M} = \mathcal{M}_{\mathcal{H}}(\mathcal{S}, v_{W'})$, a smooth scheme over $\operatorname{Spec} W'$ whose central fiber is $\overline{M}$. We have the projections

$$\begin{tikzcd} & \mathcal{S} \times_{W'} \mathcal{M} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{S} & & \mathcal{M}, \end{tikzcd}$$

and by the same argument given above, there is a quasi-universal sheaf $\widetilde{\mathcal{U}}$ on $\mathcal{S} \times_{W'} \mathcal{M}$. As long as we construct both of the quasi-universal sheaves $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ following [31, Prop. 4.6.2], we see that their pullbacks to $\overline{S} \times_{\overline{k}} \overline{M}$ must agree.

Now we can define a relative Mukai map

$$\widetilde{\theta}_v\colon v_{W'}^\perp \to H^2(\mathcal{M}, \mathbb{Z}_\ell(1)),$$

where $v_{W'}^\perp \subset \widetilde{H}(\mathcal{S}, \mathbb{Z}_\ell)$, and where $\widetilde{\theta}_v(\alpha) = \frac{1}{\rho}[\pi_{2*}(v(\widetilde{\mathcal{U}}) \cdot \pi_1^*(\alpha))]_2$. We observe that $\widetilde{\theta}_v$ restricts exactly to the map θ_v over both fibers.

Next we apply the smooth base change theorem in order to compare the cohomology groups of the geometric fibers of $\mathcal{M}$. Explicitly, we have $\mathcal{M} \to \operatorname{Spec} W'$ a proper and smooth morphism, and for all n, we have $\mu_{\ell^n}(1)$ a constructible locally constant sheaf on $\mathcal{M}$ whose torsion is prime to the characteristic of k. We conclude by [43, VI.4.2] that the cohomology groups for all geometric fibers of $\mathcal{M} \to \operatorname{Spec} W'$ are isomorphic. In particular, if we let $K := \operatorname{Frac} W'$, it follows that $H^2(M_{\overline{K}}, \mathbb{Z}_\ell(1)) \cong H^2(\overline{M}, \mathbb{Z}_\ell(1))$. The same argument shows that

$H^{2m}(S_{\overline{K}}, \mathbb{Z}_\ell(m)) \cong H^{2m}(\overline{S}, \mathbb{Z}_\ell(m))$ for $m = 0, 1$, and 2, and hence the corresponding Mukai lattices are also isomorphic.

Thus, overall the smooth base change theorem gives the following commutative diagram with horizontal isomorphisms, where the right-most vertical arrow is an isomorphism because the characteristic of K is zero:

$$\begin{tikzcd} H^2(\overline{M}, \mathbb{Z}_\ell(1)) & H^2(\mathcal{M}, \mathbb{Z}_\ell(1)) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H^2(M_{\overline{K}}, \mathbb{Z}_\ell(1)) \\ v_{\overline{k}}^\perp \arrow[u, "\theta_v"] & v_{W'}^\perp \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\widetilde{\theta}_v"'] & v_{\overline{K}}^\perp . \arrow[u, "\theta_v"', "\wr"] \end{tikzcd}$$

Therefore, the left-most vertical arrow is also an isomorphism, as desired. Again, $v(\mathcal{U})$ is defined over k, so θ_v is Galois equivariant, and it continues to respect the Mukai and Beauville-Bogomolov pairings as shown by [47, Main Thm.]. Hence θ_v is a Galois equivariant isometry. This completes the proof of (i).

The proof of (ii) follows the same argument, using the isometry $v^\perp/\langle v \rangle \xrightarrow{\sim} H^2(M, \mathbb{Z})$ [46, Thm. 1.4] for $k = \mathbb{C}$ in place of [47, Main Thm.]. □

3.3. A Galois equivariant isometry

To prove Theorem 2 for $i = 2$, it remains to show the following:

Proposition 3.3.1. *Let $v \in N(S)$ be an effective Mukai vector on a K3 surface S defined over an arbitrary field k, and consider $v^\perp \subset \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$.*

1. *When $v^2 > 0$, there is a Galois equivariant isometry*

$$v^\perp \cong H^2(\overline{S}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell,$$

where the pairing on the right side is given by the intersection form on $H^2(\overline{S}, \mathbb{Q}_\ell(1))$ *and* $-v^2$ *on the generator of* $\mathbb{Q}_\ell$*.*

2. *When* $v^2 = 0$*, there is a Galois equivariant isometry*

$$v^\perp/\langle v\rangle \cong H^2(\overline{S}, \mathbb{Q}_\ell(1)),$$

where the pairing on the right side is given by the intersection form.

Remark 3.3.2. Note that Proposition 3.3.1 need not hold when $\mathbb{Q}_\ell$ is replaced with $\mathbb{Z}_\ell$, as demonstrated by the example in Section 4.1. This difference in coefficients appears to be related to the question of whether or not the moduli space $M(v)$ is birational to the Hilbert scheme.

Proof of Proposition 3.3.1. To prove (i), let $w = (1, 0, 1-n) \in N(S)$ where $n = \frac{1}{2}(v^2+2)$. If $n > 1$ then $w^\perp = H^2(\overline{S}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell\langle(1, 0, n-1)\rangle$, so we will prove that $v^\perp \cong w^\perp$. This is done by reflecting through $v - w$ or $v + w$, as described in the two cases below.

For the first case, suppose that $(v-w)^2 \neq 0$. Then reflection through $v - w$ gives a map $\widetilde{H}(\overline{S}, \mathbb{Q}_\ell) \to \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$. It can be checked that this reflection preserves the Mukai pairing, sends v to w, and induces a map $v^\perp \xrightarrow{\sim} w^\perp$ which is Galois equivariant.

For the second case, suppose that $(v-w)^2 = 0$. Then $v^2 + w^2 = 2\langle v, w\rangle$ and $(v+w)^2 = 2v^2 + 2w^2 \neq 0$, so we consider the reflection through $v + w$. It can be checked that this gives a $\mathrm{Gal}(\overline{k}/k)$-equivariant isometry $v^\perp \xrightarrow{\sim} w^\perp$. This completes the proof of (i).

The proof of (ii) requires a few modifications to the argument above. We now consider $w = (1, 0, 0) \in N(S)$, so that $w^\perp/\langle w\rangle = H^2(\overline{S}, \mathbb{Q}_\ell(1))$. If $(v-w)^2 \neq 0$, then as above, reflection through $v - w$ restricts to a $\mathrm{Gal}(\overline{k}/k)$-equivariant isometry $v^\perp/\langle v\rangle \xrightarrow{\sim} w^\perp/\langle w\rangle$.

If instead $(v-w)^2 = 0$, then $\langle v, w\rangle = 0$ and $(v+w)^2 = 0$ as well. Let us write $v = (r, c_1, 0)$. If $r \neq 0$, then reflecting through $v - (0,0,1)$ gives that $v^\perp/\langle v\rangle \cong (0,0,1)^\perp/\langle(0,0,1)\rangle$. Then $(0,0,1)^\perp/\langle(0,0,1)\rangle \cong w^\perp/\langle w\rangle$ by reflecting through $(0,0,1) - w$.

Thus we are reduced to the case where $v = (0, c_1, 0)$. We claim that there is an ample class which pairs positively with c_1. For a rank zero sheaf $\mathcal{F}$ with $v(\mathcal{F}) = v$, $\mathcal{F}$ is supported on a union of curves:

$$(\operatorname{Supp}\mathcal{F})_{\mathrm{red}} = \cup C_i,$$

and $c_1 = c_1(\mathcal{F}) = \sum_i n_i C_i$ for some integers $n_i > 0$, since v is effective. This means for any ample divisor h on S, $c_1.h > 0$. If we let $v' = ve^h = (0, c_1, c_1.h)$, then it follows that $(v' - w)^2 = 2c_1.h \neq 0$, and so reflection through $v' - w$ is a Galois equivariant isometry $v'^\perp/\langle v'\rangle \cong w^\perp/\langle w\rangle$. Lastly, it can be checked that $v^\perp/\langle v\rangle \xrightarrow{\cdot e^h} v'^\perp/\langle v'\rangle$ is an isometry, and it is Galois equivariant because h is Galois invariant. This completes the proof of (ii). $\square$

Remark 3.3.3. In was shown in [39, Thm. 1(3)] that $H^i(\overline{M}, \mathbb{Q}_\ell) = 0$ for all i odd, so the proof of Theorem 2 is complete in the case where $\dim M_1 = \dim M_2 = 2$. In this case, $H^0(\overline{M}_1, \mathbb{Q}_\ell) \cong H^0(\overline{M}_2, \mathbb{Q}_\ell)$ and $H^4(\overline{M}_1, \mathbb{Q}_\ell) \cong H^4(\overline{M}_2, \mathbb{Q}_\ell)$ trivially (as Galois representations), and by Propositions 3.2.1 and 3.3.1,

$$H^2(\overline{M}_j, \mathbb{Q}_\ell) \cong v_j^\perp/\langle v_j\rangle \otimes \mathbb{Q}_\ell(-1) \cong H^2(\overline{S}_j, \mathbb{Q}_\ell)$$

for $j = 1$ and 2. Since by assumption $H^2(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2(\overline{S}_2, \mathbb{Q}_\ell)$, we conclude that $H^2(\overline{M}_1, \mathbb{Q}_\ell) \cong H^2(\overline{M}_2, \mathbb{Q}_\ell)$.

3.4. Reduction to the case of a single surface

In Section 3.3, we were able to conclude Theorem 2 holds for $i = 2$ by using results about a single K3 surface along with the assumption that $H^2(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2(\overline{S}_2, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations. By the following proposition, to complete the proof for $i > 2$ it is enough to show that $H^i(\overline{M}, \mathbb{Q}_\ell) \cong H^i(\overline{S}^{[n]}, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations, where $n = \frac{1}{2}\dim M$.

