RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS

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Abstract. These are survey notes on rationality of threefolds over non-closed fields, written for the Notre Dame 2023 Thematic Program on Rationality and Hyperbolicity. We focus on the intermediate Jacobian obstruction to rationality, first introduced by Clemens and Griffiths over $\mathbb{C}$. This obstruction was extended to arbitrary fields via the study of curve classes. Over non-closed fields, it was recently refined to an obstruction given by torsors under the intermediate Jacobian. We explain this refinement and discuss examples in which this refined obstruction does and does not determine rationality.

Introduction

Let X be a smooth projective variety of dimension n over a field k. We often want to understand how close X is to projective space. There are many different notions to measure this.

- X is **k-rational** if X is birational over k to projective space $\mathbb{P}^n_k$.
- X is **stably k-rational** if there is some integer m such that $X \times \mathbb{P}^m_k$ is rational.
- X is **k-unirational** if there is some integer m and a dominant rational map $\mathbb{P}^m_k \dashrightarrow X$. (By restricting this map to a general linear subspace of dimension n, we can assume $m = n$.)

Note that for each of these definitions, we can give an equivalent definition in terms of the function field $k(X)$ (since each notion is only determined up to birational isomorphism).

It is clear that a rational variety is stably rational, by taking $m = 0$. If X is stably rational, so there is a birational map $\mathbb{P}^{n+m} \overset{\sim}{\dashrightarrow} X \times \mathbb{P}^m$ for some m, this map can be post-composed with the projection $X \times \mathbb{P}^m \to X$, giving a dominant rational map from $\mathbb{P}^{n+m}$ to X. In summary, the following implications hold:

$$\text{rational} \implies \text{stably rational} \implies \text{unirational}. \tag{1}$$

In fact, all of these notions are equivalent in dimension one, and also in dimension two when k has characteristic zero (see Section 1.1). There are examples in dimension three of varieties which are stably rational but not rational, as well as those which are unirational but not stably rational.

There are other related notions, like being retract rational or rationally connected. We will focus mainly on rationality, using facts about unirational varieties along the way, so we leave out these related notions for clarity.

The goal of these lecture notes is to give an introduction to the study of rationality of geometrically rational threefolds over non-algebraically closed fields. The key tool, the intermediate Jacobian and torsors under it, was first studied as an obstruction to rationality over $\mathbb{C}$. It was shown that the intermediate Jacobian describes algebraically trivial curve

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classes (or at least provides information about such classes), and this connection has been exploited in order to generalize the rationality obstruction over $\mathbb{C}$ to non-closed fields.

Outline. In Section 1, we compare the different measures of rationality and discuss some key examples of rational and unirational varieties. In Section 2, we introduce the intermediate Jacobian and how it can obstruct rationality for smooth complex projective threefolds. We use it to show the irrationality of smooth cubic threefolds. In Section 3, in an attempt to find a replacement over fields other than $\mathbb{C}$ for the intermediate Jacobian and its rationality obstruction, we connect the intermediate Jacobian to curve classes. In Section 4, we discuss the algebraic representative for codimension 2 cycles in two key examples: the smooth complete intersection of two quadrics and smooth conic bundle threefolds. In Section 5, we finally see the intermediate Jacobian obstruction to rationality over non-closed fields, briefly introduce torsors, and then discuss a refinement of the obstruction given by torsors under the intermediate Jacobian (called the intermediate Jacobian torsor obstruction). Finally, in Section 6, we discuss various examples where the intermediate Jacobian torsor obstruction has been used to study rationality of threefolds over non-closed fields.

Relation to existing literature. There are many beautiful surveys on rationality, and inspiration for these notes was drawn heavily from those, for example [Bea16b], [Voi16], [AB17], [Pir18], [Deb23]. For a focus on conic bundles, see [Pro18]. The intermediate Jacobian has been discussed in many places, for example [Tju72], [Voi07a, Chapter 12], [GH94, Sections 2.6, 6.4].

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1. THE LÜROTH PROBLEM AND SOME RATIONALITY CONSTRUCTIONS

We assume throughout this section that k is an algebraically closed field of characteristic zero.

1.1. **Curves and surfaces.** In 1875, Lüroth proved in [Lür75] that a unirational smooth projective curve C over $k = \mathbb{C}$ is rational. This fact is now fairly standard, and holds over any field k. Indeed, given a dominant rational map $f\colon \mathbb{P}^1 \dashrightarrow C$, since $\mathbb{P}^1$ is smooth and C is projective, this extends to a surjective morphism $\mathbb{P}^1 \twoheadrightarrow C$. We can apply the Riemann-Hurwitz formula:

$$2g(\mathbb{P}^1) - 2 = d(2g(C) - 2) + (\text{ramification}),$$

where d is the degree of the morphism. The left-hand side is -2, and if $g(C) > 0$, then the right-hand side is ≥ 0. Thus, we must have $g(C) = 0$ and hence $C \cong \mathbb{P}^1$.

We see, then, that the three measures of rationality (rationality, stable rationality, and unirationality) all agree in dimension 1. This observation for curves led to what is now called *the Lüroth problem*: is every unirational variety rational?

Let's consider what happens in dimension 2. Let S be a smooth projective surface, and suppose that there is a dominant rational map $\mathbb{P}^2 \dashrightarrow S$. We claim that, as was the case for curves, this implies that S is rational. We'll make use of the following general fact. Recall that $\Omega^q_X := \bigwedge^q \Omega_X$.

Theorem 1.1. *Let X and Y be smooth projective varieties with* $\dim X = \dim Y$*, and* $X \dashrightarrow Y$ *a dominant rational map. Then*
$$h^0(X, (\Omega_X^q)^{\otimes m}) \geq h^0(Y, (\Omega_Y^q)^{\otimes m}),$$
for all $q, m \geq 0$.

Before proving this theorem, let's see how this plays out in the case of unirational surfaces with $\mathbb{P}^2 \dashrightarrow S$. On $\mathbb{P}^2$, we have $h^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) = h^0(\mathbb{P}^2, (\Omega^2_{\mathbb{P}^2})^{\otimes 2}) = 0$. By Theorem 1.1, the same vanishing must hold for S. Now, Castelnuovo's rationality criterion says that this vanishing exactly implies rationality of S (a proof of Castelnuovo's rationality criterion can be found in [Bea83, Chapter V]). Thus, any unirational smooth projective surface is rational.

These low dimensional examples give credence to Lüroth's problem, but we'll soon see examples of threefolds which are unirational but not rational. In another direction, there do exist unirational irrational surfaces in positive characteristic. Shioda in [Shi74] gives examples which are hypersurfaces in $\mathbb{P}^3$.

Proof of Theorem 1.1. The proof follows the proof given in [Har77, Theorem II.8.19], which proves that the geometric genus is a birational invariant.

Let $U \subset X$ be the largest open set on which there is a morphism $f\colon U \to Y$ which represents the rational map $X \dashrightarrow Y$. The valuative criterion of properness implies that $\operatorname{codim}_X(X \setminus U) \geq 2$. The morphism f gives an induced map $f^*\Omega_Y \to \Omega_U$, and since f is generically smooth, even after taking exterior powers and/or tensor products, we get an injection
$$f^*(\Omega_Y^q)^{\otimes m} \hookrightarrow (\Omega_U^q)^{\otimes m}.$$
This induces an inclusion on global sections $H^0(U, f^*(\Omega_Y^q)^{\otimes m}) \hookrightarrow H^0(U, (\Omega_U^q)^{\otimes m})$.

Now, on one hand, since f is dominant, it induces an inclusion
$$H^0(Y, (\Omega_Y^q)^{\otimes m}) \hookrightarrow H^0(U, f^*(\Omega_Y^q)^{\otimes m}).$$
On the other hand, since $\operatorname{codim}_X(X \setminus U) \geq 2$ and $(\Omega_X^q)^{\otimes m}$ is locally free, the restriction of global sections from X to U is an isomorphism:
$$\begin{tikzcd} H^0(U, f^*(\Omega_Y^q)^{\otimes m}) \arrow[r, hook] & H^0(U, (\Omega_U^q)^{\otimes m}) \\ H^0(Y, (\Omega_Y^q)^{\otimes m}) \arrow[u, hook] & H^0(X, (\Omega_X^q)^{\otimes m}) \arrow[u, "\wr"'] \end{tikzcd}.$$
This gives the desired inequality. □

Exercise 1.2. Explain how the valuative criterion of properness implies that a rational map $X \dashrightarrow Y$ between smooth projective varieties can be represented by a morphism $f\colon U \to Y$ with $\operatorname{codim}_X(X \setminus U) \geq 2$.

Exercise 1.3. Check that the same result holds in positive characteristic if you add the assumption that the birational map is separable. *(This is more of a remark than an exercise, but use it as an opportunity to reflect on the proof of the theorem.)*

We point out the following, which we will use frequently.

Corollary 1.4. *Suppose X is a smooth projective unirational variety. Then* $h^0(X, (\Omega_X^q)^{\otimes m}) = 0$ *for all* $q, m \geq 1$.

Finally, we remark that this gives another proof that unirational curves are rational.

1.2. **Some examples in dimension 3 (and higher).** To show that a variety is rational, we must often exhibit a birational isomorphism $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$. Similarly, to show unirationaly, we must exhibit a dominant rational map $\mathbb{P}^n \dashrightarrow X$. We give a few examples of this here.

Example 1.5. Hyperplanes in projective space are automatically rational, so consider a smooth quadric hypersurface $Q \subset \mathbb{P}^{n+1}$. For a point $x \in Q$, projection from x gives a rational map $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$ which is dominant of degree 1 because any line through x not contained in Q meets Q in a unique second point.

Example 1.6. Let $X \subset \mathbb{P}^{n+2}$ be a smooth complete intersection of two quadrics, $X = Q_0 \cap Q_1$, with $n \geq 2$. We will show that X is rational.

First, we claim that X always contains a line $\ell \subset X$ (recall that $k = \bar{k}$). To see this, we can intersect X with an appropriate $\mathbb{P}^4 \subset \mathbb{P}^{n+2}$, so that $X \cap \mathbb{P}^4$ is a smooth complete intersection of two quadric threefolds. This intersection is a smooth del Pezzo surface of degree 4 (which is the blow-up of $\mathbb{P}^2$ at 5 points in general position; see [Bea83, Proposition IV.16]), so contains 16 lines. Thus, X also contains lines.

Projection from ℓ gives a rational map $\varphi\colon X \dashrightarrow \mathbb{P}^n$, which we will show is a birational isomorphism. For a plane P containing ℓ and not contained in X, the intersection

$$P \cap X = (Q_0 \cap P) \cap (Q_1 \cap P),$$

and each $Q_i \cap P$ is (generically) a union of two lines, $\ell \cup L_i$. The lines $L_0 \cap L_1$ meet in a unique point in X, which is the preimage of P under φ. Thus, $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$.

Exercise 1.7. Rather than relying on knowledge of del Pezzo surfaces, let's show directly that a smooth complete intersection of two quadric threefolds in $\mathbb{P}^4$ contains lines.

(1) Let $Q \subset \mathbb{P}^4$ be a smooth quadric threefold. Let $F_1(Q) \subset \mathrm{Gr}(2,5)$[1] be the Fano variety of lines in Q. Show that $\dim F_1(Q) = 3$.
Hint: Use the incidence correspondence $\Psi := \{(x, L) : x \in L \subset Q\} \subset Q \times F_1(Q)$. *Show that for* $x \in Q$*, the fiber of* $\Psi \to Q$ *over* x *is* 1*-dimensional. Show that for* $L \in F_1(Q)$*, the fiber of* $\Psi \to F_1(Q)$ *over* L *is* 1*-dimensional.*

(2) Show that for quadric threefolds $Q_0, Q_1 \subset \mathbb{P}^4$ which intersect transversally, we have $\dim F_1(Q_0) \cap F_1(Q_1) = 0$, so that there are finitely many lines contained in the intersection $Q_0 \cap Q_1$.
Can you show that the degree of each $F_1(Q_i) \subset \mathrm{Gr}(2,5)$ is 4, so that we recover the 16 lines mentioned above?

Example 1.8. Let X be a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geq 2$. We will show that X is unirational.

Again, X contains a line $\ell \subset X$: intersecting X with an appropriate $\mathbb{P}^3 \subset \mathbb{P}^n$ gives a smooth cubic surface which contains 27 lines.

Let T_X be the tangent bundle, so that $\mathbb{P}(T_X|_\ell)$ is the set of lines tangent to X through a point of ℓ. If we consider such a line m which (generically) is not contained in X, the intersection $X \cap m$ will be three points, but m already meets X at a point of ℓ with

[1]Note that we use affine dimensions for the Grassmannian throughout, so that $\mathrm{Gr}(2,5)$ parametrizes lines in $\mathbb{P}^4$.

multiplicity 2. This gives a third point $x \in X$, defining a rational map

$$\begin{aligned} \varphi \colon \mathbb{P}(T_X|_\ell) &\dashrightarrow X \\ m &\mapsto x. \end{aligned}$$

We next show that this map is dominant of degree 2. For a point $x \in X$, let $P \cong \mathbb{P}^2$ be the span of ℓ with x. If P is not contained in X, it meets X in a curve of degree 3 which is $\ell \cup C$ for a conic C (which may be reducible). Then C meets ℓ in two points x_1, x_2 (generically), and the line $m_i = \langle x, x_i \rangle$ meets X only at $\{x_i\} \cup \{x\}$, with $x_i \in \ell$ (since the intersection is contained in $\ell \cup C \subset P$). Since m_i meets X at x_i in both ℓ and C, m_i must have multiplicity 2 at x_i. This means m_i is a point of $\mathbb{P}(T_X|_\ell)$ and maps to x under φ. This shows that φ is dominant of degree 2.

Since we have just constructed a dominant rational map from projective space, this shows that X is unirational.

In the case that $n = 2$: we showed in Section 1.1 that if S is a unirational surface, then S is rational, so smooth cubic surfaces are rational. We give another proof of this fact in Exercise 1.11.

In the case that $n = 3$: we will show in Section 2.5 that smooth cubic threefolds are **not** rational, which shows that the Lüroth problem is false in general. In fact, it is not known if smooth cubic threefolds are stably rational. They are essentially the only theefolds for which stable rationality is unknown [HKT16, HT19, KO20].

In the case that $n = 4$: in the study of smooth cubic fourfolds, we have some examples which are rational, and no examples which are irrational. However, it is conjectured that the very general one is irrational. This is a huge open area of research; see for example the surveys [Has16, Huy19, MS19] for more information.

In the case that $n \geq 5$: There are no known examples of odd-dimensional smooth cubic hypersurfaces. We have some examples of rational smooth cubic hypersurfaces in even dimensions (which we'll see next), but otherwise, rationality is unknown in higher dimensions.

Finally, suppose X is a smooth cubic hypersurface with $\dim X = 2m$. There exist such hypersurfaces containing two disjoint m-planes, as you'll show in Exercise 1.10. Let's show that such an X is always rational. Let $P_1 \cong \mathbb{P}^m$ and $P_2 \cong \mathbb{P}^m$ be the two projective m-planes, and define a rational map $P_1 \times P_2 \dashrightarrow X$ as follows. For $(x_1, x_2) \in P_1 \times P_2$, in the generic case, the line spanned by these two points intersects X in a unique third point x, so take $(x_1, x_2) \mapsto x$.

