Strichartz estimates for the Schrödinger equation on polygonal domains

Joint work with Matt Blair (UNM), G. Austin Ford (Northwestern U) and Sebastian Herr (U Bonn and U Düsseldorf) ... With a discussion of previous work with Andrew Hassell (Australian National University) and Luc Hillairet (Université Nantes)

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Outline

Preliminaries

Previous Results

Strichartz Estimates

Euclidean Surfaces with Conic Singularities

Outline of Proof for the Strichartz Estimates

Square Function Estimates

Possibilities for Future Work
Let $B$ be a planar, polygonal domain, not necessarily convex. Let $V$ denote the set of vertices of $B$, and let $\Delta_B$ denote the Dirichlet or the Neumann Laplacian on $L^2(B)$. 
The Nonconcentration Theorem

Theorem

Let $B$ be as above and let $U$ be any neighbourhood of $V$. Then there exists $c = c(U) > 0$ such that, for any $L^2$-normalized eigenfunction $u$ of the Dirichlet (or Neumann) Laplacian $\Delta_B$, we have

$$\int_U |u|^2 \geq c.$$

That is, $U$ is a control region for $B$. 
Applicable Billiards

Figure: Examples of polygonal billiards for which the Theorem is applicable with Dirichlet or Neumann boundary conditions on the solid lines and periodic boundary conditions on the dashed lines.
Motivation

Figure: A square billiard constructed by Crommie-Lutz-Eigler at IBM.
Motivation Cont.

- This goes back to the topic of Quantum Ergodicity, related to the question of Quantum/Classical Correspondence.
- Previous work in this area goes back to Burq-Zworski, Zelditch-Zworski, Gérard-Leichtman, Lindenstrauss, Sarnak, Melrose-Sjöstrand, de Verdière,...
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Partially Rectangular Billiards

**Theorem (Burq-Zworski)**

Let $\Omega$ be a partially rectangular billiard with the rectangular part $R \subset \Omega$, $\partial R = \Gamma_1 \cup \Gamma_2$, a decomposition into parallel components satisfying $\Gamma_2 \subset \partial \Omega$. Let $\Delta$ be the Dirichlet or Neumann Laplacian on $\Omega$. Then for any neighbourhood of $\Gamma_1$ in $\Omega$, $V$, there exists $C$ such that

$$-\Delta u = \lambda u \implies \int_V |u(x)|^2 dx \geq \frac{1}{C} \int_R |u(x)|^2 dx,$$

that is, no eigenfunction can concentrate in $R$ and away from $\Gamma_1$. 
Partially Rectangular Billiards Cont.

Figure: Control regions in which eigenfunctions have positive mass and the rectangular part for the Bunimovich stadium.
Motivation

Figure: A stadium billiard constructed by a team at IBM.
Theorem (Burq-M-Zworski)

Let \( V \) be any open neighbourhood of the convex boundary, \( \partial \mathcal{O} \), in a Sinai billiard, \( S \). If \( \Delta \) is the Dirichlet or Neumann Laplace operator on \( S \) then there exists a constant, \( C = C(V) \), such that

\[
- h^2 \Delta u = E(h)u \implies \int_V |u(x)|^2 dx \geq \frac{1}{C} \int_S |u(x)|^2 dx,
\]

for any \( h \) and \( |E(h) - 1| < \frac{1}{2} \).
Nonconcentration Proof Ideas

- Assume there exists a sequence of eigenfunctions concentrating on a periodic orbit away from the obstacle.
- This trajectory can be trapped in a periodic cylinder.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.
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Billiards with Obstacles

Figure: A maximal rectangle in a rational direction, avoiding the obstacle. Because the parallelogram is certainly periodic and our region has uniform width, it is clear that the resulting rectangle is periodic.
Theorem (M)

Let $\gamma$ be an $x$-bounded trajectory on $P = \mathbb{T}^2 \setminus S$. If $\Delta$ is the Dirichlet Laplace operator on $P$ then there exists no microlocal defect measure obtained from the eigenfunctions on $P$ such that $\text{supp} \ (d\mu) = \gamma$. 
Billiards with Slits Cont.

Figure: A pseudointegrable billiard $P$ consisting of a torus with a slit, $S$, along which we have Dirichlet boundary conditions. We would like to show that eigenfunctions of the Laplacian on this torus must have concentration in the shaded regions $V_1$ and $V_2$. 
Billiards with Slits Cont.

Above, a. and b. represent typical $x$-bounded trajectories, while c. and d. represent $x$-unbounded trajectories.