Proposition 3.4.1. *Let S_1 and S_2 be two K3 surfaces defined over an arbitrary field k such that $H^2(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2(\overline{S}_2, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations. Then $H^i(\overline{S}_1^{[n]}, \mathbb{Q}_\ell) \cong H^i(\overline{S}_2^{[n]}, \mathbb{Q}_\ell)$ as $\mathrm{Gal}(\overline{k}/k)$-representations for all $i \geq 0$.*

Proof. For a K3 surface S, de Cataldo and Migliorini show in [6, Thm. 6.2.1] that the rational Chow motive of $\overline{S}^{[n]}$ is built out of motives of symmetric products $\overline{S}^{(l(\nu))}$ where ν is a partition of n and $l(\nu)$ is the length of ν. The maps $\overline{S}^{(l(\nu))} \to \overline{S}^{[n]}$ used to give the isomorphism are induced by tautological correspondences defined over the base field, so the decomposition works over any field (see [6, Rmk. 6.2.2]). This implies the following $\mathrm{Gal}(\overline{k}/k)$-equivariant isomorphism on the level of cohomology:

$$H^*(\overline{S}^{[n]}, \mathbb{Q}_\ell) \cong \bigoplus_{\nu \in \mathfrak{P}(n)} H^*(\overline{S}^{(l(\nu))}, \mathbb{Q}_\ell)(n - l(\nu)),$$

where $\mathfrak{P}(n)$ is the set of partitions of n. Since $H^*(\overline{S}^{(m)}, \mathbb{Q}_\ell) \cong H^*(\overline{S}^m, \mathbb{Q}_\ell)^{\Sigma_m}$ for any $m \geq 1$, where $H^*(\overline{S}^m, \mathbb{Q}_\ell)^{\Sigma_m}$ is the subring of Σ_m-invariants, the result follows. $\square$

Thus the proof of Theorem 2 will be complete once we know that $H^i(\overline{M}, \mathbb{Q}_\ell) \cong H^i(\overline{S}^{[n]}, \mathbb{Q}_\ell)$ for a given K3 surface.

Remark 3.4.2. It is interesting to observe that we need not arrive at a ring isomorphism between $H^*(\overline{S}_1^{[n]}, \mathbb{Q}_\ell)$ and $H^*(\overline{S}_2^{[n]}, \mathbb{Q}_\ell)$, and in fact this appears to depend on whether or not the isomorphism $H^2(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2(\overline{S}_2, \mathbb{Q}_\ell)$ as Galois representations agrees with the cohomology ring structures. Indeed, if there is a Galois equivariant ring isomorphism $H^*(\overline{S}_1, \mathbb{Q}_\ell) \cong H^*(\overline{S}_2, \mathbb{Q}_\ell)$, then the intersection forms on the middle cohomology agree and along with Proposition 3.3.1 we get an isometry between their Mukai lattices. Following an argument akin to that given in Proposition 3.7.1 below, this implies the rings $H^*(\overline{S}_1^{[n]}, \mathbb{Q}_\ell)$ and $H^*(\overline{S}_2^{[n]}, \mathbb{Q}_\ell)$ are isomorphic.

If instead the given isomorphism $H^2(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2(\overline{S}_2, \mathbb{Q}_\ell)$ as Galois representations is not an isometry with respect to the intersection pairing, we should not expect $H^*(\overline{S}_1^{[n]}, \mathbb{Q}_\ell)$ and $H^*(\overline{S}_2^{[n]}, \mathbb{Q}_\ell)$ to be isomorphic rings. Suppose there is a ring isomorphism $\psi\colon H^*(\overline{S}_1^{[n]}, \mathbb{Q}_\ell) \xrightarrow{\sim} H^*(\overline{S}_2^{[n]}, \mathbb{Q}_\ell)$, and let $q_i\colon H^2(\overline{S}_i^{[n]}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ for $i = 1$ and 2 be the Beauville-Bogomolov form, introduced at the beginning of Section 2.7. Then for $\alpha \in H^2(\overline{S}_1^{[n]}, \mathbb{Q}_\ell)$,

$$q_1(\alpha)^n = q_2(\psi(\alpha))^n,$$

so that q_1 and q_2 agree up to an n^{th}-root of unity. The only roots of unity in $\mathbb{Q}_\ell$ are the $(\ell-1)$th roots of unity for ℓ odd and ± 1 for $\ell = 2$, so if we choose $\ell > 2$ with $\gcd(n, \ell-1) = 1$, this root must be trivial. If n is even, we can only ensure that $\gcd(n, \ell-1) = 2$, implying the root is ± 1, but we claim $q_1 \not\cong -q_2$.

Consider when n is even and $\ell = 3$ so that $\gcd(n, 2) = 2$. We will show that for the form q on $\mathbb{Q}_3^{23}$ giving $H^2(\overline{S}_1^{[n]}, \mathbb{Q}_3)$, there is no linear isomorphism of $\mathbb{Q}_3^{23}$ taking q to $-q$, and hence the Beauville-Bogomolov forms on $H^2(\overline{S}_1^{[n]}, \mathbb{Q}_3)$ and $H^2(\overline{S}_2^{[n]}, \mathbb{Q}_3)$ cannot differ by a sign. By Propositions 3.2.1 and 3.3.1, q is given by $(-E_8)^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle 2 - 2n \rangle$. In the Witt group $W(\mathbb{Q}_3)$, it can be checked that $(-E_8)^{\oplus 2} \oplus U^{\oplus 3} = (E_8)^{\oplus 2} \oplus (-U)^{\oplus 3} = 0$, so to see that $q \neq -q \in W(\mathbb{Q}_3)$, we must only check that $\langle 2 - 2n \rangle \neq \langle -(2 - 2n) \rangle$ as forms on $\mathbb{Q}_3$. The form $\langle 2 - 2n \rangle$ is equivalent to $\langle m \rangle$ for $m \in \{-3, -1, 1, 3\}$, from which it follows that $\langle m \rangle \neq \langle -m \rangle \in W(\mathbb{Q}_3)$ (see [33, Cor. VI.1.6 and Thm. VI.2.2]). We conclude that q_1 and q_2 must agree.

Therefore, again by Propositions 3.2.1 and 3.3.1, there is a Galois equivariant isometry $H^2(\overline{S}_1, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell \cong H^2(\overline{S}_2, \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell$. As in the proof of Proposition 3.3.1(i), the reflection that takes the generator of the first $\mathbb{Q}_\ell$ to the generator of the second $\mathbb{Q}_\ell$ restricts to a Galois equivariant isometry $H^2(\overline{S}_1, \mathbb{Q}_\ell) \cong H^2(\overline{S}_2, \mathbb{Q}_\ell)$, hence determining the ring structure.

3.5. Markman's surjective ring homomorphism

Because Section 3.3 completed the proof of Theorem 3.2.1 in the case where $\dim M_1 = \dim M_2 = 2$, we now assume $v^2 > 0$. Following Markman [38, p. 15], we construct a ring $R(v)$ corresponding to a geometrically primitive Mukai vector v and a ring homomorphism to the cohomology ring of $\overline{M}$. Recall the projections π_1 and π_2 from $S \times M$ to S and M, respectively, and let $\mathcal{U}$ be a quasi-universal sheaf on $S \times M$. First, define the map

$$\Phi^i_{u_v} \colon \widetilde{H}(\overline{S}, \mathbb{Q}_\ell) \to H^{2i}(\overline{M}, \mathbb{Q}_\ell(i)),$$

given by

$$\alpha \mapsto [\pi_{2*}(u_v \cdot \pi_1^*(\alpha))]_{2i}\,,$$

where u_v, defined in [40, Eq. (27)], is the pullback from $S \times M$ to $\overline{S} \times_{\overline{k}} \overline{M}$ of a normalization of $v(\mathcal{U})(\mathrm{td} M)^{-1/2}$ which Markman [40, Lem. 4.11] shows is invariant under the action of the monodromy group. This invariance will play an important role in Section 3.6. We observe that $\Phi^1_{u_v}|_{v^\perp} = \theta_v$, the Mukai map used in the proof of Proposition 3.2.1.

Definition 3.5.1. Let $R(v)$ be the graded ring freely generated by $v^\perp$ in degree 2 and by $M_{2i} \cong \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$ in degree $2i$ for $1 \leq i \leq \dim M$.

Following the notation given in Definition 2.3.17, let $\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$ denote the cohomology ring of $\overline{M}$ twisted into weight zero.

Definition 3.5.2. Let

$$h \colon R(v) \to \widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$$

be the ring homomorphism determined by $\Phi^1_{u_v}|_{v^\perp} \colon v^\perp \xrightarrow{\sim} H^2(\overline{M}, \mathbb{Q}_\ell(1))$ in degree two, and $\Phi^i_{u_v} \colon M_{2i} \to H^{2i}(\overline{M}, \mathbb{Q}_\ell(i))$ in degree $2i$ for $1 < i \leq \dim M$.

Lemma 3.5.3. *The map h is surjective.*

Proof. The case of $k = \mathbb{C}$ is proven for singular cohomology with coefficients in $\mathbb{Q}$ by [38, Lem. 10], from which it immediately follow for étale cohomology with coefficients in $\mathbb{Q}_\ell$. For k an arbitrary field, we proceed as in the proof of Proposition 3.2.1. □

3.6. The action of an orthogonal group on the cohomology

We will see in Proposition 3.6.4 that the Galois group $\mathrm{Gal}(\overline{k}/k)$ for k an arbitrary field can be seen to act on the cohomology of the moduli space of sheaves on a K3 surface through an orthogonal group. We will study a natural action of this orthogonal group on $R(v)$ and $\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$, where the representation theory is well-understood, and then recognize the Galois group acting through this orthogonal group.