For $x \in X$ not in $P_1 \cup P_2$, it is the image of the following: the span of x with P_2 gives a $\mathbb{P}^{m+1}$ which meets P_1 in a unique point $x_1 \in P_1$, and similarly the span of x with P_1 gives a unique point $x_2 \in P_2$, with the property that $(x_1, x_2) \mapsto x$. Thus, this map is dominant of degree 1, hence birational.

Exercise 1.9. Why does this construction not work if the two m-planes are distinct but not disjoint?

Exercise 1.10. Write down an explicit equation of a smooth cubic hypersurface of dimension $2m$ containing two disjoint m-planes.

Exercise 1.11. Let S be a smooth cubic surface. Show that S contains two disjoint lines, giving another proof of rationality. *Hint: Use the fact that S is the blow-up of $\mathbb{P}^2$ at 6 points in general position, see e.g.* [Har77, Section V.4].

Exercise 1.12. Let X be a cubic threefold containing a plane P. Pick coordinates on $\mathbb{P}^4$ so that $P = V(x_0, x_1)$. Then we can write $X = V(x_0q_0 + x_1q_1)$ for quadrics $q_0, q_1 \in k[x_0, x_1, x_2, x_3, x_4]$.

(1) Show that the singular locus of X contains $P \cap V(q_0) \cap V(q_1)$, which is the intersection of two conics in P.
(2) Suppose we are in the general case, so this is exactly the singular locus (four nodes contained in P). Show that projection from a singular point gives a birational isomorphism $X \overset{\sim}{\dashrightarrow} \mathbb{P}^3$.
(3) Show that X contains a line L which isn't contained in P. You might be temped to use this line for a rationality construction (e.g. writing down a rational map $L \times P \dashrightarrow X$). Why doesn't this work if L meets P?
(4) Show that X contains a line *disjoint* from P, and use this line to give a different rationality construction.

Example 1.13. We give one more example here which is a continuation of Example 1.8. Let X be a smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$ with $n \geq 2$. Projection from $\ell \subset X$ gives a rational map $X \dashrightarrow \mathbb{P}^{n-1}$ which can be resolved by blowing up ℓ:

$$\begin{tikzcd} & \mathrm{Bl}_\ell X \arrow[dl] \arrow[dr, "\pi"] & \\ X \arrow[rr, dashed] & & \mathbb{P}^{n-1} \end{tikzcd}$$

By identifying $\mathbb{P}^{n-1}$ with planes containing ℓ, the fiber of π over a point $P \in \mathbb{P}^{n-1}$ (which is not a plane contained in X) will exactly be the conic C we saw above, satisfying $P \cap X = \ell \cup C$. That is, $\pi\colon \mathrm{Bl}_\ell X \to \mathbb{P}^{n-1}$ is a **conic bundle**[2]. Generically, the fibers of a conic bundle will be smooth conics, but there will be some points in $\mathbb{P}^{n-1}$ over which the conics become singular. In fact, the conics will degenerate over a divisor $D \subset \mathbb{P}^{n-1}$ (which could be the zero divisor, although this case is not so interesting), and under some mild assumptions, the conics will only drop rank by 1 (we will make this more precise when we revisit conic bundles in Section 4.1). This conic bundle structure can also be used to give a different proof of unirationality for smooth cubic threefolds.

Definition 1.14. For a smooth conic bundle $\pi\colon Y \to \mathbb{P}^n$, we call the divisor $D \subset \mathbb{P}^n$ over which the conics degenerate the **discriminant divisor**. To D we can also associate a double cover $\varphi\colon \tilde{D} \to D$, called the **discriminant double cover**, which parametrizes the irreducible components of the fibers of $\pi|_D$.

Exercise 1.15. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold, and $\pi\colon Y := \mathrm{Bl}_\ell X \to \mathbb{P}^2$ the conic bundle from Example 1.13. We will show in this exercise that the discriminant divisor, which in this case is a plane curve, has degree 5.

Without loss of generality, we can assume $\ell = V(x_0, x_1, x_2) \subset \mathbb{P}^4_{x_0,\dots,x_4}$, and then we can write X as

$$\ell_1 x_3^2 + \ell_2 x_3 x_4 + \ell_3 x_4^2 + q_1 x_3 + q_2 x_4 + c = 0,$$

[2]Precisely, by conic bundle, we mean that π is a proper flat morphism whose generic fiber is a smooth conic.

where the ℓ_i, q_j and c are homogenenous polynomials in $k[x_0, x_1, x_2]$ of degree 1, 2, and 3, respectively. Let

$$M := \begin{bmatrix} 2\ell_1 & \ell_2 & q_1 \\ \ell_2 & 2\ell_3 & q_2 \\ q_1 & q_2 & 2c \end{bmatrix}.$$

(1) Show that Y is isomorphic to the subscheme of $\operatorname{Proj} k[y_0, y_1, y_2] \times \operatorname{Proj} k[x, y, z]$ defined by the vanishing of

$$\ell_1(x,y,z)y_0^2 + \ell_2(x,z,y)y_0y_1 + \ell_3(x,y,z)y_1^2 + q_1(x,y,z)y_0y_2 + q_2(x,y,z)y_1y_2 + c(x,y,z)y_2^3.$$

(2) Show that the bundle map $\pi \colon Y \to \mathbb{P}^2$ is the restriction of projection $\operatorname{Proj} k[y_0, y_1, y_2] \times \operatorname{Proj} k[x, y, z] \to \operatorname{Proj} k[x, y, z]$, and that the fiber over a point $(a : b : c) \in \mathbb{P}^2$ is a conic with Gram matrix given by $M(a, b, c)$.

(3) Recall that a conic is singular if and only if its Gram matrix does not have full rank (think about this if it is not something you can recall!). Show that the discriminant divisor for $Y \to \mathbb{P}^2$ is a plane quintic curve.

(4) Show that the matrix M can be interpreted as a map of vector bundles $\mathcal{E} \to \mathcal{E}^\vee \otimes L$ for $\mathcal{E} := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $L := \mathcal{O}_{\mathbb{P}^2}(1)$.

From this perspective, $Y \subset \mathbb{P}(\mathcal{E})$ and moreover $\mathcal{E} \cong \pi_*\omega_\pi^{-1}$ for ω_π the relative canonical bundle of $\pi \colon Y \to \mathbb{P}^2$. This interprets the conic bundle as a line-bundle-valued quadratic form $q \colon \mathcal{E} \to L$ (see [ABB14, Lemma 1.1.1] for more).

We will see in Section 6.4.1 that for standard conic bundles over $\mathbb{P}^2$, rationality is determined by the degree of the discriminant curve (except in degree 5, in which case it also depends on the discriminant double cover). When the degree of the discriminant curve is less than 9, the conic bundle is unirational, and it is suspected that for a general discriminant curve of large degree, the conic bundle is not unirational [Pro18, Corollary 14.3.4]. Similarly, a very general conic bundle with a discriminant curve of degree greater than 5 is not stably rational (and hence not rational) [Pro18, Corollary 14.4.4].

1.3. **Counterexamples to the standard implications in (1).** In order to show that a variety X is not rational, we must show that there is no birational isomorphism to $\mathbb{P}^n$. To do this, we should find some birational invariant which does not match when evaluated on X and $\mathbb{P}^n$. We give here the earliest counterexamples to the Lüroth problem, and more generally, threefolds which exhibit that none of the reverse implications in (1) hold:

$$\text{rational} \implies \text{stably rational} \implies \text{unirational}.$$

The birational invariants used to obstruct rationality will only be briefly introduced; the intermediate Jacobian obstruction will be the topic of Section 2 (and the focus of the remainder of the notes).

Iskovskikh–Manin: showed that smooth quartic threefolds are not rational [IM71]; however, there are examples which are known to be unirational. They show irrationality by showing that the birational automorphism group is finite, whereas for projective space the birational automorphism group is always infinite. For most smooth quartic threefolds, their unirationality is not yet known (see [Rot55, V.9] for a discussion). There are also many examples which are not stably rational [CTP16a, Théorème 1.21].

Clemens–Griffiths: showed that smooth cubic threefolds are not rational [CG72]; however, they are *all* unirational (as we showed in Example 1.8). They show irrationality by showing that the intermediate Jacobian cannot be isomorphic to a product of Jacobians of curves, which is the case for rational varieties. As previously mentioned, their stable rationality is unknown (see Remark 2.14 for a further discussion).

Artin–Mumford: exhibited a branched double cover of $\mathbb{P}^3$ which is not stably rational (and hence not rational) [AM72]; however, it is unirational. They show non-stable rationality by showing that their examples have non-trivial torsion in H^3, which on the other hand must vanish for stably rational varieties. This can also be interpreted as a non-trivial Brauer class on the variety (see [Bea16b, Section 6]).

Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer: exhibited certain conic bundle threefolds which are stably rational (X such that $X \times \mathbb{P}^3$ is rational) but not rational [BCTSSD85]. The non-rationality follows via the intermediate Jacobian obstruction.

These examples demonstrate how various birational invariants can be used to obstruct rationality, and we will refer to such an invariant as an **obstruction to rationality**. This language allows us to concisely say that if a variety is rational, then every possible obstruction must vanish (i.e. it's birational invariants must agree with that of projective space).

Definition 1.16. We will say that an obstruction to rationality **characterizes rationality** for a family of varieties if the vanishing of that obstruction implies that the family of varieties are rational.

2. The classical intermediate Jacobian obstruction

The intermediate Jacobian was classically constructed using complex algebraic geometry, so in this section we will restrict to $k = \mathbb{C}$.

2.1. **Background 1: Principally polarized abelian varieties.** A standard reference for complex abelian varieties is [BL04].

Let V be a g-dimensional complex vector space, and $\Lambda \subset V$ a full rank lattice (which means a discrete subgroup of rank $2g$, i.e. $\Lambda \cong \mathbb{Z}^{2g}$, such that $\operatorname{Span}_{\mathbb{R}} \Lambda = V$).

Definition 2.1. A **polarization** on V/Λ is a non-degenerate, skew-symmetric bilinear form $q\colon \Lambda \times \Lambda \to \mathbb{Z}$ such that

(1) $q_{\mathbb{R}}\colon V \times V \to \mathbb{R}$ satisfies $q_{\mathbb{R}}(iv, iw) = q_{\mathbb{R}}(v, w)$, and
(2) the Hermitian form $H(v, w) := q_{\mathbb{R}}(iv, w) + iq_{\mathbb{R}}(v, w)$ is positive definite.

The polarization is **principal** if q is unimodular (which means that $\Lambda^\vee := \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \cong \Lambda$).

A complex abelian variety A of dimension g is a complex torus V/Λ equipped with a polarization. There is an isomorphism $H_1(A, \mathbb{Z}) \cong \Lambda$, where $V = H^0(A, \Omega^1_A)^\vee$, and the inclusion $H_1(A, \mathbb{Z}) \hookrightarrow H^0(A, \Omega^1_A)^\vee$ is given by

$$\gamma \mapsto \left(\omega \mapsto \int_\gamma \omega\right).$$

There is an association between polarizations as defined above and ample line bundles given by $\operatorname{Hom}(\bigwedge^2 H_1(A, \mathbb{Z}), \mathbb{Z}) \cong H^2(A, \mathbb{Z})$, so that q corresponds to a divisor class θ, and q is principal if and only if $h^0(A, \mathcal{O}_A(\theta)) = 1$. In this case, the class θ gives an effective divisor, Θ, called the theta divisor, which is well-defined up to translation on A.

There is a natural homomorphism $\Lambda \to \Lambda^\vee$, and this induces a map $\lambda\colon A \to A^\vee := V^\vee/\Lambda^\vee$. We call $A^\vee$ the dual abelian variety to A.

Definition 2.2. A **principally polarized abelian variety** is an abelian variety equipped with a principal polarization.

Exercise 2.3. Show that a complex abelian variety is principally polarized if and only if it is isomorphic to its dual.

2.2. **Background 2: Hodge decomposition.** A standard reference for the Hodge decomposition is [Voi07a, Section 6.1].

While we are ultimately interested in rationality over non-closed fields, much of the theory is built on the study of rationality over $\mathbb{C}$. In the complex setting, we should use all of the tools available to us, one of which is Hodge theory. We give a brief introduction here.

Let V be a finitely generated $\mathbb{Z}$-module (or a $\mathbb{Q}$-vector space). A **Hodge structure of weight** n on V is a decomposition

$$V \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$$

with $V^{p,q}$ a complex vector space such that $V^{p,q} \cong \overline{V^{q,p}}$ (and similarly for $V \otimes_{\mathbb{Q}} \mathbb{C}$).

Let X be a smooth projective complex variety. The integral cohomology of X has a natural Hodge structure, where $H^i(X,\mathbb{Z})$ has a Hodge structure of weight i. The decomposition is given by $H^i(X,\mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$, where $H^{p,q}(X)$ is the (p,q)-Dolbeault cohomology. There is an isomorphism $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$, so you are welcome to think of these vector subspaces as sheaf cohomology groups.

Throughout, we will write $h^{p,q}(X) := \dim H^{p,q}(X)$; these dimensions are called the Hodge numbers of X. They are collected in a Hodge diamond–for example, for a surface:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Exercise 2.4 (Warm-up with Hodge numbers)**.**

(1) Make sure you understand why, for example for a surface, all other $h^{p,q} = 0$ outside of those in the diamond.
(2) Let X be a smooth projective complex surface. Write $\chi(X, \mathcal{O}_X)$ in terms of Hodge numbers.
(3) Write down the Hodge diamond for $\mathbb{P}^2$.
(4) Show that the Hodge diamond is symmetric across the vertical center axis, and also has 180° rotational symmetry. *Hint: Serre duality.*

2.3. **Jacobians of curves.** Now let C be a smooth projective complex curve of genus g. Then $H^1(C,\mathbb{Z}) \cong \mathbb{Z}^g$ and it has a Hodge decomposition

$$H^1(C,\mathbb{C}) = H^{1,0}(C) \oplus H^{0,1}(C),$$

with $H^{1,0}(C) \cong \mathbb{C}^g$ and $H^{1,0}(C) \cong \overline{H^{0,1}(C)}$. The inclusion followed by projection $H^1(C,\mathbb{R}) \hookrightarrow H^1(C,\mathbb{C}) \to H^{0,1}(C)$ is an $\mathbb{R}$-linear isomorphism, so the image of $H^1(C,\mathbb{Z})$ in $H^{0,1}(C)$, which we denote by Λ, is a full rank lattice (of rank $2g$). The quotient $J(C) := H^{0,1}(C)/\Lambda$ is a complex torus of dimension g. In fact, this torus is algebraic, since the cup product

$$H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z}) \to H^2(C,\mathbb{Z}) \cong \mathbb{Z}$$

defines a polarization on Λ with the Hermitian form $H(\alpha,\beta) = 2i\int_C \alpha \wedge \bar{\beta}$. We call $J(C)$ the **Jacobian** of C. It is principally polarized, since the cup product defines a unimodular form (i.e. the cup product is a perfect pairing, even integrally).