**Figure:** Some examples of $x$-bounded and $x$-unbounded trajectories.
Figure: This diagram describes how we "unfold" the eigenfunctions in order to derive a contradiction.
Nonconcentration Proof Ideas

- Geometric condition classifying periodic orbits that miss the control region.
- Generalize to a Euclidean Surface with Conic Singularities with the geometric condition satisfied.
- Contradiction argument using semiclassical defect measures and control theory estimates for solutions to inhomogeneous elliptic equations on rectangles.
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The Set Up for Strichartz

Suppose $u(t, x) : [-T, T] \times \Omega \rightarrow \mathbb{C}$ is a solution to the initial value problem for the Schrödinger equation on $\Omega$:

$$\begin{cases} (D_t + \Delta) u(t, x) = 0 \\ u(0, x) = f(x). \end{cases}$$

Here, $u$ satisfies either Dirichlet or Neumann homogeneous boundary conditions,

$$u|_{[-T, T] \times \partial \Omega} = 0 \quad \text{or} \quad \partial_n u|_{[-T, T] \times \partial \Omega} = 0.$$
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The Strichartz Estimates

These are a family of space-time integrability bounds of the form

$$\|u\|_{L^p([-T,T]; L^q(\Omega))} \leq C_T \|f\|_{H^s(\Omega)}$$

with $p > 2$ and $\frac{2}{p} + \frac{2}{q} = 1$. 
The Strichartz Estimates

- More precisely, this self-adjoint operator possesses a sequence of eigenfunctions forming a basis for $L^2(\Omega)$.
- We write the eigenfunction and eigenvalue pairs as $\Delta \varphi_j = \lambda_j^2 \varphi_j$, where $\lambda_j$ denotes the frequency of vibration.
- The Sobolev space of order $s$ can then be defined as the image of $L^2(\Omega)$ under $(1 + \Delta)^{-s}$ with norm

$$\|f\|_{H^s(\Omega)}^2 = \sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right)^s |\langle f, \varphi_j \rangle|^2.$$

- Here, $\langle \cdot, \cdot \rangle$ denotes the $L^2$ inner product.
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The Strichartz Estimates

- Strichartz estimates are well-established when the domain $\Omega$ is replaced by Euclidean space.
- In that case, one can take $s = 0$, and by scaling considerations, this is the optimal order for the Sobolev space; see for example Strichartz (1977), Ginibre and Velo (1985), Keel and Tao (1998), etc.
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The Strichartz Estimates

- When $\Omega$ is a compact domain or manifold, much less is known about the validity and optimality of these estimates.
- The finite volume of the manifold and the presence of trapped geodesics appear to limit the extent to which dispersion can occur.
- In addition, the imposition of boundary conditions complicate many of the known techniques for proving Strichartz estimates.
- Nonetheless, estimates on general compact domains with smooth boundary have been shown by Anton (2008) and Blair-Smith-Sogge (2008). Both of these works build on the approach for compact manifolds of Burq-Gérard-Tzvetkov (2004).
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The Theorem

**Theorem**

Let $\Omega$ be a compact polygonal domain in $\mathbb{R}^2$, and let $\Delta$ denote either the Dirichlet or Neumann Laplacian on $\Omega$. Then for any solution $u = \exp(-it\Delta)f$ to the Schrödinger IBVP with $f$ in $H^{\frac{1}{p}}(\Omega)$, the Strichartz estimates

$$\|u\|_{L^p([-T,T];L^q(\Omega))} \leq C_T \|f\|_{H^{\frac{1}{p}}(\Omega)}$$

hold provided $p > 2$, $q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$. 
Remarks

In this work, the Neumann Laplacian is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices.

In this sense, our Neumann Laplacian imposes Dirichlet conditions at the vertices and Neumann conditions elsewhere.

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Remarks

- We note that our estimates have a loss of $s = \frac{1}{p}$ derivatives as in Burq-Gérard-Tzvetkov (2004), which we believe is an artifact of our methods.

- Given specific geometries, there are results showing that such a loss is not sharp. For instance, when $\Omega$ is replaced by a flat rational torus, the Strichartz estimate with $p = q = 4$ holds for any $s > 0$, as was shown by Bourgain (1993); see also Bourgain (2007) for results in the case of irrational tori.