The Beauville-Bogomolov form q induces an action of $\mathrm{O}(q)$ on $H^2(\overline{M}, \mathbb{Q}_\ell(1))$. This gives a natural action of $\mathrm{O}(q)$ on $v^\perp$ via the isomorphism $v^\perp \cong H^2(\overline{M}, \mathbb{Q}_\ell(1))$ proven in Proposition 3.2.1, which then extends to all of $R(v)$ by defining the action to be trivial on all copies of $\mathbb{Q}_\ell\langle v\rangle \subset M_{2i}$.

Proposition 3.6.1. *The* $\mathrm{O}(q)$*-action on* $R(v)$ *descends to an action on* $\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$*.*

Proof. First, let $k = \mathbb{C}$ and consider singular cohomology with coefficients in $\mathbb{Q}_\ell$. We follow the work of Markman [40]. Let Γ_v be the subgroup of the isometry group of $H^*_{sing}(S, \mathbb{Z})$ which stabilizes v. Then by [40, Cor. 1.3], there is a group homomorphism $\gamma : \Gamma_v \to \mathrm{Aut}(H^*_{sing}(M, \mathbb{Z}))$ giving an action of $\gamma(\Gamma_v)$ on $H^*_{sing}(M, \mathbb{Z})$. Next by [40, Lem. 4.11(3)], the $\gamma(\Gamma_v)$-action extends to an action of $\mathrm{O}(q)$ on $H^*_{sing}(M, \mathbb{Q}_\ell)$ (under the choice of an inclusion $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$). In his proof of Lemma 4.11(3), Markman shows that this action extends to an action of $\mathrm{O}(q)$ by descending the action on $R(v)$ through the surjective map h. We then use the isomorphism between singular and étale cohomology to get the action on $H^*(\overline{M}, \mathbb{Q}_\ell)$.

For k an arbitrary field, we follow the argument given in Proposition 3.2.1 to arrive at an $\mathrm{O}(q)$-action on $\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$, as desired. $\square$

This $\mathrm{O}(q)$-action on the cohomology ring can be seen as arising in a different way, as developed by Verbitsky [57] and Looijenga-Lunts [37]. This allows us to give a different and interesting proof of Proposition 3.6.1.

Another proof of Prop. 3.6.1. First, suppose that $k = \mathbb{C}$, and let $n = \dim M$. For an ample class $a \in H^2_{sing}(M, \mathbb{Q}_\ell)$, we define $L_a \colon H^*_{sing}(M, \mathbb{Q}_\ell) \to H^*_{sing}(M, \mathbb{Q}_\ell)$ by $x \mapsto a \cdot x$, which is a degree 2 raising operator. By the Hard Lefschetz Theorem, the homomorphisms $L^i_a \colon H^{n-i}_{sing}(M, \mathbb{Q}_\ell) \xrightarrow{\sim} H^{n+i}_{sing}(M, \mathbb{Q}_\ell)$ for $0 \le i \le n$ are isomorphisms. This is equivalent to the existence of a degree 2 lowering operator Λ_a on $H^*_{sing}(M, \mathbb{Q}_\ell)$ such that $[L_a, \Lambda_a] = H$, where the operator H is multiplication by $k - n$ on $H^k_{sing}(M, \mathbb{Q}_\ell)$. Then the operators L_a, H, and Λ_a satisfy

$$[L_a, \Lambda_a] = H, \quad [H, L_a] = 2L_a, \quad [H, \Lambda_a] = -2\Lambda_a,$$

and so they generate a Lie subalgebra $\mathfrak{g}_a \subset \mathfrak{gl}(H^*_{sing}(M, \mathbb{Q}_\ell))$ which is isomorphic to $\mathfrak{sl}_2$. We will call such a triple a Lefschetz triple, and for any $a \in H^2_{sing}(M, \mathbb{Q}_\ell)$ such that (L_a, H, Λ_a) is a Lefschetz triple, we will say that a is of Lefschetz type. Verbisky shows in [55, Prop. 8.1] that Λ_a is uniquely determined by $a \in H^2_{sing}(M, \mathbb{Q}_\ell)$; that is, if (L_a, H, Λ_a) and (L_a, H, Λ'_a) are both Lefschetz triples, then $\Lambda_a = \Lambda'_a$. Looijenga and Lunts in [37] define a Lie algebra $\mathfrak{g}(M)$ to be the Lie algebra generated by all of the $\mathfrak{g}_a$'s for $a \in H^2_{sing}(M, \mathbb{Q}_\ell)$ of Lefschetz type. We note that these are the same as the classes which are ample for some deformation of M. In [37, Theorem 4.5(ii)], they show that the degree zero part of $\mathfrak{g}(M)$ splits as $\mathfrak{so}(q) \times \mathbb{Q}_\ell H$. The action of $\mathfrak{so}(q)$ integrates to an action of $\mathrm{Spin}(q)$ and Verbitsky [57, Cor 8.2] shows that this action factors through an action of $\mathrm{SO}(q)$

on $H^*_{sing}(M, \mathbb{Q}_\ell)$. By the comparison theorem for singular and étale cohomology, we arrive at an action of $\mathrm{SO}(q)$ on $H^*(M, \mathbb{Q}_\ell)$.

Next, we would like to show that the same action arises when k is an arbitrary field. We claim that this happens because we have the same cohomology ring for $\overline{M}$ regardless of the field k. If $\operatorname{char} k = 0$, then we saw in Proposition 3.2.1 using the Lefschetz principle that $H^2(\overline{M}, \mathbb{Q}_\ell) \cong H^2(M_{\mathbb{C}}, \mathbb{Q}_\ell) \cong H^2_{sing}(M_{\mathbb{C}}, \mathbb{Q}_\ell)$. If instead $\operatorname{char} k > 0$, in the smooth base change and lifting arguments made in Proposition 3.2.1, we observed that $H^2(\overline{M}, \mathbb{Q}_\ell) \cong H^2(M_{\overline{K}}, \mathbb{Q}_\ell) \cong H^2(M_{\mathbb{C}}, \mathbb{Q}_\ell)$. This means that regardless of the field k, there is a bijection between the classes of Lefschetz type in $H^2_{sing}(M_{\mathbb{C}}, \mathbb{Q}_\ell)$ and in $H^2(M_{\overline{k}}, \mathbb{Q}_\ell)$. Therefore, we have an isomorphism of Lie algebras $\mathfrak{g}(M_{\mathbb{C}}) \cong \mathfrak{g}(\overline{M})$. Then as above, $\mathfrak{g}(\overline{M})$ is isomorphic to $\mathfrak{so}(q) \times \mathbb{Q}_\ell H$, where this splitting still makes sense because the operator H is canonical. Lastly, applying [57, Cor 8.2] again gives the $\mathrm{SO}(q)$ action on $H^*(\overline{M}, \mathbb{Q}_\ell)$.

Lastly, since $\mathrm{O}(23) \cong \mathrm{SO}(23) \times \mathbb{Z}/2\mathbb{Z}$, we can extend the action of $\mathrm{SO}(q)$ to an action of $\mathrm{O}(q)$ by setting $-1 \in \mathbb{Z}/2\mathbb{Z}$ to act as Id on $H^{4k}(\overline{M}, \mathbb{Q}_\ell)$ and as $-\mathrm{Id}$ on $H^{4k+2}(\overline{M}, \mathbb{Q}_\ell)$, so that it acts by a ring homomorphism. □

In particular, Markman [40, Lem. 4.11(3)] proves that u_v is invariant under the extended action of $\mathrm{O}(q)$ on $\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$. This allows us to conclude the following:

Corollary 3.6.2. *The map h given in Definition 3.5.2 is equivariant for the $\mathrm{O}(q)$ action constructed in Proposition 3.6.1.*

To see explicitly that h is $\mathrm{SO}(q)$-equivariant using the description of the action given by Looijenga-Lunts and Verbitsky, we can check directly that $h(A \cdot \alpha) = A \cdot h(\alpha)$ for all $A \in \mathrm{SO}(q)$ and $\alpha \in R(v)$. We do this by checking that $\mathrm{SO}(q)$ acts trivially on u_v.

For this, we momentarily consider the case of $k = \mathbb{C}$ and singular cohomology with coefficients in $\mathbb{Q}$. In [40, Lemma 4.13], Markman proves that Verbitsky's $\mathrm{SO}(q)$ action agrees with the monodromy action on a finite index subgroup K of the monodromy group. By Lemma 3.6.3 given below, $\overline{K}$ is of finite index in $\overline{\mathrm{Mon}}$, where the Zariski closure is taken inside $\mathrm{O}(H^2(M, \mathbb{C}))$. Then [25, Ch.2, Sec. 7.3] shows that $\overline{K}$ contains $\overline{\mathrm{Mon}}^{\circ}$, and since $\overline{\mathrm{Mon}} \subset \mathrm{O}(H^2(M, \mathbb{C}))$, we see that $\overline{\mathrm{Mon}}^{\circ} = \mathrm{SO}(23, \mathbb{C})$. Since the inclusion $\overline{K}^{\circ} \subset \overline{\mathrm{Mon}}^{\circ}$ is clear, we conclude that K is Zariski dense in $\mathrm{SO}(23, \mathbb{C})$, where the proof of Lemma 3.7.2 shows that the Zariski closure of $\mathrm{O}(H^2(M, \mathbb{Z}))$ in $\mathrm{GL}(23, \mathbb{C})$ is $\mathrm{O}(H^2(M, \mathbb{C}))$.

Now, consider the set

$$\big\{(A, r) \in \mathrm{SO}(23, \mathbb{C}) \times R(v) : h(A \cdot r) - A \cdot h(r) = 0\big\},$$

which is a closed subset of $\mathrm{SO}(23, \mathbb{C}) \times R(v)$. For $A \in K$, we know by [40, Lemma 4.13] and [40, Lemma 4.15] that $A \cdot u_v = u_v$, which means h commutes with A, i.e. $K \times R(v)$ is contained in the closed subset above. Then the subset must also contain the closure of $K \times R(v)$, which by the argument above is $\mathrm{SO}(23, \mathbb{C}) \times R(v)$. In particular, the subset above contains $\mathrm{SO}(q) \times R(v)$, which means that $h\colon R(v)_{\mathbb{C}} \to \widetilde{H}_{sing}(M_{\mathbb{C}}, \mathbb{Q})$ is $\mathrm{SO}(q)$-equivariant.