2.4. **The intermediate Jacobian.** The definition of the Jacobian of a curve can be mimicked in higher dimensions for odd-dimensional varieties. Let X be a smooth projective variety of dimension $n = 2m-1$. Then $H^{2m-1}(X,\mathbb{C})$ has a Hodge decomposition, and there is a map

$$H^{2m-1}(X,\mathbb{Z}) \to \bigoplus_{i=0}^{m-1} H^{m-1-i,m+i}(X)$$

given by tensoring with $\mathbb{C}$ (notice that this step kills any torsion in the cohomology) and projecting onto the second half of the Hodge decomposition. Write Λ for the image of this map, which as above is a full rank lattice.

Definition 2.5. The **intermediate Jacobian** of X is

$$J(X) := \bigoplus_{i=0}^{m-1} H^{m-1-i,m+i}(X)/\Lambda,$$ [3]

a complex torus of dimension $\frac{1}{2}h^{2m-1}(X,\mathbb{C})$.

In general, this need not be an abelian variety, let alone principally polarized! However, in the cased we're interested in (e.g. for unirational threefolds), it will be a principally polarized abelian variety.

Example 2.6. Let's consider a unirational threefold X. By Corollary 1.4, $H^{3,0}(X) = H^0(X,\Omega_X^3) = 0$, so the Hodge decomposition gives $H^3(X,\mathbb{C}) = H^{2,1}(X) \oplus H^{1,2}(X)$ with $H^{1,2}(X) = \overline{H^{2,1}(X)}$. Then $J(X) = H^{1,2}(X)/\Lambda$ for $\Lambda = \mathrm{im}(H^3(X,\mathbb{Z}) \to H^{1,2}(X))$. In fact, in this case, $J(X)$ is a principally polarized abelian variety, with the polarization again given by the cup product

$$H^3(X,\mathbb{Z}) \otimes H^3(X,\mathbb{Z}) \to H^6(X,\mathbb{Z}) \cong \mathbb{Z}.$$

In fact, this is true even if we relax the assumption that X is unirational, as long as we assume that $H^{3,0}(X) = H^{1,0}(X) = 0$. The Hermitian form is $H(\alpha,\beta) = 2i\int_X \alpha \wedge \bar{\beta}$, and the Hodge-Riemann bilinear relations give that H is positive definite on the primitive cohomology. The vanishing of $h^{1,0}$ ensures that all of $H^{1,2}(X)$ is primitive, and the vanishing of $h^{3,0}$ thus ensures that H is everywhere positive definite. See [Voi16, Section 3.1] for more details.

The intermediate Jacobian was first introduced and studied by Griffiths, and not long after was used by Clemens and Griffiths to show that smooth cubic threefolds are irrational. To do this, they characterize $J(X)$ among all principally polarized abelian varieties when X is rational. For the proof, we'll make use of the following lemma (the proof of which can be found in [Voi07a, Theorem 7.31]).

[3] Note that in later sections, we will call this $J^m(X)$, the mth intermediate Jacobian of X.

Lemma 2.7 (Blow-up formula)**.** *Let X be a complex manifold, $Y \subset X$ a closed submanifold of codimension m, and $\mathrm{Bl}_Y X$ the blow-up of X along Y. Then for every integer i, there is an isomorphism*

$$H^i(\mathrm{Bl}_Y X, \mathbb{Z}) \cong H^i(X, \mathbb{Z}) \oplus \bigoplus_{k=1}^{m-1} H^k(Y, \mathbb{Z}).$$

Moreover, this isomorphism is compatible with the cup product and Hodge structures (if we include the appropriate twists).

Theorem 2.8 (Clemens–Griffiths [CG72])**.** *Let X be a rational smooth projective threefold over $\mathbb{C}$. Then the intermediate Jacobian $J(X)$ is isomorphic, as a principally polarized abelian variety, to a product of Jacobians of curves.*

Proof. Since X is rational, there is a birational isomorphism $\varphi\colon \mathbb{P}^3 \overset{\sim}{\dashrightarrow} X$. By Hironaka's resolution of indeterminacies [Hir64, Section 0.5], there is a threefold X' and a commutative diagram

$$\begin{tikzcd} & X' \arrow[dl, "\varepsilon"'] \arrow[dr, "f"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\varphi"'] & & X, \end{tikzcd}$$

where ε is a composition of blow-ups along smooth centers and f is a birational morphism. If any of the blow-ups in ε are blow-ups of points, we see by Lemma 2.7 that the middle cohomology is left unchanged. If, for example, a curve $C \subset \mathbb{P}^3$ is blown up, then Lemma 2.7 implies that $H^3(\mathrm{Bl}_C \mathbb{P}^3, \mathbb{Z}) \cong H^3(\mathbb{P}^3, \mathbb{Z}) \oplus H^1(C, \mathbb{Z})$. Thus, for each curve that is blown up, its H^1 will appear in $H^3(X', \mathbb{Z})$. Explicitly, we get an isomorphism

$$H^3(X', \mathbb{Z}) \cong H^3(\mathbb{P}^3, \mathbb{Z}) \oplus H^1(C_1, \mathbb{Z}) \oplus \cdots \oplus H^1(C_r, \mathbb{Z})$$

for some smooth projective curves $C_1, ..., C_r$. Since by Lemma 2.7, this isomorphism respects the cup product and Hodge structures on both sizes, it induces an isomorphism of principally polarized abelian varieties

$$J(X') \cong J(\mathbb{P}^3) \times J(C_1) \times \cdots J(C_r) \cong J(C_1) \times \cdots J(C_r),$$

since $J(\mathbb{P}^3) = 0$.

Next, we compare $J(X')$ to $J(X)$. Since $f\colon X' \to X$ is a morphism, it induces maps on cohomology,

$$f^*\colon H^3(X, \mathbb{Z}) \to H^3(X', \mathbb{Z}), \qquad f_*\colon H^3(X', \mathbb{Z}) \to H^3(X, \mathbb{Z}),$$

which moreover satisfy $f_* f^* = \mathrm{Id}$, since f is an isomorphism on a dense open subset of X'. In particular, the inclusion f^* induces an inclusion $J(X) \hookrightarrow J(X')$ as a direct factor. That is, there exists a principally polarized abelian variety A such that

$$A \times J(X) \cong J(X') \cong J(C_1) \times \cdots J(C_r),$$

where both isomorphisms are isomorphisms of principally polarized abelian varieties.

In general, such a decomposition need not be unique. However, in the category of *principally polarized abelian varieties*, there is a form of uniqueness of decomposition. We say that a principally polarized abelian variety is **indecomposable** if it is nonzero and cannot be written as a non-trivial product of principally polarized abelian varieties.

Lemma 2.9 ([CG72, Corollary 3.23])**.**

(1) The Jacobian of a curve is indecomposable.
(2) Any principally polarized abelian variety admits a unique decomposition into a product of indecomposable abelian varieties.

Finally, the lemma along with the isomorphism $A \times J(X) \cong J(C_1) \times \cdots J(C_r)$ implies that $J(X) \cong J(C_{i_1}) \times \cdots J(C_{i_s})$ for $i_1, ..., 1_s$ distinct in $\{1, 2, ..., r\}$. ☐

In fact, Clemens and Griffiths show more in [CG72, Section 3] than what we just proved: when X is a smooth complex projective threefold with $h^{3,0}(X) = h^{1,0}(X) = 0$ (so that $J(X)$ is a principally polarized abelian variety), they associate a birational invariant to $J(X)$ (note that $J(X)$ is not itself a birational invariant, as we just saw above). They define a semi-group $\mathbf{A}$ generated by the set $\{A : A \text{ is a principally polarized abelian variety}\}$ under the equivalence relation that $A \sim A'$ if there exist smooth curves C and C' such that there are morphisms $A \to A' \times J(C')$ and $A' \to A \times J(C)$. Following the proof we gave above, we can show that $J(X)$ as an element of $\mathbf{A}$ is a birational invariant.

Exercise 2.10. Let X be a smooth complex cubic threefold. Show that $J(X) \cong J(\mathrm{Bl}_\ell X)$ for ℓ a line in X.

Exercise 2.11. For a slight variation on Exercise 1.12, let $X \subset \mathbb{P}^4$ be a one nodal cubic threefold, i.e. the singular locus is one ordinary double point; see [Huy23, Sections 1.5.4, 5.5.1] for more on this setting. By a node $p \in X$, we mean that the exceptional divisor $E_p \subset \mathrm{Bl}_p X$ is a smooth quadric surface when considered as a subvariety of the exceptional divisor $E \cong \mathbb{P}^3 \subset \mathrm{Bl}_p \mathbb{P}^4$.

(1) First, show that X is rational.
(2) After a linear change of coordinates, we may assume the singular point of X is $p = [0 : 0 : 0 : 0 : 1] \in X \subset \mathbb{P}^4$. Show that X can be written as $X = V(F + x_4 G)$ for $F \in H^0(\mathbb{P}^3, \mathcal{O}(3))$ and $G \in H^0(\mathbb{P}^3, \mathcal{O}(2))$, where $\mathbb{P}^3 = V(x_4)$.
(3) Show that when you resolve the birational isomorphism from part (1) as

$$\begin{tikzcd} & \mathrm{Bl}_p X \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] & & \mathbb{P}^3 \end{tikzcd}$$

there is an isomorphism $\mathrm{Bl}_p X \cong \mathrm{Bl}_C \mathbb{P}^3$, where $C = V(F) \cap F(G)$.
(4) Conclude that $J(\mathrm{Bl}_p X) \cong J(C)$. This can be taken to be the definition of $J(X)$. Show also that $\dim J(X) = 4$, which differs from the case of a smooth cubic threefold (you will show that $\dim J(X) = 5$ in the smooth case in Exercise 2.15).

2.5. **The irrationality of smooth cubic threefolds.** Let X now be a smooth projective cubic threefold over $\mathbb{C}$. To see that X is irrational, by Theorem 2.8 it is enough to prove that $J(X)$ is not isomorphic to a product of Jacobians of curves, as principally polarized abelian varieties. To do this, we need to understand Jacobians of curves among all principally polarized abelian varieties. This is the so-called *Schottky problem*, to understand the image of the map

$$\mathcal{M}_g \to \mathcal{A}_g$$

from the moduli space of smooth genus g curves to the moduli space of principally polarized abelian varieties of dimension g, which sends (the isomorphism class of) a curve C to $J(C)$.

There are different approaches one can take here; many involve a careful study of the theta divisor. For example, if (A, Θ) is the Jacobian of a curve of genus g, then $\dim \operatorname{Sing} \Theta \geq g-4$ [AM67, Sections 1–3].

Theorem 2.12. *A smooth cubic threefold X is irrational.*

Proof. We showed in Example 1.8 that X is unirational, so $h^{1,0} = h^{3,0} = 0$, as explained in Example 2.6. Thus, $J(X)$ is a principally polarized abelian variety with theta divisor Θ. In Exercise 2.15, you'll show that $h^{1,2}(X) = 5$, so that $J(X)$ has dimension 5.

Clemens and Griffiths arrive at a contradiction by supposing that there is some smooth curve C (possibly disconnected) of genus 5 such that $J(X) \cong J(C)$. They show using technique's from Andreotti's proof of the Torelli theorem [And58, Section 3] for curves that $C^* \subset X^*$ (where C^* means the projective dual under $C \to \mathbb{P}^4$ induced by the canonical bundle and X^* means the projective dual under $X \subset \mathbb{P}^4$). They then show that C^* contains a $\mathbb{P}^2$ while X^* cannot contain such a large linear subspace.

At the same time, Mumford (see [CG72, Appendix C] and also [Bea82, Théorème, p. 190]) proved that $\Theta \subset J(X)$ has a unique singular point. For $J(X)$ to be a Jacobian of a curve, $\operatorname{Sing} \Theta$ would need to have dimension at least 1. This gives another proof that X is irrational.

Both of these arguments are quite involved, so we do not try to summarize them further here. $\square$

Remark 2.13. To A we can associate an integral cohomology class

$$\frac{1}{(g-1)!}[\Theta]^{g-1} \in H^{2g-2}(A, \mathbb{Z}).$$

Then (A, Θ) is a product of Jacobians of curves if and only if this class is an effective algebraic curve class [Mat59][4]. Indeed, if $A = J(C)$, this cohomology class is the class of the image of the Abel-Jacobi map $C \hookrightarrow J(C)$.

Unfortunately, studying the effectivity of this class is difficult in general. The theorem above shows that for smooth complex cubic threefolds, this class is not effective. Voisin showed that for smooth complex projective stably rational threefolds, this cohomology class is always algebraic, using the decomposition of the diagonal [Voi17, Theorem 1.6]. It is not known, in general, whether there exists a principally polarized abelian variety (A, Θ) of dimension g for which $[\Theta]^{g-1}/(g-1)!$ is not algebraic.

Remark 2.14. The existence of a decomposition of the diagonal (see for example [AB17, Section 3.6] for a definition) is a stable birational invariant, so if a variety does not have a decomposition of the diagonal, it is not stably rational. This is an important tool in the study of rationality, especially in showing the failure of stable rationality for many families of rationally-connected varieties [Voi15, Bea16a, CTP16a, CTP16b, HKT16, Tot16, Voi17, HT19]. While we have just seen that smooth cubic threefolds are irrational, their stable rationality is unknown. In [Voi17, Theorem 1.7], Voisin identifies a countable union of closed subvarieties with codimension ≤ 3 in the moduli space of smooth cubic threefolds for which the cubic threefolds do have a decomposition of the diagonal. Thus, this technique fails to apply, which adds to the difficulty of the problem of stable rationality for smooth cubic threefolds.

[4] See Section 3.3 below for the introduction of cycle classes and the homomorphism from cycle classes to cohomology. While out of order with the material, we include this here to point out another approach for studying when a principally polarized abelian variety is a Jacobian of a curve.

Exercise 2.15. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. Show that $h^{1,2}(X) = h^{2,1}(X) = \dim H^2(X, \Omega^1_X) = 5$. *If you aren't familiar with computing Hodge numbers, an approach is outlined below. The method relies on cohomology long exact sequences from short exact sequences of bundles on X and $\mathbb{P}^4$.*

The proof proceeds via two main equalities:

(A) $h^2(X, \Omega^1_X) = h^3(X, \mathcal{O}_X(-3))$, and
(B) $h^3(X, \mathcal{O}_X(-3)) = h^0(X, \mathcal{O}_X(1))$.

(1) Use (A) and (B) along with the divisor short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0$$

to show the desired equality, that $h^{1,2}(X) = 5$.

(2) Use adjunction to show that $\omega_X \cong \mathcal{O}_X(-2)$.

(3) Show (B) using Serre duality.

(4) From the cotangent sequence

$$0 \to \mathcal{O}_X(-3) \to \Omega_{\mathbb{P}^4}|_X \to \Omega_X \to 0,$$

where $I_X/I_X^2 \cong \mathcal{O}_X(-3)$, show that if $h^2(X, \Omega_{\mathbb{P}^2}|_X) = h^3(X, \Omega_{\mathbb{P}^4}|_X) = 0$, then (A) holds.