- However, we also point out that in certain geometries a loss of derivatives is expected due to the existence of gliding rays, as shown by Ivanovici (2008).
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Definition

A Euclidean surface with conical singularities (ESCS) is a topological space $X$ possessing a decomposition $X = X_0 \sqcup P$ for a finite set of singular points $P \subsetneq X$ such that

1. $X_0$ is an open, smooth two-dimensional Riemannian manifold with a locally Euclidean metric $g$, and
2. each singular point $p_j$ of $P$ has a neighborhood $U_j$ such that $U_j \setminus \{p_j\}$ is isometric to a neighborhood of the tip of a flat Euclidean cone $C(S^1_{\rho_j})$ with $p_j$ mapped to the cone tip.
The Real Theorem

Theorem

Let $X$ be a compact ESCS, and let $\Delta_g$ be the Friedrichs extension of $\Delta_g \big|_{C_c^\infty(X_0)}$. Then for any solution $u = \exp(-it\Delta_g) f$ to the Schrödinger IVP on $X$ with initial data $f$ in $H^{\frac{1}{p}}(X)$, the Strichartz estimates

$$\|u\|_{L^p([-T,T];L^q(X))} \leq C_T \|f\|_{H^{\frac{1}{p}}(X)}$$

hold provided $p > 2$, $q \geq 2$, and $\frac{2}{p} + \frac{2}{q} = 1$. 
Choose a nonnegative bump function $\beta$ in $C_c^\infty(\mathbb{R})$ supported in $(\frac{1}{4}, 4)$ and satisfying $\sum_{k \geq 1} \beta(2^{-k} \zeta) = 1$ for $\zeta \geq 1$.

Taking $\beta_k(\zeta) \overset{\text{def}}{=} \beta(2^{-k} \zeta)$ for $k \geq 1$ and $\beta_0(\zeta) \overset{\text{def}}{=} 1 - \sum_{k \geq 1} \beta_k(\zeta)$, we define the frequency localization $u_k$ of $u$ in the spatial variable by

$$u_k \overset{\text{def}}{=} \beta_k \left( \sqrt{\Delta_g} \right) u.$$

The operator $\beta_k \left( \sqrt{\Delta_g} \right)$ is defined using the functional calculus with respect to $\Delta_g$. Hence, $u = \sum_{k \geq 0} u_k$, and in particular, $u_0$ is localized to frequencies smaller than 1.
Littlewood-Paley

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The operator $\beta_k(\sqrt{\Delta_g})$ is defined using the functional calculus with respect to $\Delta_g$. Hence, $u = \sum_{k \geq 0} u_k$, and in particular, $u_0$ is localized to frequencies smaller than 1.
With this decomposition, we have the following square function estimate for elements $a$ of $L^q(X)$,

$$\left\| \left( \sum_{k \geq 0} |\beta_k(\sqrt{\Delta_g}) a|^2 \right)^{\frac{1}{2}} \right\|_{L^q(X)} \approx \|a\|_{L^q(X)},$$

with implicit constants depending only on $q$. 
Proof Using Square Function Estimates

Delaying the proof of the square function estimate, we have by Minkowski’s inequality that

$$
\|u\|_{L^p([-T, T]; L^q(X))} \lesssim \left( \sum_{k \geq 0} \|u_k\|_{L^p([-T, T]; L^q(X))}^2 \right)^{\frac{1}{2}}
$$

since we are under the assumption that $p, q \geq 2$.

We now claim that for each $k \geq 0$,

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\|u_k\|_{L^p([-T, T]; L^q(X))} \lesssim 2^{\frac{k}{p}} \|u_k(0, \cdot)\|_{L^2(X)}.
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Assuming this for the moment, we have by orthogonality and the localization of $\beta$ that

$$2^{\frac{2k}{p}} \| u_k(0, \cdot) \|_{L^2(X)}^2 = 2^{\frac{2k}{p}} \sum_{j=1}^{\infty} \beta_k(\lambda_j)^2 \left| \langle u(0, \cdot), \varphi_j \rangle \right|^2$$

$$\ll \sum_{j=1}^{\infty} \left(1 + \lambda_j^2\right)^{1/p} \beta_k(\lambda_j)^2 \left| \langle u(0, \cdot), \varphi_j \rangle \right|^2.$$ 

We now sum this expression over $k$; after exchanging the order of summation in $k$ and $j$, we obtain

$$\sum_{k \geq 0} 2^{\frac{2k}{p}} \| u_k(0, \cdot) \|_{L^2(X)}^2 \ll \| u(0, \cdot) \|_{H^p(X)}^2.$$ 

Combining this with Minkowski, we have reduced the proof of our Theorem to showing the dyadic Strichartz claim.
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Proof Using Square Function Estimates

- We observe that the claim follows from

\[ \| u_k \|_{L^p([0,2^{-k}];L^q(X))} \lesssim \| u_k(0, \cdot) \|_{L^2(X)}. \]