Lastly, using the lifting and specialization argument given in Lemma 3.5.3, we conclude that $h\colon R(v) \to \widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$ is also a map of $\mathrm{SO}(q)$-representations.

Lemma 3.6.3. *Let G be an algebraic group with subgroups $H < K < G$ and H is of finite index inside K. Then $\overline{H} < \overline{K}$ is also of finite index.*

Proof. We can write $K = k_1 H \cup \cdots \cup k_n H$. Since multiplication in G is a homeomorphism, it follows that $k_i\overline{H}$ is closed for $1 \leq i \leq n$, which further implies that $k_1\overline{H}\cup\cdots\cup k_n\overline{H}$ is also closed inside G. For any $k\in K$, we have $k\in k_iH\subset k_i\overline{H}$ for some i, which means $k\in k_1\overline{H}\cup\cdots\cup k_n\overline{H}$. Then $K\subset k_1\overline{H}\cup\cdots\cup k_n\overline{H}$, and hence $\overline{K}\subset k_1\overline{H}\cup\cdots\cup k_n\overline{H}$. Thus, $\overline{K}$ is contained in a finite union of cosets of $\overline{H}$, which means $\overline{H}$ is of finite index in $\overline{K}$. □

Next, we show that the Galois group can be seen as acting through $\mathrm{O}(q)$.

Proposition 3.6.4. *The action of* $\mathrm{Gal}(\overline{k}/k)$ *on* $\widetilde{H}(\overline{M},\mathbb{Q}_\ell)$ *factors through* $O(q)$.

Proof. First, we must show that $q(\sigma\alpha)=q(\alpha)$ for all $\alpha\in H^2(\overline{M},\mathbb{Q}_\ell(1))$ and $\sigma\in\mathrm{Gal}(\overline{k}/k)$. The tangent bundle on M is Galois invariant because it is defined over k, and so it follows that $\sigma\left(\sqrt{\mathrm{td}\,\overline{M}}\right)=\sqrt{\mathrm{td}\,\overline{M}}$. Then, using the definition given in Section 2.7,

$$
\begin{aligned}
q(\sigma\alpha) &= c\,(\sigma\alpha)^2.\left(\sqrt{\mathrm{td}\,\overline{M}}\right)_{4n-4}\\
&= c\sigma(\alpha^2).\sigma\left(\sqrt{\mathrm{td}\,\overline{M}}\right)_{4n-4}\\
&= \sigma\left(c\,\alpha^2.\left(\sqrt{\mathrm{td}\,\overline{M}}\right)_{4n-4}\right)\\
&= q(\alpha).
\end{aligned}
$$

Note that the second equality follows from the fact that the intersection pairing is Galois equivariant when the cohomology has been twisted into weight zero, and the last equality follows because the Galois action on $H^{4n}(\overline{M},\mathbb{Q}_\ell(2n))$ is trivial.

This shows that we have a map $\mathrm{Gal}(\overline{k}/k) \to \mathrm{O}(q)$, so we can consider the following diagram:

$$\begin{tikzcd} \mathrm{Gal}(\overline{k}/k) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)) \\ & \mathrm{O}(q) \arrow[ur] & \end{tikzcd}$$

To see that the diagram commutes, let $\sigma \in \mathrm{Gal}(\overline{k}/k)$ map to $A \in \mathrm{O}(q)$, and consider $\beta \in H^{2i}(\overline{M}, \mathbb{Q}_\ell(i))$. Since h is surjective, there exists some $\gamma \in R(v)$ such that $\beta = h(\gamma)$. Note that h is Galois equivariant because u_v is defined over the base field and hence is Galois invariant. This means $\sigma \cdot \beta = \sigma \cdot h(\gamma) = h(\sigma \cdot \gamma)$. Now, $\gamma \in R(v)$ is built out of elements of $H^2(\overline{M}, \mathbb{Q}_\ell(1))$ and elements of $\mathbb{Q}_\ell$, and so $\sigma \cdot \gamma$ is determined by $A \cdot \gamma$ where A acts on the separate components of γ. Since h is also $\mathrm{O}(q)$-equivariant, it follows that $h(A \cdot \gamma) = A \cdot h(\gamma) = A \cdot \beta$. Putting all of this together gives $\sigma \cdot \beta = A \cdot \beta$. Thus, the action of $\mathrm{Gal}(\overline{k}/k)$ on $\widetilde{H}(\overline{M}, \mathbb{Q}_\ell)$ factors through $\mathrm{O}(q)$. $\square$

3.7. An isomorphism of the cohomology rings

In this section we complete the proof of Theorem 2. Let us consider a fixed K3 surface S and a moduli space M of stable sheaves on S with an effective and geometrically primitive Mukai vector v. If $v^2 = 0$, Theorem 2 was proven in Section 3.3 (see Remark 3.3.3). Assume now that $v^2 > 0$. We will continue to use the notation introduced in the proof of Proposition 3.3.1 that $w = (1, 0, 1-n) \in N(S)$ where $n = \frac{1}{2}(v^2 + 2)$. We follow [40, Sec. 3.4] to produce an isomorphism between the cohomology rings of $\overline{M}$ and $\overline{S}^{[n]}$ by constructing a class in the middle cohomology of $\overline{M} \times \overline{S}^{[n]}$. This class will depend on the choice of an isometry

$g\colon \widetilde{H}(\overline{S}, \mathbb{Q}_\ell) \xrightarrow{\sim} \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$ such that $g(v) = w$, and we will specifically use the reflection constructed in Proposition 3.3.1.

We outline here what Markman does to produce the desired ring isomorphism, where he starts with a complex projective K3 surface and an isometry on $H^*_{sing}(S, \mathbb{Z})$. By considering an integral isometry, the cohomology class produced is an element of $H^{2n}(M \times S^{[n]}, \mathbb{Z})$, and then Markman shows that this class induces a ring isomorphism

$$H^*_{sing}(M, \mathbb{Z}) \xrightarrow{\sim} H^*_{sing}(S^{[n]}, \mathbb{Z}).$$

Since we will start with an isometry over $\mathbb{Q}_\ell$, the resulting class, and hence the map on cohomology, will also be defined over $\mathbb{Q}_\ell$. We will make a density argument to show that this map on $\mathbb{Q}_\ell$-cohomology is also an isomorphism.

In order to produce a map $H^*(\overline{M}, \mathbb{Q}_\ell) \to H^*(\overline{S}^{[n]}, \mathbb{Q}_\ell)$, we would like to compose cohomological Fourier-Mukai transforms with the isometry g. First, we have the map $H^*(\overline{M}, \mathbb{Q}_\ell) \to H^*(\overline{S}, \mathbb{Q}_\ell)$ induced by the class u_v in the cohomology of $\overline{S} \times_{\overline{k}} \overline{M}$, where u_v is the pullback from $S \times_k M$ to $\overline{S} \times_{\overline{k}} \overline{M}$ of a normalization of $v(\mathcal{U})(\operatorname{td} M)^{-1/2}$, defined in [40, Eq. (3.4)]. This is followed by $g\colon \widetilde{H}(\overline{S}, \mathbb{Q}_\ell) \to \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$, and the last map $H^*(\overline{S}, \mathbb{Q}_\ell) \to H^*(\overline{S}^{[n]}, \mathbb{Q}_\ell)$ is induced by the class u_w, defined analogously to u_v. The resulting morphism can be described using a cohomology class given below.

For a projective variety X, consider the universal polynomial map

$$l\colon \oplus_i H^{2i}(X, \mathbb{Q}_\ell) \to \oplus_i H^{2i}(X, \mathbb{Q}_\ell)$$

taking the Chern character of a sheaf to its total Chern class. That is,

$$l(r + a_1 + a_2 + \cdots) = 1 + a_1 + \left(\frac{1}{2}a_1^2 - a_2\right) + \cdots .$$

Let π_{ij} be the projection from $\overline{M} \times \overline{S} \times \overline{S}^{[n]}$ onto the product of the i^{th} and j^{th} factors. We define

$$\gamma_g := c_{2n}\big(l(-\pi_{13*}[\pi_{12}^*((1 \otimes g)(u_v))^\vee \pi_{23}^*(u_w)])\big),$$

so that $\gamma_g \in H^{4n}(\overline{M} \times \overline{S}^{[n]}, \mathbb{Q}_\ell(2n))$, the middle cohomology group. For further discussion on this choice of cohomology class, see [40, Sec. 3.4].