(5) Finally, show the necessary vanishing in part (4) using the long exact sequence on cohomology from the Euler sequence:

$$0 \to \Omega_{\mathbb{P}^4}|_X \to \mathcal{O}_X(-1)^{\oplus 5} \to \mathcal{O}_X \to 0.$$

To show the vanishing of the cohomology groups of $\mathcal{O}_X(-1)$, use the divisor short exact sequence from (1) along with Serre duality.

3. The connection to curve classes

We are interested in rationality over non-closed fields, but the construction of the intermediate Jacobian above depends heavily on the complex structure. We would like to find a replacement for this construction: for X a smooth threefold over a field different from $\mathbb{C}$, is there a principally polarized abelian variety associated to X which, if X is rational, must take on a specific form?

To see how this should work, we need to connect the intermediate Jacobian to other algebraic data associated to X. To do this, we will continue to work over $\mathbb{C}$, and we'll first revisit the construction of the intermediate Jacobian.

3.1. **The intermediate Jacobians.**
Let X be a smooth complex projective variety of dimension n.

Definition 3.1. The mth intermediate Jacobian of X is

$$J^m(X) := \bigoplus_{i=0}^{m-1} H^{m-1-i,m+i}(X)/H^{2m-1}(X, \mathbb{Z}).$$

Note that we are now conflating $H^{2m-1}(X, \mathbb{Z})$ with its image in $\bigoplus_{i=0}^{m-1} H^{m-1-i,m+i}(X)$.

Example 3.2. For $m = 2$, we have $J^m(X) = H^{1,2}(X) \oplus H^{0,3}(X)/H^3(X, \mathbb{Z})$, the complex torus that we referred to in Section 2 simply as the intermediate Jacobian. This is the case we are hoping to better understand!

Example 3.3. For $m = 1$, we have $J^1(X) = H^{0,1}(X)/H^1(X,\mathbb{Z})$, which we can study via the exponential short exact sequence:

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

Taking the long exact sequence on comology gives

$$\cdots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \cdots,$$

where the boundary map $c_1\colon H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$ is the first Chern class map after identifying $H^1(X,\mathcal{O}_X^*) \cong \operatorname{Pic} X$. Recalling that $H^{0,1}(X) \cong H^1(X,\mathcal{O}_X)$, we see that $J^1(X) \cong \ker(c_1) \subset \operatorname{Pic} X$. We will call $\ker(c_1)$ the **Picard variety** of X, denoted $\operatorname{Pic}^0 X$.

In fact, there is a scheme called the **Picard scheme** of X, $\mathbf{Pic}_X$, which is a group scheme for which $\mathbf{Pic}_X(\mathbb{C}) = \operatorname{Pic} X$. This is not a scheme of finite type, but the connected component containing $\mathcal{O}_X$, $\mathbf{Pic}_X^0$, is a variety, also called the Picard variety, and satisfies $\mathbf{Pic}_X^0(\mathbb{C}) = \operatorname{Pic}^0 X$.

The Picard scheme can be defined for a smooth projective variety X over any algebraically closed field k[5]. There is a functor $\operatorname{Pic}_{X/k}\colon (\mathrm{Sch}/k) \to (\mathrm{Ab})$ called the relative Picard functor, and $\mathbf{Pic}_X$ represents this functor. See [CTS21, Section 2.5] or [Kle14] for details.

Exercise 3.4. Fill in the details showing that $J^1(X) \cong \ker c_1$.

Example 3.5. For $m = n = \dim X$, we have $J^n(X) = H^{n-1,n}(X)/H^{2n-1}(X,\mathbb{Z})$. We can identify $J^n(X)$ with the **Albanese variety** of X, $\operatorname{Alb} X$. This is purely formal; by the exercises below, $J^n(X) \cong J^1(X)^\vee$ and $\operatorname{Alb} X \cong \operatorname{Pic}^0(X)^\vee$.

In general, the Albanese variety $\mathbf{Alb}_X$ is defined as the unique abelian variety satisfying the following universal property: for any point $x_0 \in X$, there exists a morphism $\mathrm{alb}\colon X \to \mathbf{Alb}_X$ sending $x_0 \mapsto 0$ such that for any morphism $f\colon X \to B$ with B an abelian variety and $x_0 \mapsto 0_B$, there exists a unique morphism $\mathbf{Alb}_X \to B$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{alb}"'] & & B \\ & \mathbf{Alb}_X \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

That is, the Albanese variety is the "closest" abelian variety to X. As with the Picard variety, this variety always exists over an algebraically closed field.

Exercise 3.6. Recall the definition of the dual complex abelian variety in Section 2.1. Show that $J^n(X) \cong J^1(X)^\vee$.

Exercise 3.7. Show that for a complex abelian variety A, the dual abelian variety satsifies $A^\vee \cong \operatorname{Pic}^0 A$. *Hint: Use the fact that* $A \cong H^0(A,\Omega_A^1)^\vee/H_1(A,\mathbb{Z})$.

Exercise 3.8. Note that given an abelian variety A, there is a canonical isomorphism $(A^\vee)^\vee \cong A$. In fact, this is true over any algebraically closed field if we use $\mathbf{Pic}_A^0$ as the definition of $A^\vee$.

In this exercise, you'll show that $(\mathbf{Pic}_X^0)^\vee$ is $\mathbf{Alb}_X$ for any smooth complex projective variety X.

[5]More precisely, it can be defined over any field, but then $\operatorname{Pic}_X(k)$ can be larger than $\operatorname{Pic} X$. See the suggested references for more.

(1) Fix a point $x_0 \in X$. Show that there is always a morphism $X \to (\mathbf{Pic}^0_X)^\vee$ sending x_0 to $\mathcal{O}_{\mathbf{Pic}^0_X}$.

Hint: By the representability of the relative Picard functor, there is a Poincaré bundle $\mathcal{P}$ on $X \times \mathbf{Pic}^0_X$ which satisfies:

- *For all $[L] \in \mathbf{Pic}^0_X$, $\mathcal{P}|_{X\times\{[L]\}} \cong L$, and*
- *$\mathcal{P}$ is normalized such that $\mathcal{P}|_{\{x_0\}\times\mathbf{Pic}^0_X} \cong \mathcal{O}_{\mathbf{Pic}^0_X}$.*

(2) Let $f\colon X \to B$ be a morphism to an abelian variety B such that $f(x_0) = 0_B$. Show that this induces a morphism of group schemes $f^{\vee\vee}\colon (\mathbf{Pic}^0_X)^\vee \to B$.

(3) Finally, show that the morphism from (2) is the unique morphism making the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & B \\ & (\mathbf{Pic}^0_X)^\vee \arrow[ur, "f^{\vee\vee}"'] & \end{tikzcd}$$

3.2. **The situation in dimension 3.** If $\dim X = 3$, we have identified the following:

$$\begin{aligned} J^1(X) &\cong \operatorname{Pic}^0 X = \mathbf{Pic}^0_X(\mathbb{C}) \\ J^2(X) &\cong \ ?? \\ J^3(X) &\cong \operatorname{Alb} X = \mathbf{Alb}_X(\mathbb{C}). \end{aligned}$$

We have natural algebraic objects, indeed algebraic varieties, which describe the first and third intermediate Jacobians of the threefold, and which have definitions which make sense over any field $k = \bar{k}$. It is natural, then, to hope that there is some abelian variety which is isomorphic to $J^2(X)$, and which can be defined over any algebraically closed field.

3.3. **Cycle classes and Abel-Jacobi maps.** When Griffiths first studied the intermediate Jacobians of a smooth complex projective variety X, he also introduced a map from codimension m cycles to $J^m(X)$ for each $0 \le m \le n$. This connection will lead us to the algebraic object we should associate to $J^2(X)$, so we now take the time to introduce cycles and various equivalence relations on them.

Definition 3.9. The group of codimension m cycles on X is

$$Z^m(X) := \left\{ \sum n_Y Y : Y \subset X \text{ is a subvariety of codimension } m \right\}.$$

The subgroup $Z^m_{\mathrm{rat}}(X)$ is generated by

$$[V \cap \{t_0\} \times X] - [V \cap \{t_1\} \times X],$$

where $V \subset \mathbb{P}^1 \times X$ is flat over $\mathbb{P}^1$ and $t_0, t_1 \in \mathbb{P}^1$.

Two cycles $Z_1, Z_2 \in Z^m(X)$ are **rationally equivalent** if $Z_1 - Z_2 \in Z^m_{\mathrm{rat}}(X)$, and the mth **Chow group** of X is

$$\mathrm{CH}^m X := Z^m(X)/Z^m_{\mathrm{rat}}(X).$$

The elements are rational equivalence classes of codimension m cycles on X.

Since X is assumed to be smooth throughout, we can also write $\mathrm{CH}_k X := \mathrm{CH}^{n-k} X$ and discuss dimension k cycles on X.

Example 3.10. For $m = 1$, we have $\mathrm{CH}^1 X = \mathbb{Z}\{\text{prime Weil divisors}\}/\text{rational equivalence}$. Since rational equivalence of divisors agrees with linear equivalence (see Exercise 3.11), it follows that $\mathrm{CH}^1 X \cong \operatorname{Pic} X$, via the usual identification of Weil divisors and Cartier divisors.

Exercise 3.11. Show that two divisors are rationally equivalent if and only if they are linearly equivalent.

Definition 3.12. The **cycle class map** $\mathrm{cl}_m \colon \mathrm{CH}^m X \to H^{2m}(X, \mathbb{Z})$ is the group homomorphism sending a class $[Y]$ to the Poincaré dual of the fundamental class of $Y \subset X$. Let $(\mathrm{CH}^m X)_{\mathrm{hom}} := \ker \mathrm{cl}_m$, the subgroup of **homologically trivial** classes.

Using complex geometry, one can define the **Abel–Jacobi map**

$$\mathrm{AJ}^m_X \colon (\mathrm{CH}^m X)_{\mathrm{hom}} \to J^m(X),$$

which, when interpreted correctly, can be written as

$$[Y] \mapsto \left(\omega \mapsto \int_{C_Y} \omega \right),$$

where C_Y is such that $d(C_Y) = Y$ (using that Y becomes trivial in cohomology).

Exercise 3.13 (For those with a background in complex geometry)**.** Make clear how the Abel–Jacobi map can be given by integration over subvarieties. For example, why must we restrict to homologically trivial classes? Why do rationally equivalent subvarieties give the same value in $J^m(X)$?

I haven't given the explicit definition of the Abel–Jacobi map since I'm interested in a more algebraic interpretation. Let's see what we can say in the examples we have already considered: that of the Picard variety and the Albanese variety.

Example 3.14. We saw above that $\mathrm{CH}^1 X \cong \operatorname{Pic} X$. The cycle class map

$$\mathrm{cl}_1 \colon \mathrm{CH}^1 X \to H^2(X, \mathbb{Z})$$

sends a prime divisor $[D]$ to $c_1(D)$, which is to say that $\mathrm{cl}_1 = c_1$ under the identification $\mathrm{CH}^1 X \cong \operatorname{Pic} X$. Thus, $\ker \mathrm{cl}_1 \cong \ker c_1$, and so $(\mathrm{CH}^1 X)_{\mathrm{hom}} \cong \operatorname{Pic}^0(X)$, and the Abel-Jacboi map is an isomorphism.

In fact, the situation for divisors is the nicest possible case. For the case of codimension n cycles, we also have a good understanding.

Example 3.15. To better match the literature, we will write $\mathrm{CH}^n X = \mathrm{CH}_0 X$, and consider the codimension n cycles as 0 cycles. Elements are formal sums of points on X, and $\mathrm{cl}_n \colon \mathrm{CH}_0 X \to H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z}$ is given by

$$\sum n_x x \mapsto \sum n_x,$$

the degree map. Then $(\mathrm{CH}_0 X)_{\mathrm{hom}}$ is the subset of degree 0 cycles, often written $(\mathrm{CH}_0 X)_0$. The Abel–Jacobi map

$$\mathrm{AJ}^n_X \colon \mathrm{CH}_0(X)_0 \to \operatorname{Alb} X$$

can be interpreted as follows. Recall that $\mathrm{alb} \colon X \to \operatorname{Alb} X$ depends on the choice of a basepoint, $x_0 \in X$. This choice allows us to write down a map $X \to (\mathrm{CH}_0 X)_0$ by sending $x \mapsto [x] - [x_0]$. Note that this map is really just a map of sets, since $(\mathrm{CH}_0 X)_0$ does not have the structure of a variety, and X need not have the structure of a group.

These maps along with AJ_X^n fit into the following diagram:

$$\begin{tikzcd} X \arrow[rr, "\mathrm{alb}"] \arrow[dr] & & \operatorname{Alb} X \\ & \mathrm{CH}_0(X)_0 \arrow[ur, "\mathrm{AJ}_X^n"'] & \end{tikzcd}$$

Since this is true for any choice of basepoint, we see that AJ_X^n is independent of this choice.

It is not in general the case that AJ_X^n is an isomorphism (although it is always surjective, see [Blo80, Lecture 1]), but it is when restricted to torsion subgroups. For an abelian group G and a prime number ℓ, let $G(\ell)$ denote the ℓ-primary component of G.

Theorem 3.16 (Roitman [Roj80, Theorem 3.1])**.** *For all primes ℓ, the Abel–Jacobi map restricts to an isomorphism on ℓ-primary subgroups $(\mathrm{CH}_0 X)_0(\ell) \xrightarrow{\sim} \operatorname{Alb} X(\ell)$.*

In fact, Roitman proved this for X a smooth projective variety defined over any algebraically closed field, as long as $\ell \neq \operatorname{char} k$. It was extended by Milne to hold also when $\ell = \operatorname{char} k$ [Mil82, Theorem 0.1].

It would be nice to have a characterization for when the Abel-Jacobi map AJ_X^n is an isomorphism. The following conjecture has been put forth by Bloch:

Conjecture 3.17 (Bloch [Blo80, Lecture 6])**.** *Let X be a smooth complex projective variety. The following are equivalent:*

(1) $\mathrm{AJ}_X^n \colon (\mathrm{CH}_0 X)_0 \to \operatorname{Alb} X$ is an isomorphism.
(2) The Hodge numbers $h^{i,0}(X) = 0$ for $i \geq 2$.

Bloch shows that (1) implies (2) for surfaces (and in fact, Mumford shows for surfaces that when (2) doesn't hold, the kernel of AJ_X^2 can be quite large [Mum68]). The reverse implication has been verified for surfaces of Kodaira dimension less than 2 [BKL76], and some surfaces of general type [PW16, Voi14]. For an example in higher dimensions, see [Lat18].

We'd like to now return to the case $m = 2$, where we have

$$\mathrm{AJ}_X^2 \colon (\mathrm{CH}^2 X)_{\mathrm{hom}} \to J^2(X).$$

To say something about this map, and to understand what $J^2(X)$ tells us about codimension 2 cycles, we need a couple more definitions.