- Indeed, if this estimate holds, then time translation and mass conservation imply the same estimate holds with the time interval \([0, 2^{-k}]\) replaced by \([2^{-k}m, 2^{-k}(m + 1)]\). Taking a sum over all such dyadic intervals in \([-T, T]\) then yields the desired estimate.
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Proof Using Square Function Estimates

We localize our solution in space using a finite partition of unity $\sum_\ell \psi_\ell \equiv 1$ on $X$ such that $\text{supp}(\psi_\ell)$ is contained in a neighborhood $U_\ell$ isometric to either an open subset of the plane $\mathbb{R}^2$ or a neighborhood of the tip of a Euclidean cone $C(S^1_\rho)$.

It now suffices to see that if $\psi$ is an element of this partition and $U$ denotes the corresponding open set in $\mathbb{R}^2$ or $C(S^1_\rho)$, then

$$\|\psi u_k\|_{L^p([0,2^{-k}];L^q(U))} \lesssim \|u_k(0,\cdot)\|_{L^2(U)}.$$
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Proof Using Square Function Estimates

- Observe that $\psi u_k$ solves the equation

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(D_t + \Delta_g) (\psi u_k) = [\Delta_g, \psi] u_k
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over $\mathbb{R}^2$ or $C(S^1_\rho)$.

- Letting $S(t)$ denote the Schrödinger propagator either on Euclidean space or the Euclidean cone, depending on which space $U$ lives in, we have for $t \geq 0$ that

\[
\psi u_k(t, \cdot) = S(t) (\psi u_k(0, \cdot)) + \\
\int_0^{2^{-k}} 1_{\{t > s\}}(s) S(t - s) ([\Delta_g, \psi] u_k(s, \cdot)) \, ds.
\]
Proof Using Square Function Estimates

Observe that $\psi u_k$ solves the equation

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- On the plane, estimates on the Schrödinger operator are well known.
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Proof of Square Function Estimates

The estimate is actually valid for any exponent $1 < q < \infty$. If $X_0$ were compact, the estimate in Seeger-Sogge (1989) would suffice for our purpose.

Extra care must be taken in our case, however, as $X_0$ is an incomplete manifold. Thus, we take advantage of a spectral multiplier theorem that allows us to employ a classical argument appearing in Stein (1970). This method is also treated in Ivanovici-Planchon (2008) and in the thesis of Blair (2005).
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Proof of Square Function Estimates

- The multiplier theorem we use is due to Alexopolous (2004) and treats multipliers defined with respect to the spectrum of a differential operator on a manifold, see also the work of Duong, Ouhabaz, and Sikora (2002).

- It requires that the Riemannian measure is doubling and that the heat kernel \( P(t, x, y) \) generated by \( \Delta_g \) should satisfy a Gaussian upper bound of the form

\[
P(t, x, y) \lesssim \frac{1}{\left| B(x, \sqrt{t}) \right|} \exp \left( - \frac{b \, \text{dist}_g(x, y)^2}{t} \right),
\]

where \( \left| B(x, \sqrt{t}) \right| \) is the volume of the ball of radius \( \sqrt{t} \) about \( x \) and \( b > 0 \) is a constant.

- We prove that this estimate holds on any ESCS.
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Proof of Square Function Estimates

- We use a theorem of Grigor’yan (1997) that establishes Gaussian upper bounds on arbitrary Riemannian manifolds.

- His result implies that if \( P(t, x, y) \) satisfies on-diagonal bounds

\[
P(t, x, x) \lesssim \max \left( \frac{1}{t}, C \right)
\]

for some constant \( C > 0 \) then there exists \( b > 0 \) such that

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- Since \( |B(x, \sqrt{t})| \approx t \) for bounded \( t \), this is equivalent to the heat kernal bound.
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- The proof relies on an argument of Cheeger (1983) relating the heat kernel of a model space to that of an intrinsic kernel on $X_0$.
- Then, on the Euclidean cone, we bound the heat kernel using an explicit formula for the heat kernel derived in Cheeger-Taylor I,II (1982) and specifically a form of the heat kernel written down in Li (2003).
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The machinery presented here is generally applicable to any problem with known Strichartz estimates on the Euclidean cone. Hence, the authors hope to extend the result of Ford (2009) from the Schrödinger equation to the Wave equation, which our result then allows us to extend to any ESCS and hence any polygonal domain.
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Hence, the authors hope to extend the result of Ford (2009) from the Schrödinger equation to the Wave equation, which our result then allows us to extend to any ESCS and hence any polygonal domain.