# Sheaves on a K3 Surface and Galois Representations

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## Motivating Goal

Let $S$ be a K3 surface defined over a finite field, and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be fine moduli spaces of stable sheaves on $S$ of equal dimension. We want to establish that the zeta function of $\mathcal{M}_1$ is equal to that of $\mathcal{M}_2$:

$$Z(\mathcal{M}_1, t) = Z(\mathcal{M}_2, t).$$

## Overview

- Moduli spaces $\mathcal{M}$ of stable sheaves on a K3 surface $S$ are deformation equivalent to the Hilbert scheme $S^{[n]}$ parameterizing dimension zero subschemes of length $n = \frac{1}{2}\dim \mathcal{M}$ in $S$.
- While two such moduli spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are deformation equivalent, they need not be birational, so it is surprising that these two varieties should have the same zeta function.
- The proof of the main result below is straightforward for $i = 1$ [1]. We transport the result to $i > 1$ by showing:

> The Galois action on $H^2(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell)$ and the ring structure on $H^*(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell)$ determine the Galois action on all of $H^*(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell)$.

- This result parallels a result of Looinjenga and Lunts [2] on Hodge structures:

> For a compact hyperkähler manifold $X$, the algebra structure on $H^*(X, \mathbb{Q})$ and the Hodge structure on $H^2(X)$ determine the Hodge structure on all of $H^*(X)$.

## The moduli space

- **The moduli space**

$$\mathcal{M} := \mathcal{M}_H(S, v)$$

parameterizes Gieseker $H$-semistable sheaves on $S$ with Mukai vector $v$.

- When $v$ is primitive and $H$ is generic, $\mathcal{M}$ is a smooth projective variety over $k$ [1, Theorem 2.4].

## Main Result

We deduce the motivating goal from the following more general result.

> **Theorem.** *Suppose that $S$ is a K3 surface defined over an arbitrary field $k$, and let $\ell$ be a prime number which is invertible in $k$. Then for all $i$,*
>
> $$H^{2i}(\mathcal{M}_{1,\bar{k}}, \mathbb{Q}_\ell) \cong H^{2i}(\mathcal{M}_{2,\bar{k}}, \mathbb{Q}_\ell)$$
>
> *as $\mathrm{Gal}(\bar{k}/k)$-representations.*

## Mukai lattice and vectors

- The **$\ell$-adic Mukai lattice** of $S$ is the $\mathrm{Gal}(\bar{k}/k)$-module

$$\widetilde{H}^*(S_{\bar{k}}, \mathbb{Z}_\ell) := H^0(S_{\bar{k}}, \mathbb{Z}_\ell) \oplus H^2(S_{\bar{k}}, \mathbb{Z}_\ell(1)) \oplus H^4(S_{\bar{k}}, \mathbb{Z}_\ell(2))$$

endowed with the Mukai pairing.

- Let $\omega$ be the numerical equivalence class of a point on $S_{\bar{k}}$. A **Mukai vector** on $S$ is an element

$$v \in N(S) := \mathbb{Z} \oplus \mathrm{NS}(S) \oplus \mathbb{Z}\omega.$$

We consider this as an element of $\widetilde{H}^*(S_{\bar{k}}, \mathbb{Z}_\ell)$ under the natural inclusion.

- The **Mukai vector** of a coherent sheaf $\mathscr{F}$ on S is

$$v(\mathscr{F}) := \mathrm{ch}(\mathscr{F})\sqrt{\mathrm{td}(S)} \in \widetilde{H}^*(S_{\bar{k}}, \mathbb{Z}_\ell).$$

- Charles [1] shows

$$\theta_{v,\ell} \colon v^\perp \to H^2(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell(1)),$$

given by $\theta_{v,\ell}(\alpha) = \pi_{2*}(v(\mathcal{U}) \cdot \pi_1^*(\alpha))$ is a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism.

## Sketch of the proof

**Step 1:** Given two Mukai vectors $v_1, v_2 \in N(S)$, we show that $v_1^\perp \cong v_2^\perp$ as Galois representations.