Definition 3.18. The subgroup $Z^m_{\mathrm{alg}}(X) \leqslant Z^m(X)$ is generated by

$$[V \cap \{t_0\} \times X] - [V \cap \{t_1\} \times X],$$

where $V \subset C \times X$ for C an integral curve and V flat over C, and $t_0, t_1 \in C$.

Two cycles $Z_1, Z_2 \in Z^m(X)$ are **algebraically equivalent** if $Z_1 - Z_2 \in Z^m_{\mathrm{alg}}(X)$, and the subgroup of algebraically trivial classes is

$$(\mathrm{CH}^m X)^0 := Z^m_{\mathrm{alg}}(X)/Z^m_{\mathrm{rat}}(X) \leqslant \mathrm{CH}^m X.$$

Let $\mathrm{NS}^m(X) := Z^m(X)/Z^m_{\mathrm{alg}}(X) \cong \mathrm{CH}^m X/(\mathrm{CH}^m X)^0$, sometimes called the Neron-Severi group, the group of algebraic equivalence classes[6].

[6]Sometimes the term Neron-Severi group is reserved specifically for the case of $m = 1$.

The subgroup of algebraically trivial classes is sometimes also written $A^m X$. Unfortunately, the Chow groups $\mathrm{CH}^m X$ are also sometimes written $A^m X$. We will use the notation $(\mathrm{CH}^m X)^0$ both to avoid confusion and to better match the literature in the case of non-algebraically closed fields.

Exercise 3.19. We say that two cycles $Z_1, Z_2 \in Z^m(X)$ are **homologically equivalent** if $Z_1 - Z_2 \in (\mathrm{CH}^m X)_{\mathrm{hom}}$, or equivalently, $\mathrm{cl}_m([Z_1]) = \mathrm{cl}_m([Z_2])$. Show that

$$\text{rationally equivalent} \implies \text{algebraically equivalent} \implies \text{homologically equivalent.}$$

(You might compare to [Har77, Exercise V.1.7] *for the case of surfaces.)*

In fact, we have that $(\mathrm{CH}^1 X)^0 = (\mathrm{CH}^1 X)_{\mathrm{hom}}$ (algebraic equivalence and homological equivalence agree for divisors, see [Ful84, Section 19.3.1]), and $(\mathrm{CH}^n X)^0 = (\mathrm{CH}_0 X)_0$ (algebraic equivalence and homological equivalence agree for 0-cycles, see [Voi07b, Section 8.2.1]).

3.4. **The situation in dimension 3, revisited.** We can now revisit the case where $\dim X = 3$. We saw that $J^1(X) \cong \mathrm{Pic}^0 X$, and this is further isomorphic to $(\mathrm{CH}^1 X)^0$, so $J^1(X)$ exactly tells us about algebraically trivial divisor classes.

Similarly, we saw that $J^3(X) \cong \mathrm{Alb}\, X$, and the torsion subgroup is isomorphic to $(\mathrm{CH}^3 X)^0_{\mathrm{tors}}$.

Naturally, we ask: *is there an abelian variety which provides information about algebraically trivial curve classes on X?*

3.5. **An algebraic representative for curve classes.** As our first step in moving away from $k = \mathbb{C}$, let's now assume k is any algebraically closed field $k = \bar{k}$.

Let X be a smooth projective variety over k. Let T be a variety and pick a point $t_0 \in T$ and a cycle $W \in \mathrm{CH}^m(T \times X)$ such that $\{W_{\{t\}\times X}\} \in \mathrm{CH}^m X$ for all $t \in T$. Then W gives an algebraic family of cycle classes on X parametrized by T. This family gives a map $T \to (\mathrm{CH}^m X)^0$ given by $t \mapsto W_{\{t\}\times X} - W_{\{t_0\}\times X}$.

Let A be an abelian variety over k.

Definition 3.20. A homomorphism $\varphi \colon (\mathrm{CH}^m X)^0 \to A$ is a **regular homomorphism** if for every algebraic family (T, W) as above, the composition

$$T \to (\mathrm{CH}^m X)^0 \to A$$

is a morphism of varieties.

A pair (A_0, φ_0), with A_0 an abelian variety and $\varphi_0 \colon (\mathrm{CH}^m X)^0 \to A_0$ a regular homomorphism, is an **algebraic representative** for $(\mathrm{CH}^m X)^0$ if it is universal among such pairs: for every (A, φ), with A an abelian variety and φ a regular homomorphism, there is a morphism $f \colon A_0 \to A$ making the following diagram commute:

$$\begin{tikzcd}
(\mathrm{CH}^m X)^0 \arrow[r, "\varphi_0"] \arrow[dr, "\varphi"'] & A_0 \arrow[d, dashed, "\exists f"] \\
& A
\end{tikzcd}$$

Exercise 3.21.

(1) Show that if (A_0, φ_0) exists, then φ_0 is surjective. *Hint: You may use as fact (or try to prove) that the image of a regular homomorphism is an abelian subvariety* [Mur85, Lemma 1.6.2]*.*
(2) Show that once f exists, it must be unique.

Note that there is no guarantee that φ_0 is an isomorphism. Rather, we should interpret the definition as saying that the algebraic representative is the abelian variety whose structure is "closest" to that of $(\mathrm{CH}^m X)^0$.

Exercise 3.22.

(1) Show that $\mathrm{Pic}^0 X$ is the algebraic representative for $(\mathrm{CH}^1 X)^0$.
(2) Show that $\mathrm{Alb}\, X$ is the algebraic representative for $(\mathrm{CH}^n X)^0$.

Finally, we can state the result of Murre which gives algebraic meaning to $J^2(X)$.

Theorem 3.23 (Murre [Mur85, Theorem A], [Kah21, Theorem 1])**.** *There exists an abelian variety* $\mathrm{Ab}^2 X$ *which is the algebraic representative for* $(\mathrm{CH}^2 X)^0$*.*

Moreover, suppose $k = \mathbb{C}$ *and let* $J_a^2(X) := \mathrm{im}(\mathrm{AJ}_X^2 |_{(\mathrm{CH}^2 X)^0}) \subset J^2(X)$*. Then* $J_a^2(X)$ *is the algebraic representative for* $(\mathrm{CH}^2 X)^0$*.*

So there we have it! There is an abelian variety $\mathrm{Ab}^2 X$, which makes sense over any algebraically closed field, and which suggests a replacement for $J^2(X)$ when $k \neq \mathbb{C}$. In fact, the following result implies that in many cases, the algebraic representative $(\mathrm{CH}^2 X)^0 \to J_a^2(X)$ is an isomorphism, and $J_a^2(X)$ is isomorphic to $J^2(X)$ over $\mathbb{C}$ (see also [Voi13, Section 1]).

Theorem 3.24 ([BS83, Theorem 1])**.** *Suppose* X *is a smooth projective variety over* $k = \bar{k}$ *such that* $\mathrm{CH}_0 X$ *is supported on a curve*[7]*.*

(1) If $\mathrm{char}\, k = 0$*, then* $(\mathrm{CH}^2 X)^0 \xrightarrow{\sim} \mathrm{Ab}^2 X$*.*
(2) If $k = \mathbb{C}$*, then* $\mathrm{AJ}_X^2 \colon (\mathrm{CH}^2 X)_{\mathrm{hom}} = (\mathrm{CH}^2 X)^0 \xrightarrow{\sim} J^2(X)$*.*

For example, if $\mathrm{CH}_0 X = \mathbb{Z}$ (e.g. if X is rationally connected), then $\mathrm{CH}_0 X$ is supported on a point (hence also on a curve). Bloch and Srinivas prove this result by proving a decomposition of the diagonal. Again, see for example [AB17, Sections 3, 5] for a discussion on this technique and its role in the study of rationality.

To summarize, for X a smooth projective variety over a field $k = \bar{k}$, there exists an algebraic representative for $(\mathrm{CH}^m X)^0$ when $m = 1, 2, n$. Unfortunately, when $m \neq 1, 2, n$, the existence of algebraic representatives is unknown; see e.g. [ACMV23, Section 3].

If you've been following along closely, you'll recall that we really need a *principally polarized* abelian variety to make the obstruction to rationality argument following Theorem 2.8. Benoist and Wittenberg upgrade Murre's abelian variety to a principally polarized one [BW20, Section 2]. They later give a new construction of a principally polarized abelian variety in [BW23, Section 2.3], which will be discussed in Section 5 and used for studying rationality over non-closed fields.

[7]That is, there exists a curve $C \subset X$ such that $\mathrm{CH}_0(X \setminus C) = 0$.

4. Two key examples

Here, we explore the geometry in two key examples which we will revisit over non-algebraically closed fields. We saw in Section 3 that over any algebraically closed field, we have an abelian variety that we can think of as a replacement for the intermediate Jacobian. It remains to be seen if we can still use it for a rationality obstruction as we did in Section 2, but first, let's see how this circle of ideas plays out in two examples.

Throughout this section, we will work over an algebraically closed field k of characteristic not 2.

4.1. **Smooth conic bundle threefolds.** Let $\pi\colon Y \to \mathbb{P}^2$ be a smooth conic bundle threefold, which was first introduced in Example 1.13. Let $\Delta \subset \mathbb{P}^2$ be the discriminant curve and $\varpi\colon \tilde{\Delta} \to \Delta$ the discriminant double cover (recall Definition 1.14). We will assume that $\pi\colon Y \to \mathbb{P}^2$ is **ordinary**, which means that Δ is smooth and irreducible, and that $\pi\colon Y \to \mathbb{P}^2$ is **standard**, which means that $\operatorname{Pic} Y = \pi^* \operatorname{Pic} \mathbb{P}^2 \oplus \mathbb{Z} = \mathbb{Z}^{\oplus 2}$. These assumptions imply that $\tilde{\Delta}$ is smooth and irreducible, the map ϖ is étale, and the fibers of π have at worst simple degeneration.

In general, there is not a pushforward map on divisors, but for covers of curves, there is: the norm map $\varpi_*\colon \mathbf{Pic}^0_{\tilde{\Delta}} \to \mathbf{Pic}^0_{\Delta}$, which is given by $\sum n_i p_i \mapsto \sum n_i \varpi(p_i)$. Using this pushforward, we can naturally associate to Y a principally polarized abelian variety.

Definition/Theorem 4.1. *The* **Prym variety** *of* $\tilde{\Delta}/\Delta$ *is*

$$\mathbf{Prym}_{\tilde{\Delta}/\Delta} := (\ker \varpi_*)^0,$$

the connected component of the identity, which is a principally polarized abelian variety defined over k.

In our study of the intermediate Jacobian of Y, we'll want to use the following characterization of $\mathbf{Prym}_{\tilde{\Delta}/\Delta}$.

Theorem 4.2 (Mumford [Mum74, Sections 2, 3])**.** *There is an isomorphism* $\mathbf{Prym}_{\tilde{\Delta}/\Delta} \cong \mathbf{Pic}^0_{\tilde{\Delta}}/\varpi^*\mathbf{Pic}^0_{\Delta}$.

Not surprisingly, this principally polarized abelian variety naturally associated to the discriminant double cover of Y is isomorphic to the intermediate Jacobian of Y.

Theorem 4.3 (Mumford, Beauville [Bea77, Théorème 2.1, 3.1])**.** *There is an isomorphism* $(\mathrm{CH}^2\, Y)^0 \cong \mathbf{Prym}_{\tilde{\Delta}/\Delta}$. *Moreover, if* $k = \mathbb{C}$*, we also have* $J^2(Y) \cong \mathbf{Prym}_{\tilde{\Delta}/\Delta}$.

Sketch of the proof. To obtain such an isomorphism, we need to figure out how to map between $\tilde{\Delta}$ and Y. To do so, first consider the fiber diagram

$$\begin{tikzcd} Y_\Delta \arrow[r, hook] \arrow[d] & Y \arrow[d, "\pi"] \\ \Delta \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd}$$

and notice that $Y_\Delta \to \Delta$ has a section $\delta\colon \Delta \to Y_\Delta$, given by sending a point $p \in \Delta$ to the intersection point of the two components of the fiber Y_p. Let $Y' := \mathrm{Bl}_{\delta(\Delta)} Y$, and S be the proper transform of Y_Δ. Then S is a $\mathbb{P}^1$-bundle over $\tilde{\Delta}$. We summarize this in the following diagram.

$$\begin{tikzcd}
 & S \arrow[r, hook, "j"] \arrow[dl, "p"'] \arrow[d] & Y' \arrow[d, "\varepsilon"] \\
\tilde{\Delta} \arrow[dr, "\varpi"'] & Y_\Delta \arrow[r, hook] \arrow[d] & Y \arrow[d, "\pi"] \\
 & \Delta \arrow[r, hook] \arrow[u, bend left, "\delta"] & \mathbb{P}^2
\end{tikzcd}$$

If $k = \mathbb{C}$, we can consider the following homomorphisms on cohomology:

$$H^1(\tilde{\Delta}, \mathbb{Z}) \xrightarrow{p^*} H^1(S, \mathbb{Z}) \xrightarrow{j_*} H^3(Y', \mathbb{Z}) \xrightarrow{\varepsilon_*} H^3(Y, \mathbb{Z}).$$

This induces a map $J(\tilde{\Delta}) \to J^2(Y)$ which is surjective and has kernel $\varpi^* J(\Delta)$. The work goes into proving these last two claims; see [Bea77, Chapitre II].

For any $k = \bar{k}$, we do the same thing but on Chow groups:

$$\mathrm{CH}^1 \tilde{\Delta} \xrightarrow{p^*} \mathrm{CH}^1 S \xrightarrow{j_*} \mathrm{CH}^2 Y' \xrightarrow{\varepsilon_*} \mathrm{CH}^2 Y,$$

and you can check that this composition is not mysterious: it sends a point $[p] \in \mathrm{CH}^1 \tilde{\Delta}$ to the curve class $[L_p]$, where L_p is the line in Y_p parametrized by $p \in \tilde{\Delta}$. This induces a map on algebraically trivial classes. See [Bea77, Chapitre III] for the details. $\square$

Exercise 4.4. Let X be a smooth complex cubic threefold. Show that $J^2(X) \cong \mathbf{Prym}_{\tilde{\Delta}/\Delta}$ for Δ a plane quintic curve. *Hint: recall that we saw this curve back in Exercise 1.15.*

We saw in Section 2.5 that a smooth complex cubic threefold X is irrational because $J^2(X)$ is not a product of Jacobians of curves. In fact, rather than studying $J^2(X)$ directly, Mumford showed that for the principally polarized $(\mathbf{Prym}_{\tilde{\Delta}/\Delta}, \Theta)$, the theta divisor Θ has a unique singular point (which is not of large enough dimension to be a Jacobian of a curve).

4.2. **Smooth complete intersections of two quadrics in $\mathbb{P}^5$.** Let $X = Q_0 \cap Q_1 \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics, and recall that we showed in Example 1.6 that X is rational (the rationality construction made use of a line $\ell \subset X$). Thus we know by Theorem 2.8 that when $k = \mathbb{C}$, $J^2(X)$ is a product of Jacobians of curves. In Reid's thesis [Rei72, Theorem 4.14], he shows that $J^2(X)$ is actually the Jacobian of a curve which is naturally associated to X.