Now consider the projections from $\overline{M} \times \overline{S}^{[n]}$:

$$\begin{tikzcd} & \overline{M} \times \overline{S}^{[n]} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \overline{M} & & \overline{S}^{[n]}. \end{tikzcd}$$

We also let γ_g denote the induced map $H^*(\overline{M}, \mathbb{Q}_\ell) \to H^*(\overline{S}^{[n]}, \mathbb{Q}_\ell)$ given by

$$\alpha \mapsto p_*(q^*(\alpha) \cdot \gamma_g).$$

Proposition 3.7.1. *Let S be a K3 surface defined over an arbitrary field k and $v \in N(S)$ an effective and geometrically primitive Mukai vector of length $v^2 > 0$ with a v-generic polarization $H \in \mathrm{NS}(S)$. Let $g\colon \widetilde{H}(\overline{S}, \mathbb{Q}_\ell) \to \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$ denote the isometry produced in the proof of Proposition 3.3.1. Then the map $\gamma_g\colon H^*(\overline{M}, \mathbb{Q}_\ell) \to H^*(\overline{S}^{[n]}, \mathbb{Q}_\ell)$ is a Galois equivariant ring isomorphism.*

Proof. We begin by assuming that $k = \mathbb{C}$ and the cohomology is singular cohomology. Let $I := \mathrm{Isom}(\widetilde{H}(S), v, w)$ be the subvariety of $\mathbb{A}^{24\times 24}_{\mathbb{Q}_\ell}$ consisting of isometries $\widetilde{H}(S) \to \widetilde{H}(S)$ which send v to w. Similarly, let $\mathrm{Hom}(H^*(M), H^*(S^{[n]}))$ be the affine variety of graded vector space homomorphisms from $H^*(M)$ to $H^*(S^{[n]})$. Then we get a map of varieties

$$\Psi\colon I \to \mathrm{Hom}(H^*(M), H^*(S^{[n]}))$$

sending an isometry g to the map γ_g defined above. Consider the subspace Z of I containing all those isometries g such that γ_g is a ring homomorphism. We will show that $Z = I$. Observe that γ_g being a ring homomorphism is a closed condition so $\Psi(Z) \subset \mathrm{Hom}(H^*(M), H^*(S^{[n]}))$ is closed. Since Z is the preimage under Ψ of a closed subspace, $Z \subset I$ is closed.

Given a $\mathbb{Z}$-point $g\colon \widetilde{H}(S,\mathbb{Z}) \to \widetilde{H}(S,\mathbb{Z})$ of I, by [40, Thm. 3.10] the map $\gamma_g\colon H^*(M,\mathbb{Z}) \to H^*(S^{[n]},\mathbb{Z})$ is a ring homomorphism, so Z contains all of the $\mathbb{Z}$-points of I, $I(\mathbb{Z})$. By Lemma 3.7.2 below, we see that the $\mathbb{Z}$-points of I are Zariski dense in I, i.e $\overline{I(\mathbb{Z})} = I$. Since Z is closed, $\overline{I(\mathbb{Z})} = I \subset Z$. Thus, we conclude that every morphism γ_g for $g \in I$ is a ring homomorphism.

Next, we claim that in fact every homomorphism in $\mathrm{Im}(\Psi)$ is a ring isomorphism. We consider the algebraic map

$$I \to \mathrm{Hom}(H^*(M), H^*(M))$$

sending $g \mapsto \gamma_{g^{-1}}\gamma_g$, where $\gamma_{g^{-1}}$ is defined analogously to γ_g for $g^{-1} \in \mathrm{Isom}(\widetilde{H}(S), w, v)$. Then the subspace

$$\{g : \gamma_{g^{-1}}\gamma_g - \mathrm{Id} = 0\} \subset I$$

is again closed because it is the preimage of a closed subspace in $\mathrm{Hom}(H^*(M), H^*(M))$. When g is a $\mathbb{Z}$-point of I, by [40, Lem. 3.12] we know that $\gamma_{g^{-1}}\gamma_g = \gamma_{g^{-1}g} = \gamma_{\mathrm{Id}} = \mathrm{Id}$. Thus this closed subspace contains all of the $\mathbb{Z}$-points of I. Again using Lemma 3.7.2, the $\mathbb{Z}$-points of I are Zariski dense in I, so we conclude that $\gamma_{g^{-1}}\gamma_g = \mathrm{Id}$ for all $g \in I$. The same argument shows that $\gamma_f\gamma_{f^{-1}} = \gamma_{ff^{-1}} = \mathrm{Id}$ for all $f \in \mathrm{Isom}(\widetilde{H}(S), w, v)$, and hence every such γ_g is an isomorphism. In particular, the isometry g constructed in Proposition 3.3.1 is a $\mathbb{Q}_\ell$-point of I and therefore γ_g is an isomorphism. Lastly, the comparison theorem for singular and étale cohomology gives the ring isomorphism on étale cohomology, $\gamma_g \colon H^*(M, \mathbb{Q}_\ell) \xrightarrow{\sim} H^*(S^{[n]}, \mathbb{Q}_\ell)$.

For k an arbitrary field, we proceed as in the proof of Proposition 3.2.1, using the Lefschetz principle and the lifting argument for fields of characteristic zero and $p > 0$, respectively, to conclude that γ_g remains an isomorphism.

To see that γ_g is Galois equivariant, we observe that both u_v and u_w are Galois invariant, and all of the other operations in the construction of the class

γ_g are Galois equivariant. That is, for $\sigma \in \mathrm{Gal}(\overline{k}/k)$ and $\alpha \in H^*(M, \mathbb{Q}_\ell)$,

$$\begin{aligned}
\sigma\gamma_g &= \sigma c_{2n}\big(l(-\pi_{13*}[\pi_{12}^*((1\otimes g)(u_v))^\vee \pi_{23}^*(u_w)])\big) \\
&= c_{2n}\big(l(-\pi_{13*}[\pi_{12}^*((1\otimes g)(\sigma u_v))^\vee \pi_{23}^*(\sigma u_w)])\big) \\
&= c_{2n}\big(l(-\pi_{13*}[\pi_{12}^*((1\otimes g)(u_v))^\vee \pi_{23}^*(u_w)])\big) \\
&= \gamma_g,
\end{aligned}$$

and so

$$\begin{aligned}
\gamma_g(\sigma\alpha) &= p_*(q^*(\sigma\alpha)\cdot\gamma_g) \\
&= p_*(q^*(\sigma\alpha)\cdot\sigma\gamma_g) \\
&= \sigma p_*(q^*(\alpha)\cdot\gamma_g).
\end{aligned}$$

Hence the resulting morphism γ_g is equivariant. □

Lemma 3.7.2. *Using the notation introduced above, the $\mathbb{Z}$-points of* $\mathrm{Isom}(\widetilde{H}(S), v, w)$ *are Zariski dense.*

Proof. Let $I = \mathrm{Isom}(\widetilde{H}(S), v, w) \subset \mathbb{A}_{\mathbb{C}}^{24\times 24}$. We claim that $\overline{I(\mathbb{Z})} = I$. Consider $I_{\mathbb{R}} \subset \mathbb{A}_{\mathbb{R}}^{24\times 24}$, which is a torsor over $\mathrm{Stab}(v)_{\mathbb{R}} := \{A \in \mathrm{O}(\widetilde{H}(S)) : Av = v\} \subset \mathbb{A}_{\mathbb{R}}^{24\times 24}$, and observe that $\mathrm{Stab}(v)_{\mathbb{R}}$ is isomorphic to $\mathrm{O}(v^\perp) \cong \mathrm{O}(3,20) \subset \mathbb{A}_{\mathbb{R}}^{23\times 23}$ as group schemes over $\mathbb{Q}$. By [44, Thm. 5.1.11] and discussed in Section 2.8, the $\mathbb{Z}$-points, which we let $\mathrm{Stab}(v)_{\mathbb{Z}}$ denote, form a lattice in $\mathrm{Stab}(v)_{\mathbb{R}}$. The proof of [44, Cor. 4.5.6] shows that the connected component of the Zariski closure of $\mathrm{Stab}(v)_{\mathbb{Z}}$ is equal to the connected component of $\mathrm{Stab}(v)_{\mathbb{R}}$, which we denote by $\mathrm{Stab}(v)_{\mathbb{R}}^\circ$. So $\overline{\mathrm{Stab}(v)_{\mathbb{Z}}}^\circ = \mathrm{Stab}(v)_{\mathbb{R}}^\circ$. Since $\mathrm{Stab}(v)_{\mathbb{R}} \cong \mathrm{O}(3,20)$, it follows that $\mathrm{Stab}(v)_{\mathbb{R}}^\circ \cong \mathrm{SO}(3,20) \subset \overline{\mathrm{Stab}(v)_{\mathbb{Z}}}$. Recall that $v^\perp \cong (-E_8)^{\oplus 2} \oplus U^{\oplus 3} \oplus \langle 2-2n\rangle$, so $\mathrm{Stab}(v)_{\mathbb{Z}}$

also contains a point of determinant -1. The smallest algebraic group containing both $\mathrm{Stab}(v)^{\circ}_{\mathbb{R}}$ and this point of determinant -1 is $\mathrm{Stab}(v)_{\mathbb{R}}$, since $\mathrm{SO}(3,20)$ is the only index two subgroup of $\mathrm{O}(3,20)$. Thus in fact $\mathrm{Stab}(v)_{\mathbb{R}} \subset \overline{\mathrm{Stab}(v)_{\mathbb{Z}}}$. Finally, we observe that $\mathrm{Stab}(v)_{\mathbb{R}}$ is Zariski dense in its complexification $\mathrm{Stab}(v)_{\mathbb{C}} \subset \mathbb{A}^{24\times 24}_{\mathbb{C}}$ (see [44, Rmk. 18.1.8(3)]), and so $\mathrm{Stab}(v)_{\mathbb{Z}}$ is Zariski dense in $\mathrm{Stab}(v)_{\mathbb{C}} \cong \mathrm{O}(23,\mathbb{C})$. Since I is a torsor over $\mathrm{Stab}(v)_{\mathbb{C}}$, when we consider $I \subset \mathbb{A}^{24\times 24}_{\mathbb{C}}$, we see that $\overline{I(\mathbb{Z})} = I$. Then under a choice of inclusion $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, we have $I(\mathbb{Q}_\ell) \subset \overline{I(\mathbb{Z})}$. $\square$

CHAPTER IV

EXAMPLES AND COMPUTATIONS

In this chapter, we give a variety of examples and computations which provide additional context and applications of the results in Chapter III. In Section 4.1, we give an explicit example that demonstrates why Proposition 3.3.1 cannot be strengthened and interpret the example in terms of birationality. In Section 4.2, we show that we should not expect the moduli space of sheaves $M_H(v)$ to be defined as a variety over a field k if H is not also defined over that field. In Section 4.3, we do not directly apply any new results, but we demonstrate how to compute the zeta function of $S^{[3]}$ for S a K3 surface defined over a finite field $\mathbb{F}_q$. Then in Sections 4.4 and 4.5, we use the results from Chapter III to explicitly show that the zeta function of a six-dimensional moduli space of sheaves on a K3 surface defined over $\mathbb{F}_q$ has the same zeta function as $S^{[3]}$.