**Step 2:** Markman [3] constructs a graded ring $R(v)$ generated by $v^\perp$ in degree 2 and by $M_{2i} \cong \widetilde{H}^*(S_{\bar{k}}, \mathbb{Q}_\ell) = v^\perp \oplus \langle v \rangle$ in degree $2i$, and a ring homomorphism

$$h : R(v) \to \widetilde{H}^*(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell)$$

which sends $v^\perp$ in degree 2 isomorphically onto $H^2(\mathcal{M}, \mathbb{Q}_\ell(1))$ via $\theta_{v,\ell}$, and for $\alpha \in M_{2i}$,

$$h(\alpha) = [\pi_{2*}(u_v \cdot \pi_1^*(\alpha))]_{2i},$$

where $u_v$, defined in [3, Eq. (27)], is a normalization of $v(\mathcal{U})$ which Markman shows is invariant under the action of a finite-index subgroup of the monodromy group. The maps $\pi_1$ and $\pi_2$ are the projections from $S \times \mathcal{M}$ onto $S$ and $\mathcal{M}$, respectively, and $\mathcal{U}$ is the universal sheaf on $S \times \mathcal{M}$.
Extending the results of Markman to étale cohomology, we show that $h$ is surjective. This requires lifting $S$ to characteristic zero and applying the smooth base change theorem to the relative moduli space.

**Step 3:** The Beauville-Bogomolov form $q$ on $H^2(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell)$ gives rise to a natural action of $\mathrm{SO}(q)$ on $\widetilde{H}^*(S_{\bar{k}}, \mathbb{Q}_\ell)$ via the isomorphism $v^\perp \cong H^2(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell(1))$ and by extending the action trivially to $\langle v \rangle$. Looijenga and Lunts [2], following Verbitsky [4], construct a natural action of $\mathfrak{so}(q)$ on the entire cohomology ring $\widetilde{H}^*(\mathcal{M}_{\bar{k}}, \mathbb{Q}_\ell)$ which integrates and descends to an action of $\mathrm{SO}(q)$.

**Remark:** Over $\mathbb{C}$, $\mathrm{SO}(q)$ is the connected component of the Zariski closure of the monodromy group of $\mathcal{M}$. Markman [3] shows that these two $\mathrm{SO}(q)$ actions agree.

**Step 4:** Decompose $R(v_1)$ into irreducible $\mathrm{O}(q)$ representations. For each $1 \leq i \leq \dim \mathcal{M}_1 = \dim \mathcal{M}_2$, there exists an $\mathrm{O}(q)$-invariant subspace $V$ such that:

$$\begin{tikzcd}
V \arrow[dr] \arrow[drr, bend left, "\sim"] \arrow[ddrr, bend right, "\sim"'] & & \\
& R(v_1) \arrow[r, two heads, "h_1"] \arrow[d, "\sim"] \arrow[dr, two heads, "\tilde{h}_2"] & H^{2i}(\mathcal{M}_{1,\bar{k}}, \mathbb{Q}_\ell(i)) \\
& R(v_2) \arrow[r, two heads, "h_2"] & H^{2i}(\mathcal{M}_{2,\bar{k}}, \mathbb{Q}_\ell(i))
\end{tikzcd}$$

Since $\mathrm{Gal}(\bar{k}/k) \subset \mathrm{O}(q)$, $V$ is also Galois invariant, and hence the $\mathrm{Gal}(\bar{k}/k)$-actions on $H^{2i}(\mathcal{M}_{1,\bar{k}}, \mathbb{Q}_\ell(i))$ and on $H^{2i}(\mathcal{M}_{2,\bar{k}}, \mathbb{Q}_\ell(i))$ are equal to the action on $V$.

## References


[1] François Charles. Birational boundedness for holomorphic symplectic varieties, Zarhin's trick for $K3$ surfaces, and the Tate conjecture. *Ann. of Math. (2)*, 184(2):487–526, 2016.

[2] Eduard Looijenga and Valery A. Lunts. A Lie algebra attached to a projective variety. *Invent. Math.*, 129(2):361–412, 1997.

[3] Eyal Markman. On the monodromy of moduli spaces of sheaves on $K3$ surfaces. *J. Algebraic Geom.*, 17(1):29–99, 2008.

[4] Mikhail Verbitsky. Cohomology of compact hyperkähler manifolds and its applications. *Geometric and Functional Analysis*, 6(4):601–611, 1996.


## Future Research

We aim to prove the corresponding result for moduli spaces of sheaves on an abelian surface. In this case, there is an added difficulty because the Galois action on the second cohomology no longer determines the Galois action on the higher cohomology groups.