Let $Q_i = V(q_i)$ with q_i homogeneous degree 2 polynomials for $i = 1, 2$, and

$$\mathcal{Q} := V(t_0 q_0 + t_1 q_1) \subset \mathbb{P}^1 \times \mathbb{P}^5$$

be the pencil of quadrics containing X. Then projection to $\mathbb{P}^1$ gives a map $\mathcal{Q} \to \mathbb{P}^1$ which is a quadric fourfold bundle. The fibers of $\mathcal{Q} \to \mathbb{P}^1$ are singular if and only if

$$\det(t_0 M_0 + t_1 M_1) = 0,$$

where M_i is the Gram matrix for q_i. Since X is smooth, this implies $\det(t_0 M_0 + t_1 M_1)$, which is a degree 6 polynomial, vanishes at 6 distinct points $x_1, .., x_6$.

Exercise 4.5. Let $Q \subset \mathbb{P}^5$ be a smooth quadric 4-fold.

(1) Show that Q contains 2-planes.
Here's the idea: For any point $x \in Q$, $T_xQ \cap Q$ is a quadric of dimension 3. Check that it is singular, and it is the cone over a smooth quadric surface Q' (See [Har92, Lecture 22] *for a nice introduction to quadrics). Now use lines on Q' to produce 2-planes on Q.*

(2) Let $F := F_2(Q)$, the Fano variety of 2-planes on Q. Show that $\dim F = 3$.
Hint: Use the incidence correspondence $\Psi := \{(x, \Lambda) : x \in \Lambda \subset Q\} \subset Q \times F$. Show that for $x \in Q$, the fiber of $\Psi \to Q$ over x is 1-dimensional. Show that for $\Lambda \in F$, the fiber of $\Psi \to F$ over Λ is 2-dimensional. (cf. Exercise 1.7)

(3) What can you say about 2-planes if Q is not smooth, but is rather a cone over a smooth quadric 3-fold?

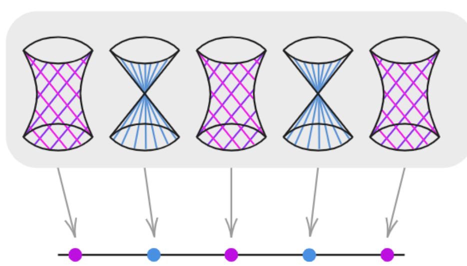
(4) Show that if Q is smooth, $F_2(Q)$ has two connected components, and if Q is a cone over a smooth quadric 3-fold, $F_2(Q)$ has only one connected component.

Now, let

$$\mathcal{F}_2(\mathcal{Q}/\mathbb{P}^1) := \{(\lambda, W) : W \text{ 2-plane}, W \subset Q_\lambda\} \subset \mathbb{P}^1 \times \mathrm{Gr}(3, 6),$$

the relative Fano variety of 2-planes in the fibers of $\mathcal{Q} \to \mathbb{P}^1$. A point of $\mathcal{F}_2(\mathcal{Q}/\mathbb{P}^1)$ parametrizes a 2-plane in some quadric Q_λ in the pencil $\mathcal{Q} \to \mathbb{P}^1$.

This object may not be familiar to you, but you should think of it as a higher-dimensional analogue of what happens for a pencil of quadric surfaces:



Most fibers will be smooth quadric surfaces, which have two rulings of lines. Finitely many of the fibers will degenerate to singular quadrics, and assuming only simply degeneration, they will have just one ruling of lines (simple degeneration will follow automatically if the quadric surface fibration comes from a pencil of quadrics containing a smooth complete intersection; see [Rei72, Proposition 2.1]).

For a quadric fourfold fibration such as $\mathcal{Q} \to \mathbb{P}^1$, the smooth fibers will contain two families of 2-planes. Over the points $x_1, ..., x_6$, the quadrics drop rank and contain just one family of 2-planes. This information is captured by taking the Stein factorization of $\varphi \colon \mathcal{F}_2(\mathcal{Q}/\mathbb{P}^1) \to \mathbb{P}^1$:

$$\begin{tikzcd} \mathcal{F}_2(\mathcal{Q}/\mathbb{P}^1) \arrow[dd, "\varphi"'] \arrow[dr, "p"] & \\ & C \arrow[dl, "q"] \\ \mathbb{P}^1 & \end{tikzcd}$$

The Stein factorization gives that:

- C is a smooth curve,
- the fibers of p are connected, and
- q is a double cover ramified over $x_1, \dots, x_6$.

That is, C is a hyperelliptic curve of genus 2. In his thesis, Reid shows the following:

Theorem 4.6 (Reid [Rei72, Theorem 4.8, 4.14]). *When $k = \mathbb{C}$, there is an isomorphism $J^2(X) \cong J(C)$. For any $k = \bar{k}$ of characteristic not 2, there is an isomorphism $\mathbf{Pic}^0_C \cong F_1(X)$, where $F_1(X)$ is the Fano variety of lines on X.*

The following exercise is meant to give some ideas behind the proof (at least of the first part). For another perspective on the abelian variety structure of $F_1(X)$, see [Don80, Section 2], where Donagi explicitly exhibits the group law on $F_1(X)$.

Exercise 4.7. Let $k = \mathbb{C}$ and fix a line $s \subset X$. Since X is the base locus of $\mathcal{Q} \to \mathbb{P}^1$, this gives a line in Q_λ for all $\lambda \in \mathbb{P}^1$. Let $\tilde{C}_s := \{(\lambda, \Lambda) : s \subset \Lambda \subset Q_\lambda\} \subset \mathcal{F}_2(\mathcal{Q}/\mathbb{P}^1)$, so that $\tilde{C}_s$ parametrizes the 2-planes in the fibers of $\mathcal{Q} \to \mathbb{P}^1$ which contain the line s.

(1) Show that $p|_{\tilde{C}_s} \colon \tilde{C}_s \to C$ is an isomorphism, concluding that $\tilde{C}_s$ is a smooth curve.
(2) Show that there is a morphism $r \colon \tilde{C}_s \to F_1(X)$. *Hint: Given a 2-plane $\Lambda \subset Q_\lambda$, consider $\Lambda \cap X$.*
    Let $r' \colon C \to F_1(X)$ be r precomposed with the isomorphism $C \xrightarrow{\sim} \tilde{C}_s$, and $\Gamma_{r'} \in \mathrm{CH}^2(C \times F_1(X))$ the cycle class of the graph of r'.
(3) Show that $\Gamma_{r'}$ induces a map $J(C) \to \mathrm{Alb}(F_1(X))$.
    Next, let $T := \{(s, x) : x \in s\} \subset F_1(X) \times X$, and consider $T \in \mathrm{CH}^2(F_1(X) \times X)$.
(4) Show that T induces a map $\mathrm{Alb}(F_1(X)) \to J^2(X)$.

Reid shows that the two morphisms in (3) and (4) are isomorphisms. Finally, Reid shows that $F_1(X)$ is an abelian variety, so $\mathrm{Alb}\, F_1(X) \cong F_1(X)$.

5. Rationality criteria over non-closed fields

We are finally in a place to discuss rationality over non-closed fields. Let X be a smooth projective variety over a field k and fix an algebraic closure $\bar{k}$ of k. We say that X is **geometrically rational** if $X_{\bar{k}}$ is rational. If X is rational (that is, k-rational), then X is geometrically rational.

We will be interested in understanding when a variety X is rational, so we can assume X is geometrically rational. By the Lang-Nishimura Theorem (see [Poo17, Theorem 3.6.11]), the existence of a k-point is a birational invariant for projective varieties, so we may also assume $X(k) \neq \emptyset$. Note that if X is unirational, this condition is automatically satisfied.

Exercise 5.1. Let C be a geometrically rational curve over a field k. Show that C is rational if and only if $C(k) \neq \emptyset$.

The Lang-Nishimura Theorem is overkill here; you can show that C being rational implies $C(k) \neq \emptyset$ without it.

This exercise says that the existence of a k-point characterizes rationality for geometrically rational curves.

For surfaces, the story is more complicated, but completely understood. See, for example, [Has09] for a discussion of the classification of rational surfaces over non-closed fields. There exist non-rational smooth cubic surfaces (even ones containing a point!) [CT88, Example 3.3], whereas a cubic surface is always geometrically rational (as we saw in Section 1.1).

For threefolds, the story is still unfolding. This will be the focus of the remainder of these notes. In this section, we let X be a smooth projective threefold over a field k. Fix an algebraic closure $\bar{k}$ of k and let k^p be the perfect closure of k in $\bar{k}$. Let $G_k := \mathrm{Gal}(\bar{k}/k^p)$.

5.1. **A rationality obstruction for threefolds.** In an effort to study rationality, we would like a replacement for the intermediate Jacobian, and in particular, a criterion along the lines of Theorem 2.8 to use as a rationality obstruction. We saw in Section 3 that, over algebraically closed fields, there is an abelian variety which can play the role of $J^2(X)$: Murre's algebraic representative for $(\mathrm{CH}^2 X)^0$ (Theorem 3.23). The following result shows that often the situation is even better: this abelian variety *descends to* k.

Theorem 5.2 (Achter–Casalaina-Martin–Vial)**.**
(1) [ACMV17, Theorem B] *If $k \subset \mathbb{C}$, then there is an abelian variety J over k such that $J_{\mathbb{C}} \cong J^2_a(X)$.*
(2) [ACMV17, Theorem 4.4] *If k is perfect, then there is an abelian variety A over k such that $A_{\bar{k}} \cong \mathrm{Ab}^2(X_{\bar{k}})$, and $A = J$ when $k \subset \mathbb{C}$.*

The issue is that when k is imperfect, the abelian variety $\mathrm{Ab}^2(X_{\bar{k}})$ does not give $\bar{k}/k$-descent data. In this setting, Achter, Casalaina-Martin and Vial prove the existence of an algebraic representative for $(\mathrm{CH}^2 X)^0$, but it is unknown whether this abelian variety is isomorphic over $\bar{k}$ to $\mathrm{Ab}^2(X_{\bar{k}})$ [ACMV23, Theorem 2].

To remedy this issue, Benoist and Wittenberg give a new construction of an abelian variety, via algebraic K-theory, which can serve in the role of the intermediate Jacobian over *any* field k. The trade-off is that their construction only works for *geometrically rational* threefolds. Luckily, since we're interested in the rationality of threefolds, there is no harm in restricting our study to those which are geometrically rational.

Theorem 5.3 (Benoist–Wittenberg [BW23, Theorem 3.1])**.** *Let X be a smooth projective geometrically rational threefold over k. There exists a group scheme $\mathbf{CH}^2_X$ over k such that:*
(1) The identity component $(\mathbf{CH}^2_X)^0$ is a principally polarized abelian variety, which agrees with those in Theorem 5.2 when k is perfect.
(2) There is a G_k-equivariant isomorphism $\mathrm{CH}^2 X_{\bar{k}} \xrightarrow{\sim} (\mathbf{CH}^2_X)(\bar{k})$ which restricts to an isomorphism $(\mathrm{CH}^2 X_{\bar{k}})^0 \cong (\mathbf{CH}^2_X)^0(\bar{k})$.
(3) The G_k-invariant algebraic curve classes are given by

$$(\mathrm{NS}^2 X_{\bar{k}})^{G_k} = \left(\mathbf{CH}^2_X/(\mathbf{CH}^2_X)^0\right)(k).$$

(4) If X is rational over k, then there exist smooth curves $C_1, ..., C_r$ over k such that

$$(\mathbf{CH}^2_X)^0 \cong \mathbf{Pic}^0_{C_1} \times \cdots \times \mathbf{Pic}^0_{C_r}$$

as principally polarized abelian varieties.

This is excellent! Part (4) exactly gives the statement of Theorem 2.8 over any field, and hence gives a rationality obstruction over non-closed fields. We will continue to call this the intermediate Jacobian obstruction to rationality. We will call the group scheme $\mathbf{CH}^2_X$ the **codimension 2 Chow scheme**.

We will skip the proof of this result here but point out that part (4) is proven analogously to the proof of Theorem 2.8, by comparing the codimension 2 Chow schemes of X and $\mathbb{P}^3$ via the resolution of the rational parameterization.

Notice that this rationality obstruction only takes into account the identity component of the group scheme $\mathbf{CH}^2_X$. What do the other components of $\mathbf{CH}^2_X$ tell us? We will see below that they can be used for a refinement of the intermediate Jacobian obstruction to rationality. But first, we need a digression on the theory of torsors.

5.2. **Torsors.** The main reference for this section is [Poo17, Section 5.12]. Let k^s be the separable closure of k in $\bar{k}$ and $G_{k^s} := \mathrm{Gal}(k^s/k)$.

Let's start with C a smooth projective curve over a field $k = \bar{k}$. We've worked with $\mathbf{Pic}^0_C$ already, but we can also consider the other components of the Picard scheme. For curves, these components are of the form $\mathbf{Pic}^d_C$ for $d \in \mathbb{Z}$, parametrizing degree d line bundles on C. There is an action of $\mathbf{Pic}^0_C$ on $\mathbf{Pic}^d_C$, given by the group structure on $\mathbf{Pic}_C$, and $\mathbf{Pic}^0_C$ acts freely and transitively.

Since $k = \bar{k}$, we can pick a point $M \in \mathbf{Pic}^d_C$, and this choice determines an isomorphism

$$\begin{aligned} \mathbf{Pic}^0_C &\xrightarrow{\sim} \mathbf{Pic}^d_C \\ L &\mapsto L \otimes M. \end{aligned}$$

If instead $k \neq \bar{k}$, then we still have that $\mathbf{Pic}^0_C$ and $\mathbf{Pic}^d_C$ are defined over k, but it could be the case that $\mathbf{Pic}^d_C(k) = \emptyset$ (in particular, they're no longer isomorphic, since we always have $\mathcal{O}_C \in \mathbf{Pic}^0_C(k)$). Regardless, there is still an action of $\mathbf{Pic}^0_C$ on $\mathbf{Pic}^d_C$, and there is an isomorphism $\mathbf{Pic}^0_{C,k^s} \cong \mathbf{Pic}^d_{C,k^s}$ respecting the action of $\mathbf{Pic}^0_{C,k^s}$. We call $\mathbf{Pic}^d_C$ a $\mathbf{Pic}^0_C$-torsor, or principal homogeneous space.

Definition 5.4. Let G be a smooth algebraic group scheme over a field k. A G-**torsor over** k (or a **torsor under** G or a **principal homogeneous space for** G) is a k-variety X equipped with a right action of G such that X_{k^s} equipped with the right action of G_{k^s} is isomorphic to G_{k^s} (equipped with the right action of translation).

This is equivalent to saying that there is a morphism $\mu\colon G \times X \to X$ (giving the action of G on X) for which $\mu(k^s)\colon G(k^s) \times X(k^s) \to X(k^s)$ is a free and transitive action of $G(k^s)$ on $X(k^s)$.

Exercise 5.5. Check that $\mathbf{Pic}^d_C$ satisfies the definition of being a $\mathbf{Pic}^0_C$-torsor.