4.1. Two moduli spaces which are likely not birational

We observe that Proposition 3.3.1 need not hold when $\mathbb{Q}_\ell$ is replaced with $\mathbb{Z}_\ell$, as demonstrated by the following example. We consider the degree two K3 surface S defined over $\mathbb{F}_2$ in [21, Ex. 6.1], which is defined by the vanishing of

$$w^2 + w(x^2y + y^3 + y^2z) + x^5z + x^3y^2z + x^2y^3z + x^3yz^2 + x^2y^2z^2 + y^2z^4 + xz^5 + yz^5 + z^6$$

in $\mathbb{P}_{\mathbb{F}_2}(1,1,1,3)$, weighted projective space. By the proof of [21, Prop. 6.3] has $\operatorname{rank}(\mathrm{NS}(\overline{S})) = 2$. In particular, Hassett, Várilly-Alvarado, and Varilly find two independent classes in $\mathrm{NS}(S)$ on which the intersection pairing has discriminant

-5, which means the classes span $\mathrm{NS}(S)$ (a similar example was discussed in more detail in Example 3.1.4).

In this case, Mukai vectors are elements of $N(S) \cong \mathbb{Z}^4$, and we can consider the geometrically primitive and effective Mukai vector $v = (5, 2, 3, 0)$ in $N(S)$. Since $\mathrm{rank}(\mathrm{NS}(S)) = \mathrm{rank}(\mathrm{NS}(\overline{S}))$, there is a polarization which is generic with respect to v, and hence $M(v)$ is a 12-dimensional smooth projective variety. There is no $u \in N(S)$ such that $\langle u, v \rangle = 1$, so we do not expect $M(v)$ to be a fine moduli space. If there is an isometry $v^\perp \cong H^2(\overline{S}, \mathbb{Z}_\ell(1)) \oplus \mathbb{Z}_\ell$, then we can restrict it to the subspace of Galois invariants. The proof of [21, Prop. 6.3] also shows that the only invariant classes in $H^2(\overline{S}, \mathbb{Z}_\ell(1))$ are those in $\mathrm{NS}(S)$, and so the sublattice $H^2(\overline{S}, \mathbb{Z}_\ell(1))^{\mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)} \oplus \mathbb{Z}_\ell$ has discriminant 50. It can be checked that the pairing on $(v^\perp)^{\mathrm{Gal}(\overline{\mathbb{F}}_2/\mathbb{F}_2)}$ is

$$\begin{pmatrix} -2 & 3 & -1 \\ 3 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

which has discriminant 2. For these lattices to be isomorphic, the discriminants must differ by the square of a unit, but when $\ell = 5$, this is not the case. So $v^\perp \ncong H^2(\overline{S}, \mathbb{Z}_5(1)) \oplus \mathbb{Z}_5$ as sublattices of $\widetilde{H}(\overline{S}, \mathbb{Z}_5)$. By Proposition 3.3.1, it is only after tensoring with $\mathbb{Q}_5$ that these lattices become isomorphic.

This difference in coefficients is related to the question of whether the corresponding moduli space $M(v)$ is birational to the Hilbert scheme. If $w = (1, 0, 0, -5)$ in $N(S)$, then $M(w) = S^{[6]}$ and it is clear that $w^\perp = H^2(\overline{S}, \mathbb{Z}_\ell(1)) \oplus \mathbb{Z}_\ell\langle(1, 0, 0, 5)\rangle$. For hyperkähler varieties defined over the complex numbers, the Beauville-Bogomolov form and hence the resulting discriminant group is a birational invariant. While this result has not been proved over arbitrary fields, our calculations suggest that we have found two moduli spaces that are not birational.

4.2. Defining the polarization over a finite field extension

We show here that it is necessary for the polarization H to be defined over the base field k in order for the moduli space $M_H(v)$ to be a quasiprojective variety. In particular, we revisit the K3 surface discussed in Example 2.2.7. We saw that for $0 < \epsilon < \frac{1}{2}$, the sheaf $\mathcal{L}$ was geometrically stable with respect to $H' = H + \epsilon C_1$. Recall that in that example, the K3 surface and the polarization H are defined over $\mathbb{F}_3$.

Recall that the preimage $C \subset S$ of the tritangent line in $\mathbb{P}^2_{\mathbb{F}_3}$ is defined by

$$w^2 = 2y^2(x^2 + 2xy + 2y^2)^2,$$

and the right-hand side cannot be written as a square, since 2 is not a square modulo 3. This means that C does not split as C_1 and C_2 over $\mathbb{F}_3$. However, over $\mathbb{F}_9 = \mathbb{F}_3[\alpha]/(\alpha^2 + 1)$ we can write

$$w^2 = \big(\alpha y(x^2 + 2xy + 2y^2)\big)^2,$$

so that C_1 and C_2 are both defined over $\mathbb{F}_9$. Then by [35, Thm. 0.2], cited in Theorem 2.3.15, $M_{H'}(v)$ is a projective variety over $\mathbb{F}_9$.

We will argue that, on the other hand, $M_{H'}(v)$ cannot possibly be a quasi-projective scheme defined over $\mathbb{F}_3$. If it were, then after taking the base change, we would have

$$M_{H'}(v)_{\overline{\mathbb{F}}_3} \subset \mathbb{P}^N_{\overline{\mathbb{F}}_3},$$

for some N, and the Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_3/\mathbb{F}_3)$ would permute the points of $M_{H'}(v)_{\overline{\mathbb{F}}_3}$. Recall that by looking at the eigenvalues of Frobenius given in the proof of Proposition 5.5 in [20], we see that the Frobenius action swaps C_1 and C_2. When we apply σ^* to the sequence

$$0 \to \mathcal{O}_{C_2} \to \mathcal{L} \to \mathcal{O}_{C_1} \to 0,$$

we get

$$0 \to \mathcal{O}_{C_1} \to \sigma^* \mathcal{L} \to \mathcal{O}_{C_2} \to 0.$$

Note that $p_{H'}(\sigma^* \mathcal{L}) = p_{H'}(\mathcal{L})$, since $v(\sigma^* \mathcal{L}) = v(\mathcal{L})$. However, the calculations in Example 2.2.7 show that $p_{H'}(\mathcal{O}_{C_2}) < p_{H'}(\mathcal{L}) = p_{H'}(\sigma^* \mathcal{L})$, which makes $\mathcal{O}_{C_2}$ a destabilizing quotient of $\sigma^* \mathcal{L}$. Since $\sigma^* \mathcal{L}$ is unstable with respect to H', the Galois action moves points of the subscheme out of the scheme—a contradiction.

4.3. The zeta function of $\mathrm{Hilb}^3 S$ by counting points

The zeta function of the Hilbert scheme of points on a variety over a finite field $\mathbb{F}^q$ can be computed explicitly by understanding the types of points in $\mathrm{Hilb}^m S = S^{[m]}$. We demonstrate this by computing $Z(S^{[3]}, t)$ for S a K3 surface over $\mathbb{F}_q$, where $q = p^n$ for some prime p. By the Weil conjectures for K3 surfaces, discussed in Section 2.4.7, the eigenvalues of the Frobenius map $f := F^n \times \mathrm{id}\colon \overline{S} \to \overline{S}$ are $\{1\}, \{\alpha_1, ..., \alpha_{22}\}$, and $\{q^2\}$ on $H^0(S, \mathbb{Q}_\ell), H^2(S, \mathbb{Q}_\ell)$, and $H^4(S, \mathbb{Q}_\ell)$ respectively (see [28, 4.4.1]). This means

$$N_r := \#S(\mathbb{F}_{q^r}) = 1 + \sum_{i=1}^{22} \alpha_i^r + q^{2r}.$$

Let us consider what length 3 subschemes of S look like, so that we can count the points in $S^{[3]}(\mathbb{F}_{q^r})$. We can have the following:

1. three distinct points from $S(\mathbb{F}_{q^r})$; there are $\binom{N_r}{3}$ such points;

2. one point from $S(\mathbb{F}_{q^r})$ along with a distinct non-reduced point, i.e. a point of $S(\mathbb{F}_{q^r})$ along with a tangent direction; there are $N_r(N_r-1)(q^r+1)$ points of this type;

3. one point from $S(\mathbb{F}_{q^r})$ along with one point from $S(\mathbb{F}_{q^{2r}})$; there are $\dfrac{N_r(N_{2r}-N_r)}{2}$ such points, since the points from $S(\mathbb{F}_{q^{2r}})$ come in Galois-conjugate pairs when considered as length two subschemes of S, and we don't want to count the points already defined over $\mathbb{F}_{q^r}$ again;

4. one point from $S(\mathbb{F}_{q^{3r}})$; there are $\dfrac{N_{3r}-N_r}{3}$ points of this type, for the same reasoning as above;

5. one curvilinear point, i.e. a point from $S(\mathbb{F}_{q^r})$ along with tangent direction information of orders one and two; there are $N_r(q^r+1)q^r$ such points, by [3, Sec. IV.2];

6. one non-local complete intersection supported on a point from $S(\mathbb{F}_{q^r})$; there are N_r points of this kind.

The reference for types of points arising in $S^{[n]}$ which are supported on a single point of S is [3].