Exercise 5.6 (From [Poo17, Examples 5.5.3 and 5.12.8])**.** Let $T := V(x^2 + 2y^2 - 1) \subset \mathbb{A}^2_{\mathbb{Q}}$ and $X := V(x^2 + 2y^2 + 3) \subset \mathbb{A}^2_{\mathbb{Q}}$.

(1) Show that T is a group scheme with multiplication given by

$$\begin{aligned} m\colon T \times T &\to T \\ ((x_1, y_1), (x_2, y_2)) &\mapsto (x_1x_2 - 2y_1y_2, x_1y_2 + y_1x_2). \end{aligned}$$

(In fact $T_{\bar{\mathbb{Q}}} \cong \mathbb{G}_{m,\bar{\mathbb{Q}}}$.)

(2) Show that X is a T-torsor over $\mathbb{Q}$.

(In fact, it is a non-trivial torsor since $X(\mathbb{Q}) = \emptyset$.)

Exercise 5.7. Let C be a smooth projective genus one curve over k (and note that $C(k)$ may be empty). Show that C is a torsor under the elliptic curve $\mathbf{Pic}^0_C$.

The collection of all G-torsors up to isomorphism is parametrized by the cohomology set $H^1(k, G(\bar{k}))$. When G is commutative, this cohomology set is an *abelian group*.

Definition/Theorem 5.8. *For an abelian variety A, the* **Weil-Châtelet group**

$$\mathrm{WC}(A) := \{\textit{torsors under } A\}/\textit{isomorphism}$$

is an abelian group.

Exercise 5.9. The group operation on the Weil-Châtelet group of an abelian variety A can be described as follows. Let T_1 and T_2 be A-torsors, and $T_1 \times^A T_2$ the quotient of $T_1 \times T_2$ by the A-action where $a \cdot (t_1, t_2) := (a + t_1, [-1]a + t_2)$ for $a \in A$. Here, $[-1]\colon A \to A$ is the standard involution on A (i.e. it is the inverse morphism for the group structure on A).

(1) Show that the action of A on $T_1 \times T_2$ (given by acting on the first factor) descends to an action on $T_1 \times^A T_2$, making $T_1 \times^A T_2$ into an A-torsor.
(2) Let $[T_1] + [T_2] := [T_1 \times^A T_2]$. Show that this operation gives a group law on $\mathrm{WC}(A)$, where the inverse of a torsor T is T with the action $a \cdot t := [-1]a + t$.

For an A-torsor X, we will write $[X]$ for its class in $\mathrm{WC}(A)$. We list here some of the nice properties of $\mathrm{WC}(\mathbf{Pic}^0_C)$ for C a smooth projective curve:

(1) $[\mathbf{Pic}^0_C] = 0$.
(2) $[T] = 0$ if and only if $T(k) \neq \emptyset$ if and only if $T \cong \mathbf{Pic}^0_C$.
(3) For all $d \in \mathbb{Z}$, $[\mathbf{Pic}^d_C] = d[\mathbf{Pic}^1_C]$.

In particular, we will make use of these properties in the next section. We remark that the first two properties hold more generally for any smooth algebraic group G.

Exercise 5.10. Use the structure of $\mathbf{Pic}_C$ and the description of the group operation in Exercise 5.9 to prove property (3).

Exercise 5.11. Let C be a smooth projective curve over k of genus $g \geq 2$, and let $m = 2g-2$. Show that for all $t \in \mathbb{Z}$, $[\mathbf{Pic}^{tm}_C] = [\mathbf{Pic}^0_C] = 0$.

This shows that the subgroup $\langle \mathbf{Pic}^1_C \rangle \leqslant \mathrm{WC}(\mathbf{Pic}^0_C)$ is a finite cyclic group. The order of this subgroup is called the period of C.

Finally, we remark that not all torsors under $\mathbf{Pic}^0_C$ are of the form $\mathbf{Pic}^d_C$; the Weil-Châtelet group is often infinite (e.g. for number fields and function fields of curves) [Sha57, CL19].

5.3. **The refined rationality obstruction.** Let's again return to the case where X is a smooth projective geometrically rational threefold over a field k and recall that $G_k = \mathrm{Gal}(\bar{k}/k^p)$. We are now in a position to understand how the other components of $\mathbf{CH}^2_X$ give a rationality obstruction.

For $\alpha \in (\mathrm{NS}^2 X_{\bar{k}})^{G_k}$, let $(\mathbf{CH}^2_X)^\alpha$ be the preimage of α under the quotient $\mathbf{CH}^2_X \to \mathbf{CH}^2_X/(\mathbf{CH}^2_X)^0$. By Theorem 5.3, there is a G_k-equivariant isomorphism

$$(\mathrm{CH}^2 X_{\bar{k}})^\alpha \cong (\mathbf{CH}^2_X)^\alpha(\bar{k}),$$

so this component parametrizes codimension 2 cycles algebrically equivalent to α. Since α is fixed by Galois, the component $(\mathbf{CH}^2_X)^\alpha$ is defined over k, and is in fact a $(\mathbf{CH}^2_X)^0$-torsor.

The theorem below was proved first by Hassett and Tschinkel over $k = \mathbb{R}$ [HT21a, Proposition 34], then extended to fields $k \subset \mathbb{C}$ [HT21b, Theorem 6.3], and then proved by Benoist and Wittenberg over arbitrary fields [BW23, Theorem 3.11][8].

[8]The results are more general than what we state in Theorem 5.12; we add these assumptions to simplify, and hopefully clarify, the situation.

Theorem 5.12 (Hassett–Tschinkel, Benoist–Wittenberg)**.** *Suppose that X is rational over k, and is such that $(\mathbf{CH}^2_X)^0 \cong \mathbf{Pic}^0_C$ for C a smooth projective geometrically connected curve of genus ≥ 2. Then for all $\alpha \in (\mathrm{NS}^2 X_{\bar{k}})^{G_k}$, there exists some $d \in \mathbb{Z}$ such that $(\mathbf{CH}^2_X)^\alpha \cong \mathbf{Pic}^d_C$ and this isomorphism respects the actions of $(\mathbf{CH}^2_X)^0$ and $\mathbf{Pic}^0_C$ via the isomorphism $(\mathbf{CH}^2_X)^0 \cong \mathbf{Pic}^0_C$.*

Another way to say this is that, via the isomorphism $(\mathbf{CH}^2_X)^0 \cong \mathbf{Pic}^0_C$, we can consider $(\mathbf{CH}^2_X)^\alpha$ as a $\mathbf{Pic}^0_C$-torsor. Then we must have an equality

$$[(\mathbf{CH}^2_X)^\alpha] = [\mathbf{Pic}^d_C] \in \mathrm{WC}(\mathbf{Pic}^0_C).$$

As discussed at the end of Section 5.2, not all $\mathbf{Pic}^0_C$-torsors are of the form $\mathbf{Pic}^d_C$ for some $d \in \mathbb{Z}$, so this result is placing a restriction on what these torsors can look like when X is rational. Thus, Theorem 5.12 gives a refinement to the intermediate Jacobian obstruction to rationality. We will call this the **intermediate Jacobian torsor (IJT) obstruction**: If a geometrically rational threefold X is such that $(\mathbf{CH}^2_X)^0 \cong \mathbf{Pic}^0_C$ for C a smooth projective geometrically connected curve of genus ≥ 2, but there is some $\alpha \in (\mathrm{NS}^2 X_{\bar{k}})^{G_k}$ for which $(\mathbf{CH}^2_X)^\alpha \ncong \mathbf{Pic}^d_C$ for any $d \in \mathbb{Z}$, then X must be irrational over k.

Observe that this is the first obstruction we have encountered so far that only exists over non-closed fields. If $k = \bar{k}$, all of the $(\mathbf{CH}^2_X)^0$-torsors are isomorphic to $(\mathbf{CH}^2_X)^0$, and there is no content to Theorem 5.12 beyond that of the intermediate Jacobian obstruction.

We will see in the next section how the IJT obstruction has been used to characterize rationality for many families of geometrically rational threefolds.

6. Rationality results for threefolds over non-closed fields

Here, we see how the IJT obstruction has been used in the study of rationality for threefolds over non-algebraically closed fields.

6.1. **Intersections of two quadrics.** Let $X := Q_0 \cap Q_1 \subset \mathbb{P}^5$ be smooth over an arbitrary field k. We have seen that X is geometrically rational, and moreover, Example 1.6 actually shows that if X contains a line over k, then X is rational over k.

Theorem 6.1 (Hassett–Tschinkel, Benoist–Wittenberg)**.** *A smooth threefold complete intersection of two quadrics X is rational if and only if X contains a line over k.*

Proof. The reverse direction was discussed above, so let's show the forward direction. Suppose that X is rational. To show that X contains a line over k, it is equivalent to show that $F_1(X)(k) \neq \emptyset$.

Recall that when $k = \bar{k}$, we saw in Theorem 4.6 that $F_1(X) \cong \mathbf{Pic}^0_C$, where C is the double cover of $\mathbb{P}^1$ parametrizing families of 2-planes in the pencil of quadrics containing X.

When $k \neq \bar{k}$, the isomorphism does not necessarily descend, since it could be the case that $F_1(X)$ has no k-points. However, Wang shows in [Wan18, Theorem 1.1] that $F_1(X)$ is a $\mathbf{Pic}^0_C$-torsor and satisfies

$$2[F_1(X)] = [\mathbf{Pic}^1_C] \tag{2}$$

in $\mathrm{WC}(\mathbf{Pic}^0_C)$.

On the other hand, you can show that there is some $\alpha \in (\mathrm{NS}^2 X_{\bar{k}})^{G_k}$ such that

$$F_1(X) \cong (\mathbf{CH}^2_X)^\alpha,$$

where the isomorphism sends a point $t \in F_1(X)$ to the class $[L_t] \in (\mathbf{CH}^2_X)^\alpha$, where $L_t \subset X$ is the line parametrized by the point t. This isomorphism implies that

$$\mathbf{Alb}_{F_1(X)} \cong \mathbf{Alb}_{(\mathbf{CH}^2_X)^\alpha}.$$

We need to be a bit careful here–our first introduction to the Albanese variety in Example 3.5 made use of the existence of a point (cf. Exercise 3.8). There is a theory of Albanese varieties for torsors (see [Poo17, Example 5.12.11] and Exercise 6.4), from which it follows that $\mathbf{Alb}_{F_1(X)} \cong \mathbf{Pic}^0_C$ and $\mathbf{Alb}_{(\mathbf{CH}^2_X)^\alpha} \cong (\mathbf{CH}^2_X)^0$.

Thus, we are in the setting that X is rational and $(\mathbf{CH}^2_X)^0 \cong \mathbf{Pic}^0_C$ with C a smooth curve of genus 2. By Theorem 5.12, since we're assuming X is rational, it follows that

$$F_1(X) \cong (\mathbf{CH}^2_X)^\alpha \cong \mathbf{Pic}^d_C$$

for some $d \in \mathbb{Z}$. Using (2) along with the properties of the Weil-Châtelet group introduced in Section 5.2, this means

$$2d[\mathbf{Pic}^1_C] = 2[\mathbf{Pic}^d_C] = [\mathbf{Pic}^1_C] \in \mathrm{WC}(\mathbf{Pic}^0_C). \tag{3}$$

Now, since $g(C) = 2$, $\deg K_C = 2g(C) - 2 = 2$, and the canonical divisor gives an isomorphism $\mathbf{Pic}^0_C \cong \mathbf{Pic}^{2t}_C$ for all $t \in \mathbb{Z}$ (cf. Exercise 5.11). Again using the properties of the Weil-Châtelet group, it follows that

$$2[\mathbf{Pic}^1_C] = [\mathbf{Pic}^2_C] = 0. \tag{4}$$

Now, (3) together with (4) imply that

$$0 = d \cdot 2[\mathbf{Pic}^1_C] = [\mathbf{Pic}^1_C],$$

from which it follows that $[\mathbf{Pic}^e_C] = 0$ for all $e \in \mathbb{Z}$. In particular, $[F_1(X)] = 0$, which means $F_1(X)(k) \neq \emptyset$, so X contains a line over k. □

Exercise 6.2.

(1) Show that all lines in X are algebraically equivalent.
(2) Show that any two distinct lines in X are not rationally equivalent.

Exercise 6.3. Show that $\mathrm{NS}^2 X_{\bar{k}} \cong \mathbb{Z}$, thus showing that the codimension 2 Chow scheme $\mathbf{CH}^2_X$ has a $\mathbb{Z}$ grading.

Exercise 6.4. [Poo17, Example 5.12.11] Let X be a geometrically integral variety over k, and $\mathcal{C}_X$ the category of triples (A, T, f) where A is an abelian variety, T is an A-torsor, and $f\colon X \to T$ is a morphism. A morphism (A, T, f) to (A', T', f') is a homomorphism $\alpha\colon A \to A'$ and a morphism $\tau\colon T \to T'$ such that the following diagrams commute:

$$\begin{tikzcd} T \times A \arrow[r] \arrow[d, "{(\tau,\alpha)}"'] & T \arrow[d, "\tau"] \\ T' \times A' \arrow[r] & T' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & T \arrow[d, "\tau"] \\ & T' \end{tikzcd}$$

It is a theorem that this category has an initial object $(\mathbf{Alb}_X, \mathbf{Alb}^1_X, \iota)$; $\mathbf{Alb}_X$ is the **Albanese variety** of X, and $\mathbf{Alb}^1_X$ is the **Albanese torsor** of X.

(1) Show that, if X has a k-point $x \in X(k)$, this definition of the Albanese variety agrees with the one introduced in Example 3.5.

(2) Let A be an abelian variety and T an A-torsor. Show that $\mathbf{Pic}^0_T \cong \mathbf{Pic}^0_A$.
Hint: Use a functorial approach. Show that for any scheme S over k, the set $T(S)$ is in bijection with A_S-torsor isomorphisms $A_S \cong T_S$. Use pushforward along these isomorphisms to get a map $\mathbf{Pic}^0_A \to \mathbf{Pic}^0_T$. See [Ols08, Section 2.1].

(3) Let C be a smooth projective (geometrically integral) curve. Show that $\mathbf{Alb}_{\mathbf{Pic}^d_C} \cong \mathbf{Pic}^0_C$. *Hint: You will need to use that $\mathbf{Pic}^0_C$ is principally polarized.*

Before looking at other examples, we point out that Theorem 6.1 says that for smooth threefold complete intersections of two quadrics, the IJT obstruction characterizes rationality.

6.2. **Fano threefolds with geometric Picard rank 1.** Shortly after the work of Hassett–Tschinkel and Benoist–Wittenberg, Kuznetsov and Prokhorov in [KP23, Theorem 1.1] characterized rationality for geometrically rational Fano threefolds with geometric Picard rank one (working only in characteristic zero). There are 8 families which make up the complete classification of such threefolds, and their results for the various families are of the form:

$$X \text{ is rational} \iff X(k) \neq \emptyset,$$

or

$$X \text{ is rational} \iff X(k) \neq \emptyset \text{ and } X \text{ contains a genus 0 curve of degree } d \text{ (for a specific } d).$$

The first case says that for some families, rationality is characterized by the existence of a k-point.

In the second case, they show that the existence of such a curve can be used for a rationality construction (for example, one of the 8 families is complete intersections of two quadrics where $d = 1$). For the forward implication, they show that such curves are parametrized by a $(\mathbf{CH}^2_X)^0$-torsor, and that the rationality of X implies that torsor is trivial. Thus, in these cases, the IJT obstruction (plus the existence of a k-point) characterizes rationality.

6.3. **Cubic threefolds containing a plane.** We showed in Theorem 2.12 that smooth complex cubic threefolds are irrational, and saw in Section 4.1 that Mumford's analysis of the singularities of the theta divisor used the isomorphism $J^2(X) \cong \mathbf{Prym}_{\tilde{\Delta}/\Delta}$ (Exercise 4.4). Over any algebraically closed field k with $\operatorname{char} k \neq 2$, Murre showed that if a smooth cubic threefold is rational, then the associated Prym variety is isomorphic to a product of Jacobians of curves [Mur73, Theorem p. 63]. Since Mumford's study of theta divisors of Prym varieties holds over $k = \bar{k}$, $\operatorname{char} k \neq 2$, it follows that a smooth cubic threefold is irrational over any field k of characteristic not 2.

On the other hand, we saw in Exercise 1.12 that general cubic threefolds containing a plane (which have 4 isolated singularities on the plane) are geometrically rational over fields k of characteristic not 2. We actually gave two rationality constructions: projection from a node or projection from a line disjoint from the plane. Over a non-closed field, we get rationality if either a node or a line disjoint from the plane is defined over the ground field. Brooke recently used the IJT obstruction to show that the converse is also true.

Theorem 6.5 (Brooke [Bro, Theorem 1.2])**.** *Let X be a cubic threefold containing exactly one plane P over a field of characteristic not 2, and suppose that X has isolated singularities, all of which are confined to P. Then X is rational over k if and only if X contains a node defined over k or a line defined over k in $X \setminus P$.*

Again, this says that the IJT obstruction characterizes rationality for general cubic threefolds containing a plane. The proof strategy is similar to the previous cases, where a careful analysis of the Fano variety of lines on X is necessary to show that rationality implies the existence of the desired node or line over k.

6.4. **Conic bundle threefolds.** Let $\pi\colon Y \to \mathbb{P}^2$ be a smooth conic bundle threefold over a field k with $\operatorname{char} k \neq 2$, and $G_k = \operatorname{Gal}(\bar{k}/k^p)$ for k^p the perfect closure of k in $\bar{k}$. Let $\Delta \subset \mathbb{P}^2$ be the discriminant curve and $\varpi\colon \tilde{\Delta} \to \Delta$ the discriminant double cover. As in Section 4.1, we will assume that $\pi\colon Y \to \mathbb{P}^2$ is geometrically ordinary and geometrically standard. We will start with a description of the curve classes on Y, and then see how this description can be used for rationality results on conic bundle threefolds.

When $k = \bar{k}$, we saw in Theorem 4.3 that $(\mathrm{CH}^2 Y)^0 \cong \mathbf{Prym}_{\tilde{\Delta}/\Delta}$. It follows that, for any $\alpha \in \mathrm{NS}^2 Y$, the coset $(\mathrm{CH}^2 Y)^\alpha$ of rational curve classes algebraically equivalent to α is isomorphic to $(\mathrm{CH}^2 Y)^0$ (where the isomorphism is given by picking a base point of $(\mathrm{CH}^2 Y)^\alpha$). Thus, all curve classes on Y are described by the Prym variety.

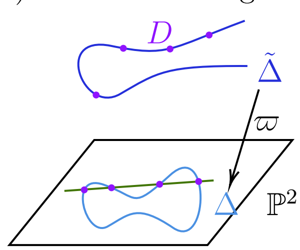
If k is arbitrary, it could be that there are no classes in $(\mathrm{CH}^2 Y)^\alpha$–these sets instead become torsors under $(\mathrm{CH}^2 Y)^0$. This is exactly analogous to the case of $F_1(X)$ for X a smooth threefold complete intersection of two quadrics. It would be nice to have torsors under $\mathbf{Prym}_{\tilde{\Delta}/\Delta}$ which describe these families of curve classes. In [FJS+23, Section 4], we introduce a group scheme whose components give exactly such torsors.

Definition 6.6. The **polarized Prym scheme of** $\tilde{\Delta}/\Delta$ is

$$\mathbf{PPrym}_{\tilde{\Delta}/\Delta} := \{\mathcal{O}_{\tilde{\Delta}}(D) : \mathcal{O}_{\Delta}(\varpi_* D) \cong \mathcal{O}_{\Delta}(m) \text{ for some } m \in \mathbb{Z}\} \subset \mathbf{Pic}_{\tilde{\Delta}}.$$

The scheme $\mathbf{PPrym}_{\tilde{\Delta}/\Delta}$ is a group scheme over k with identity component the usual $\mathbf{Prym}_{\tilde{\Delta}/\Delta} = (\ker \varpi_*)^0$.

Example 6.7. Suppose that $\deg \Delta = 4$, so that a line in $\mathbb{P}^2$ intersects Δ in 4 points (counted with multiplicity). Let $D \in \mathbf{Pic}^4_{\tilde{\Delta}}(k)$ be the following collection of 4 points on $\tilde{\Delta}$:



Then $\mathcal{O}_{\Delta}(\varpi_* D) \cong \mathcal{O}_{\Delta}(1)$, so $D \in \mathbf{PPrym}_{\tilde{\Delta}/\Delta}(k)$.

Proposition 6.8 (Frei–Ji–Sankar–Viray–Vogt [FJS+23, Theorem 5.1])**.** *There exists a surjective G_k-equivariant group homomorphism*

$$\mathrm{CH}^2 Y_{\bar{k}} \to \mathbf{PPrym}_{\tilde{\Delta}/\Delta}(\bar{k})$$

that induces an isomorphism between $(\mathrm{CH}^2 Y_{\bar{k}})^\alpha$ and a component of $\mathbf{PPrym}_{\tilde{\Delta}/\Delta}$ for all $\alpha \in (\mathrm{NS}^2 Y_{\bar{k}})^{G_k}$.

Moreover, if Y is geometrically rational, there exists a surjective morphism

$$\mathbf{CH}^2_Y \to \mathbf{PPrym}_{\tilde{\Delta}/\Delta}$$

giving an isomorphism on components.

Note that the theorem does not give a global isomorphism. Rather, multiple $(\mathrm{CH}^2 Y_{\bar{k}})^0$-cosets map to the same component of $\mathbf{PPrym}_{\tilde{\Delta}/\Delta}$. However, when restricted to any coset $(\mathrm{CH}^2 Y_{\bar{k}})^{\alpha}$, the homomorphism is an isomorphism (and every component of $\mathbf{PPrym}_{\tilde{\Delta}/\Delta}$ is in the image). The same happens when the statement can be upgraded to a statement about the codimension 2 Chow scheme.

We will skip the proof of Proposition 6.8 here, but just mention that we use the same maps that we saw in the proof of Theorem 4.3. However, in the case that Y is geometrically rational, some finesse is required to upgrade these maps to morphisms of schemes.

6.4.1. *Geometric rationality.* We continue to let k be a field with $\operatorname{char} k \neq 2$. Geometrically, the rationality of conic bundles over $\mathbb{P}^2$ is understood. We have that $\pi\colon Y \to \mathbb{P}^2$ is geometrically rational if and only if $\deg \Delta \leq 4$ or $\deg \Delta = 5$ and $\tilde{\Delta} \to \Delta$ is an even theta characteristic (see [Pro18, Section 3.9.2] for a definition). The result for $\deg \Delta \leq 4$ was proved by Iskovskikh [Isk96] and follows from rationality results for conic bundle surfaces, [Pro18, Corollary 5.6.1]. The rationality in degree 5 with even theta characteristic was shown by Panin (in characteristic zero) [Pan80], and the irrationality in degree 5 with odd theta characteristic is due to Shokurov [Sho83], [Pro18, Theorem 7.5, Proposition 8.1]. The irrationality when $\deg \Delta \geq 6$ is due to Beauville [Bea77, Théorème 4.9], who showed that $\mathbf{Prym}_{\tilde{\Delta}/\Delta}$ is not a product of Jacobians of curves, using the ideas discussed in Sections 2 and 4 about the singularities of the theta divisor. In fact, all of these results together say that geometric rationality of conic bundle threefolds over $\mathbb{P}^2$ is characterized by the intermediate Jacobian obstruction [Pro18, Theorem 9.1].

Remark 6.9. You showed in Exercise 1.15 that the conic bundle coming from a smooth cubic threefold has a discriminant curve of degree 5. It also has $\tilde{\Delta} \to \Delta$ corresponding to an odd theta characteristic.

6.4.2. *Rationality over non-closed fields.* Now suppose k is an arbitrary field ($\operatorname{char} k \neq 2$), $Y(k) \neq \emptyset$, and $\deg \Delta \leq 5$. These last two assumptions are necessary for rationality. A slight modification to the work of Iskovskikh shows that if $\deg \Delta \leq 3$, then Y is rational over k. So the first case of interest is when $\deg \Delta = 4$. Let's specialize to that case. The study of curve classes on $Y \to \mathbb{P}^2$ discussed above allows us to better understand the group scheme $\mathbf{CH}^2_Y$. This in turn allows us to construct examples of conic bundle threefolds where the IJT obstruction does and does not vanish.

Theorem 6.10 (FJSVV [FJS+23, Theorem 1.4])**.** *There exists a smooth conic bundle* $\pi\colon Y \to \mathbb{P}^2$ *over* $\mathbb{Q}$*, which is geometrically ordinary and geometrically standard, with* $\deg \Delta = 4$*, for which the intermediate Jacobian torsor obstruction vanishes, but* Y *is irrational over* $\mathbb{Q}$*.*

Exercise 6.11. Let $Y \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched along a smooth $(2,2)$-divisor.

(1) Show that Y has the structure of a conic bundle over $\mathbb{P}^2$ and the structure of a quadric surface bundle over $\mathbb{P}^1$.
(2) Show that the discriminant curve of the conic bundle $Y \to \mathbb{P}^2$ has degree 4.

The examples we construct in [FJS+23, Theorem 1.4, Example 1.6] with interesting IJT behavior, including the one mentioned in Theorem 6.10, are constructed as these double covers, and the quadric surface fibration is a key ingredient in our understanding of the intermediate Jacobian torsors.

Recall that the IJT obstruction vanishing means that $(\mathrm{CH}^2_Y)^0 \cong \mathbf{Pic}^0_C$ for a smooth projective curve C with $g(C) \geq 2$, and that moreover $(\mathbf{CH}^2_Y)^\alpha \cong \mathbf{Pic}^d_C$ for some $d \in \mathbb{Z}$, for all $\alpha \in (\mathrm{NS}^2\, Y_{\bar{k}})^{G_k}$. If this obstruction did not vanish, Theorem 5.12 would imply that Y is irrational. To show the vanishing of this obstruction, we make crucial use of the description of the codimension 2 Chow scheme given in Proposition 6.8.

For irrationality, we show that the set $Y(\mathbb{R})$ is disconnected, which cannot happen for a rational variety: for any smooth projective variety X over $\mathbb{R}$, the number of connected components of $X(\mathbb{R})$ is a birational invariant [DK81, Theorem 13.3][9]. Thus, Y is not rational over any subfield of $\mathbb{R}$.

Theorem 6.10 says that the IJT obstruction does **not** characterize rationality for this family of conic bundle threefolds. In fact, for this family of conic bundles (i.e. those which are geometrically ordinary and geometrically standard with degree 4 discriminant curve), since all of the classical obstructions to rationality (unirationality, Brauer group, birational automorphism group, intermediate Jacobian) vanish, there is currently no known strongest rationality obstruction. In addition to the example of Theorem 6.10, we give an example for which the real points are connected and the ITJ obstruction is used to detect irrationality, and we give an example for which the real points are connected and the IJT obstruction vanishes; the rationality over $\mathbb{Q}$ is unknown.

Remark 6.12. We point out that Theorem 6.10 is not the first observation that the IJT obstruction can fail to determine rationality. Indeed, [BW20, Theorem 5.7] gives a conic bundle threefold over $\mathbb{R}$ (with geometrically reducible discriminant cover) for which the intermediate Jacobian is trivial. The authors use the Brauer group to detect irrationality.

Why does the IJT obstruction fail to capture rationality here? The issue is that the codimension 2 Chow scheme depends on less data than the birational isomorphism class of $Y \to \mathbb{P}^2$. Proposition 6.8 tells us that the $(\mathbf{CH}^2_Y)^0$-torsors $(\mathbf{CH}^2_Y)^\alpha$ are isomorphic to torsors under $\mathbf{Prym}_{\tilde{\Delta}/\Delta}$, and the description depends only on the discriminant double cover $\varpi\colon \tilde{\Delta} \to \Delta$ [Pro18, Proposition 3.10]. When $k = \bar{k}$, the birational isomorphism class of $Y \to \mathbb{P}^2$ also only depends on the cover $\varpi\colon \tilde{\Delta} \to \Delta$. However, when $k \neq \bar{k}$, it *also* depends on a class in $(\mathrm{Br}\, k)[2]$. When this extra information not seen by $\mathbf{CH}^2_Y$ vanishes, then the IJT obstruction does characterize rationality.

Theorem 6.13 (FJSVV [FJS+23, Theorem 1.5])**.** *Let $\pi\colon Y \to \mathbb{P}^2$ be a smooth conic bundle over a field k of characteristic not 2, which is geometrically ordinary and geometrically standard, with $\deg \Delta = 4$, and assume $(\mathrm{Br}\, k)[2] = 0$. Then Y is rational over k if and only if the IJT obstruction vanishes.*

6.5. **How to use the IJT obstruction to study rationality.** We end with a brief summary of how the intermediate Jacobian torsor obstruction is used in all of the examples above to study rationality over non-closed fields. Let X be a smooth projective geometrically rational threefold over a field k.

Step 1: Understand some (or one, or all) of the components of $\mathbf{CH}^2_X$.

For example, in Theorem 6.1, we identify a component as $F_1(X)$. In Proposition 6.8, we identify all of the components as isomorphic to specific torsors under the Prym variety.

[9]This birational invariant was extended to smooth quasiprojective geometrically connected varieties over $\mathbb{R}$ in [CTP90, Main Theorem] using unramified cohomology.

Step 2: Answer the following: does an isomorphism $(\mathbf{CH}^2_X)^\alpha \cong \mathbf{Pic}^d_C$ for some smooth projective curve C and some $d \in \mathbb{Z}$ provide a rationality construction for X?

For example, such an isomorphism implies the existence of a k-point on the torsor $F_1(X)$ when X is a smooth complete intersection of two quadrics. On the other hand, for conic bundles like the example in Theorem 6.10, such an isomorphism need not imply a rationality construction. In some cases, it implies the existence of a geometric curve class which is Galois invariant, and which, if it descends to the ground field, gives a rationality construction; however, the geometric curve class need not descend.

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