Overall, we find that

$$\begin{aligned}
\#S^{[3]}(\mathbb{F}_{q^r}) &= \binom{N_r}{3} + N_r(N_r-1)(q^r+1) + \frac{N_r(N_{2r}-N_r)}{2} \\
&\quad + \frac{N_{3r}-N_r}{3} + N_r(q^r+1)q^r + N_r \\
&= 1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r} \\
&\quad + (1+2q^r+2q^{2r}+2q^{3r}+q^{4r})\sum_{i=1}^{22}\alpha_i^r + (1+q^r+q^{2r})\sum_{i=1}^{22}\alpha_i^{2r} \\
&\quad + (1+2q^r+q^{2r})\sum_{i<j}\alpha_i^r\alpha_j^r + \sum_{i\le j\le k}\alpha_i^r\alpha_j^r\alpha_k^r
\end{aligned}$$

By the Weil conjectures (see Theorem 2.4.5), we know that

$$Z(S^{[3]},t) = \frac{P_1(t)P_3(t)\cdots P_{11}(t)}{P_0(t)P_2(t)\cdots P_{12}(t)},$$

where each $P_i(t) \in \mathbb{Z}[t]$, and each $P_i(t)$ factors as $\prod_j(1-\alpha_{ij}t)$ for some $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{i,j}| = q^{i/2}$ for all $1 \le i \le 2n-1$ and all j. This equivalently tells us that

$$\#S^{[3]}(\mathbb{F}_{q^r}) = \sum_{i=0}^{12}(-1)^i\sum_{j=1}^{b_i}\alpha_{i,j}^r,$$

where b_i is the degree of $P_i(t)$. Thus we should rather collect the terms in the sum above based on their length, and we will be able to write down the polynomials

$P_i(t)$. Doing this, we find that

$$\begin{aligned}\#S^{[3]}(\mathbb{F}_{q^r}) = 1 + q^r + \sum_{i=1}^{22} \alpha_i^r + 2q^{2r} + 2q \sum_{i=1}^{22} \alpha_i^r + \sum_{i\leq j} \alpha_i^r \alpha_j^r \\ + 2q^{3r} + 2q^{2r} \sum_{i=1}^{22} \alpha_i^r + 2q^r \sum_{i<j} \alpha_i^r \alpha_j^r + q^r \sum_{i=1}^{22} \alpha_i^{2r} + \sum_{i\leq j\leq k} \alpha_i^r \alpha_j^r \alpha_k^r \\ + 2q^{4r} + 2q^{3r} \sum_{i=1}^{22} \alpha_i^r + q^2 \sum_{i\leq j} \alpha_i^r \alpha_j^r + q^{5r} + q^{4r} \sum_{i=1}^{22} \alpha_i^r + q^{6r}\end{aligned}$$

Since every term in this expression appears with a positive sign, we can conclude that $H^i_{\acute{e}t}(\overline{S}^{[3]}, \mathbb{Q}_\ell) = 0$ and $P_i(t) = 1$ for all $1 \leq i \leq 11$ odd. Then we have the following:

- $P_0(t) = 1 - t$,
- $P_2(t) = (1 - qt) \prod_{i=1}^{22} (1 - \alpha_i t)$,
- $P_4(t) = (1 - q^2 t)^2 \prod_{i=1}^{22} (1 - q\alpha_i t)^2 \prod_{i\leq j} (1 - \alpha_i \alpha_j t)$,
- $P_6(t) = (1-q^3 t)^2 \prod_{i=1}^{22} (1-q^2 \alpha_i t)^2 \prod_{i<j} (1-q\alpha_i\alpha_j t)^2 \prod_{i=1}^{22} (1-q\alpha_i^2 t) \prod_{i\leq j\leq k} (1-\alpha_i\alpha_j\alpha_k t)$,
- $P_8(t) = (1 - q^4 t)^2 \prod_{i=1}^{22} (1 - q^3 \alpha_i t)^2 \prod_{i\leq j} (1 - q^2 \alpha_i \alpha_j t)$,
- $P_{10}(t) = (1 - q^5 t) \prod_{i=1}^{22} (1 - q^4 \alpha_i t)$, and
- $P_{12}(t) = 1 - q^6 t$.

Finally, we have that

$$Z(S^{[3]}, t) = \frac{1}{P_0(t) P_2(t) P_4(t) P_6(t) P_8(t) P_{10}(t) P_{12}(t)}.$$

4.4. The Galois representations arising from a six-dimensional moduli space

In Section 4.5, we will verify Theorem 1 explicitly for a six-dimensional moduli space on a fixed K3 surface S over $\mathbb{F}_q$ by showing that the zeta function of the moduli space is equal to that of $S^{[3]}$ computed in Section 4.3. In order to carry out this verification, we first give a concrete example of the Galois representations arising in the cohomology of the moduli space when S is defined over an arbitrary field k. This will also help illuminate and give context to the results presented in Chapter III. We follow the strategy laid out in [38, Ex. 14], where Markman decomposes $H^*(S^{[3]}, \mathbb{Q})$ into irreducible representations of the monodromy group. We can do the same for a general M of the same dimension.

Suppose $v^2 = 4$ so that $\dim M = 6$. Recall from Proposition 3.2.1 that $v^\perp \cong H^2(\overline{M}, \mathbb{Q}_\ell(1))$, and following the notation presented in Definition 3.5.1, $M_{2i} \cong \widetilde{H}(\overline{S}, \mathbb{Q}_\ell)$. As $G := \mathrm{SO}(q)$ representations, $H^2(\overline{M}, \mathbb{Q}_\ell(1))$ is the standard representation, which we will write as V, and $M_{2i} \cong V \oplus \mathbf{1}_G$. Then $R(v)$ is generated by the following in the given gradings:

Grading:	Generators:
2	V
4	$\mathrm{Sym}^2 V \oplus V \oplus \mathbf{1}_G$
6	$\mathrm{Sym}^3 V \oplus \mathrm{Sym}^2 V \oplus \bigwedge^2 V \oplus V^{\oplus 2} \oplus \mathbf{1}_G$

Note that in degree-six grading, we have generators $\mathrm{Sym}^3 V \oplus V \otimes M_4 \oplus M_6$, and so

$$V \otimes M_4 = V \otimes (V \oplus \mathbf{1}_G) = \mathrm{Sym}^2 V \oplus \bigwedge^2 V \oplus V$$

contributes one of the standard representations.

When it is convenient, we will continue to write $\mathrm{Sym}^2 V$ and $\mathrm{Sym}^3 V$, but we remark that these are not irreducible representations. If we let $u \in \mathrm{Sym}^2 V$ be the inverse form of the bilinear form q on V, then the related irreducible representations are $V(d) = \mathrm{Sym}^d V / u\,\mathrm{Sym}^{d-2} V$ [13, Ex. 19.21].

For the sake of notation, we write H^{2i} for $H^{2i}(\overline{M}, \mathbb{Q}_\ell(i))$. We can use Poincaré Duality to determine the Galois action on higher cohomology groups, so we need only consider the generators of $R(v)$ up to degree six. We see that

$$h_2 \colon R(v)_2 \xrightarrow{\sim} H^2,$$

$$h_4 \colon \mathrm{Sym}^2 V \oplus V \oplus \mathbf{1}_G \twoheadrightarrow H^4,$$

and

$$h_6 \colon \mathrm{Sym}^3 V \oplus \mathrm{Sym}^2 V \oplus \bigwedge^2 V \oplus V^{\oplus 2} \oplus \mathbf{1}_G \twoheadrightarrow H^6.$$

We can decompose the sources of h_4 and h_6 into irreducible G-representations, and then use a dimension-counting argument to determine which of these representations inject into H^4 and H^6, respectively. The dimensions of H^{2i} come from Göttsche's formula [15], and a table of dimensions for $\dim M \leq 18$ can be found in [38, Sec. 6].

By Verbitsky [56], $\mathrm{Sym}^2 V$ injects into $H^4(\overline{M}, \mathbb{Q}_\ell(2))$, which makes up 276 dimensions of the 299-dimensional $H^4(\overline{M}, \mathbb{Q}_\ell(2))$. Under h_4, V and $\mathbf{1}_G$ must separately inject into $H^4(\overline{M}, \mathbb{Q}_\ell(2))$ or map to zero. Since these along with $\mathrm{Sym}^2 V$ must jointly surject onto $H^4(\overline{M}, \mathbb{Q}_\ell(2))$, a simple dimension-counting argument shows that V (23-dimensional) must inject into the remaining 23 dimensions–its image must intersect the image of $\mathrm{Sym}^2 V = V(2) \oplus \mathbf{1}_G$ trivially, since they are different irreducible representations. Thus we have found a G-subrepresentation of $R(v)_4$ which maps isomorphically under h_4:

$$\mathrm{Sym}^2 V \oplus V \xrightarrow{\sim} H^4.$$

Next we consider

$$h_6\colon \mathrm{Sym}^3 V \oplus \mathrm{Sym}^2 V \oplus \bigwedge^2 V \oplus V^{\oplus 2} \oplus \mathbf{1}_G \twoheadrightarrow H^6.$$

Again by Verbitsky [56] we know that $\mathrm{Sym}^3 V \hookrightarrow H^6$, which accounts for 2300 of the 2554 dimensions of H^6. The remaining G-representations can be written as

$$V(2) \oplus \bigwedge^2 V \oplus V^{\oplus 2} \oplus \mathbf{1}_G^{\oplus 2},$$

which are irreducible representations of dimensions 275, 253, 23, and 1, respectively. Again by counting dimensions, the only way for the remaining irreducible representations to map injectively onto the remaining 254 dimensions of H^6 is for $\bigwedge^2 V$ and a copy of $\mathbf{1}_G \subset \mathbf{1}_G^{\oplus 2}$ to inject. Thus we have found a G-subrepresentation of $R(v)_6$ which maps isomorphically under h_6:

$$\mathrm{Sym}^3 V \oplus \bigwedge^2 V \oplus \mathbf{1}_G \xrightarrow{\sim} H^6.$$

Now, by Proposition 3.6.4, we know that $\mathrm{Gal}(\overline{k}/k)$ acts through $\mathrm{O}(q)$, so it follows that the Galois action is determined independently of the choice of the moduli space.

Note that $\mathbf{1}_G^{\oplus 2} \subset R(v)_6$, and hence there is a copy of $\mathbf{1}_G \subset \mathbf{1}_G^{\oplus 2}$ surjecting onto a one-dimensional subspace of $H^6(\overline{M}, \mathbb{Q}_\ell(3))$. It is plausible that when considering two such moduli spaces M_1 and M_2 on two K3 surfaces S_1 and S_2, there are different one-dimensional subspaces of $\mathbf{1}_G$ injecting into $H^6(\overline{M}_1, \mathbb{Q}_\ell(3))$ and $H^6(\overline{M}_2, \mathbb{Q}_\ell(3))$, respectively. Moreover, the inclusions of $\mathrm{Gal}(\overline{k}/k) \subset \mathrm{O}(q)$ may be different for these two moduli spaces, so there may be concern that $H^6(\overline{M}_1, \mathbb{Q}_\ell(3))$ and $H^6(\overline{M}_2, \mathbb{Q}_\ell(3))$ are not isomorphic as Galois representations. However, by the results of Chapter III, we see that the resulting representations do not depend on these possible differences.

For similar examples of determining the cohomology as $\mathrm{SO}(q)$-representations, we direct the reader to [18] for computations in dimensions four and six, and [1] for a computation in dimension eight.

4.5. The zeta function of a six-dimensional moduli space by studying Galois representations

With the work of Section 4.4 in hand, we are now ready to compute the zeta function $Z(M_H(v), t)$ for any choice of $v \in \{w \in \widetilde{H}^*(\overline{S}, \mathbb{Z}_\ell) : w^2 = 4\}$ such that $M = M_H(v)$ is a smooth projective variety over $\mathbb{F}_q$. In order to do this, we will use what we have determined above about the cohomology of M as Galois representations. Recall from the discussion in Section 2.5 that the action of the Frobenius f^*, whose inverse is in the Galois group, determines $Z(M, t)$. Thus, we claim that the eigenvalues of

$$f^*\colon H^i(\overline{M}, \mathbb{Q}_\ell) \to H^i_{\acute{e}t}(\overline{M}, \mathbb{Q}_\ell),$$

for each i do not depend on the choice of the Mukai vector v. To see this, we will compute the eigenvalues, and hence the polynomials $P_i(t)$ so as to write

$$Z(M,t) = \frac{1}{P_0(t)P_2(t)P_4(t)P_6(t)P_8(t)P_{10}(t)P_{12}(t)}.$$

As noted in Remark 3.3.3, $H^i(\overline{M},\mathbb{Q}_\ell) = 0$ for odd i, which is why the numerator of $Z(M,t)$ is one.

For $i = 0$, $f^* = \mathrm{id}$ on $H^0(\overline{M},\mathbb{Q}_\ell)$, and so $P_0(t) = 1 - t$. For $i = 12$, $f^* = q^6 \cdot \mathrm{id}$ on $H^{12}(\overline{M},\mathbb{Q}_\ell)$ because F^n (where $q = p^n$ and F is the absolute Frobenius) is a finite morphism of degree q^6. This means $P_{12}(t) = 1 - q^6 t$. Note that both of these polynomials are actually prescribed by the Weil conjectures (see Theorem 2.4.5).

Next we consider $i = 2$, for which we know by Proposition 3.2.1 that $v^\perp \cong H^2(\overline{M},\mathbb{Q}_\ell(1))$. By Proposition 3.3.1, there is an isomorphism $v^\perp \cong H^2(\overline{S},\mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell$, and so

$$H^2(\overline{M},\mathbb{Q}_\ell) \cong v^\perp(-1) \cong H^2(\overline{S},\mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell(-1).$$

By the Weil conjectures for K3 surfaces, discussed in Example 2.4.7, the eigenvalues of f^* on $H^2(\overline{S},\mathbb{Q}_\ell)$ are $\{\alpha_1, ..., \alpha_{22}\}$, and the eigenvalue of f^* on $\mathbb{Q}_\ell(-1)$ is q. Thus,

$$P_2(t) = (1 - qt)\prod_{i=1}^{22}(1 - \alpha_i t).$$

Now let $i = 4$. We saw in Section 4.4 that

$$H^4(\overline{M},\mathbb{Q}_\ell(2)) \cong \mathrm{Sym}^2(H^2(\overline{M},\mathbb{Q}_\ell(1))) \oplus H^2(\overline{M},\mathbb{Q}_\ell(1)).$$

This means the eigenvalues of f^* on $H^4(\overline{M}, \mathbb{Q}_\ell(2))$ are $\left\{1, \frac{\alpha_i\alpha_j}{q^2}, \frac{\alpha_i}{q}\right\}$, for $1 \leq i \leq j \leq 22$, coming from $\mathrm{Sym}^2 H^2(\overline{M}, \mathbb{Q}_\ell(1))$, and $\left\{\frac{\alpha_i}{q}, 1\right\}$, for $1 \leq i \leq 22$, coming from $H^2(\overline{M}, \mathbb{Q}_\ell(1))$. We scale all of these values by q^2 to get the eigenvalues on $H^4(\overline{M}, \mathbb{Q}_\ell)$, which gives

$$P_4(t) = (1 - q^2t)^2 \prod_{i=1}^{22} (1 - q\alpha_i t)^2 \prod_{i \leq j} (1 - \alpha_i\alpha_j t).$$

To compute the eigenvalues of f^* on $H^6(\overline{M}, \mathbb{Q}_\ell)$, we again use the work from Section 4.4, which gave that

$$H^6(\overline{M}, \mathbb{Q}_\ell(3)) \cong \mathrm{Sym}^3(H^2(\overline{M}, \mathbb{Q}_\ell(1))) \oplus \bigwedge^2 H^2(\overline{M}, \mathbb{Q}_\ell(1)) \oplus \mathbb{Q}_\ell.$$

Then the eigenvalues of f^* on $H^6(\overline{M}, \mathbb{Q}_\ell(3))$ are $\left\{\frac{\alpha_i\alpha_j\alpha_k}{q^3}, \frac{\alpha_i\alpha_j}{q^2}, \frac{\alpha_i}{q}, 1\right\}$, for $1 \leq i \leq j \leq k \leq 22$, coming from $\mathrm{Sym}^3 H^2(\overline{M}, \mathbb{Q}_\ell(1))$, $\left\{\frac{\alpha_i\alpha_j}{q^2}, \frac{\alpha_i}{q}\right\}$, for $1 \leq i < j \leq 22$, coming from $\bigwedge^2 H^2(\overline{M}, \mathbb{Q}_\ell(1))$, and $\{1\}$ coming from $\mathbb{Q}_\ell$. To get the eigenvalues on $H^6(\overline{M}, \mathbb{Q}_\ell)$, we scale all of these by q^3 and thus get that

$$P_6(t) = (1 - q^3t)^2 \prod_{i=1}^{22} (1 - q^2\alpha_i t)^2 \prod_{i<j} (1 - q\alpha_i\alpha_j t)^2 \prod_{i=1}^{22} (1 - q\alpha_i^2 t) \prod_{i \leq j \leq k} (1 - \alpha_i\alpha_j\alpha_k t).$$

For $i = 8$ and $i = 10$, the functional equation from the Weil conjectures gives the following equality of eigenvalues from f^*:

$$\{\beta_{i,j}\}_j = \left\{\frac{q^6}{\beta_{12-i,j}}\right\}_j,$$

for each $0 \leq i \leq 12$ and j running over the eigenvalues of f^* on $H^i_{\acute{e}t}(\mathcal{M}_H(v), \mathbb{Q}_\ell)$. This completely determines the eigenvalues of f^* when $i = 8$ and $i = 10$, since the sets $\{\beta_{4,j}\}_j$ and $\{\beta_{2,j}\}_j$ are determined above, respectively. Now, by the Weil conjectures for K3 surfaces, there is an equality of sets

$$\{\alpha_1, ..., \alpha_{22}\} = \left\{ \frac{q^2}{\alpha_1}, ..., \frac{q^2}{\alpha_{22}} \right\}.$$

This allows us to write the eigenvalues of f^* on $H^8(\overline{M}, \mathbb{Q}_\ell)$ and $H^{10}(\overline{M}, \mathbb{Q}_\ell)$ nicely, in the following sense. We know, for example, that $\{q\alpha_1, ..., q\alpha_{22}\}$ is a subset of the set of eigenvalues of f^* on $H^4(\overline{M}, \mathbb{Q}_\ell)$, which means $\left\{ \frac{q^6}{q\alpha_1}, ..., \frac{q^6}{q\alpha_{22}} \right\}$ is a subset of the eigenvalues of f^* on $H^8(\overline{M}, \mathbb{Q}_\ell)$. We can rewrite this as

$$\left\{ \frac{q^5}{\alpha_1}, ..., \frac{q^5}{\alpha_{22}} \right\} = \{q^3\alpha_1, ..., q^3\alpha_{22}\}.$$

Doing this for all of the eigenvalues for $i = 8$ and $i = 10$, we find that

$$P_8(t) = (1 - q^4 t)^2 \prod_{i=1}^{22} (1 - q^3 \alpha_i t)^2 \prod_{i \leq j} (1 - q^2 \alpha_i \alpha_j t),$$

and

$$P_{10}(t) = (1 - q^5 t) \prod_{i=1}^{22} (1 - q^4 \alpha_i t).$$

We have now computed $P_{2i}(t)$ for all $i = 0, ..., 6$ and the polynomials all agree with those computed in Section 4.3. This allows us to conclude that

$$Z(M, t) = Z(S^{[3]}, t),$$

regardless of the choice of $v \in \{w \in \widetilde{H}^*(\overline{S}, \mathbb{Z}_\ell) : w^2 = 4\}$.

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