Eigenfunction L^p Estimates on Manifolds of Constant Negative Curvature

Melissa Tacy

Department of Mathematics
Australian National University

melissa.tacy@anu.edu.au

July 2010

Joint with Andrew Hassell

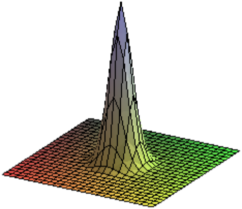
Concentration of Eigenfunctions

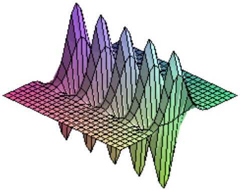
Let u_j be a L^2 normalised eigenfunction of the Laplace-Beltrami Operator on a n dimensional smooth compact manifold M.

$$-\Delta u_j = \lambda_j^2 u_j$$

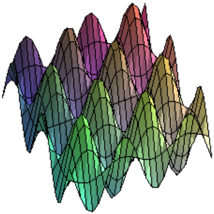
$$\sqrt{-\Delta} u_j = \lambda_j u_j$$

Concentrated





Dispersed



Eigenfunction Estimates

- Concentrated eigenfunctions usually have large L^p norm for $p > 2$.
- Suggests we study the L^p norms of eigenfunctions.
- Seek estimates of the form

$$\|u_j\|_{L^p} \lesssim f(\lambda_j, p) \|u_j\|_{L^2}$$

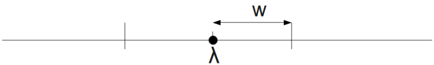
- Not easy to study eigenfunctions directly. Therefore we will study sums (clusters) of eigenfunctions.

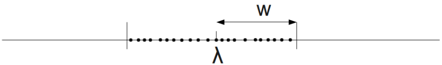
Spectral Windows

We study norms of spectral clusters on windows of width w

$$E_\lambda = \sum_{\lambda_j \in [\lambda - w, \lambda + w]} E_j$$

E_j projection onto λ_j eigenspace.





Obviously include eigenfunctions but also can include sums of eigenfunctions if w is large enough.

Spectral Window of Size One

Easier to work with an approximate spectral cluster.
Pick χ smooth such that $\chi(0) = 1$ and $\hat{\chi}$ is supported in $[\epsilon, 2\epsilon]$.
We will study

$$\chi_\lambda = \chi(\sqrt{-\Delta} - \lambda)$$

Write

$$\chi_\lambda = \int_\epsilon^{2\epsilon} e^{it\sqrt{-\Delta}} e^{-it\lambda} \hat{\chi}(t) dt$$

If we can write $e^{it\sqrt{-\Delta}}$ as an integral operator with kernel $e(x, y, t)$ we can write

$$\chi_\lambda u = \int_\epsilon^{2\epsilon} \int_M e(x, y, t) e^{-it\lambda} \hat{\chi}(t) u(y) dt dy$$

Half Wave Kernel Method

The operator $e^{it\sqrt{-\Delta}}$ is the fundamental solution to

$$\begin{cases} (i\partial_t + \sqrt{-\Delta})U(t) = 0 \\ U(0) = \delta_y \end{cases}$$

We can build a parametrix for this propagator and write its kernel as

$$e(x, y, t) = \int_0^\infty e^{i\theta(d(x,y)-t)} a(x, y, t, \theta) d\theta$$

where $a(x, y, t, \theta)$ has principal symbol

$$\theta^{\frac{n-1}{2}} a_0(x, y, t)$$

Expression for χ_λ

Substituting into the expression for χ_λ

$$\chi_\lambda u = \int_\epsilon^{2\epsilon} \int_M \int_0^\infty e^{i\theta(d(x,y)-t)} e^{-it\lambda} \theta^{\frac{n-1}{2}} \tilde{a}(x,y,t,\theta) u(y) d\theta dy dt$$

Change of variables $\theta \to \lambda\theta$

$$\chi_\lambda u = \lambda^{\frac{n+1}{2}} \int_\epsilon^{2\epsilon} \int_M \int_0^\infty e^{i\lambda\theta(d(x,y)-t)} e^{-it\lambda} \theta^{\frac{n-1}{2}} \tilde{a}(x,y,t,\theta) u(y) d\theta dy dt$$

Now use stationary phase in (t,θ). Nondegenerate critical points when

$$d(x,y) = t \quad \theta = 1$$

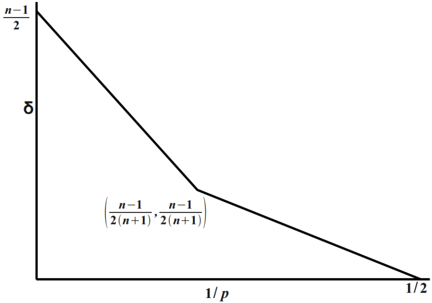
$$\chi_\lambda = \lambda^{\frac{n-1}{2}} \int_M e^{i\lambda d(x,y)} a(x,y) u(y) dy$$

where $a(x,y)$ is supported away from the diagonal.

Sogge's Result

Sogge's result on windows of width 1 gives a complete sharp (for clusters) set of L^p estimates.

$$\|\chi_\lambda u\|_{L^p} \lesssim \lambda^{\delta(n,p)} \|u\|_{L^2}$$

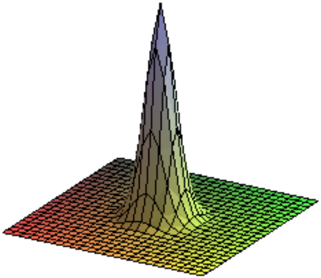


$$\delta(n,p) = \begin{cases} \frac{n-1}{2} - \frac{n}{p} & \frac{2(n+1)}{n-1} \leq p \leq \infty \\ \frac{n-1}{4} - \frac{n-1}{2p} & 2 \leq p \leq \frac{2(n+1)}{n-1} \end{cases}$$

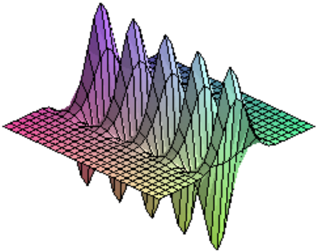
Sharpness for clusters

Estimates sharp for spectral clusters and also sharp on the sphere.
Two regimes for sharp estimates

Point



Tube



Sharpness for Eigenfunctions

Can find spherical harmonics for both regimes. However geodesic flow on a sphere is the antithesis of chaotic.

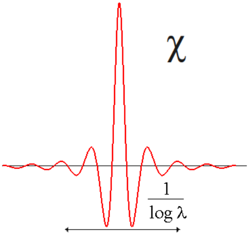
- Sphere has many stable invariant sets under the flow.
- Every point has a conjugate point.
- Large multiplicity of eigenvalues so a width one window is efficient.
- Expect improvements on multiplicities and eigenfunction estimates for "chaotic" systems
- For $n = 2$ and negative curvature conjectured $C_\epsilon \lambda^\epsilon$ growth.

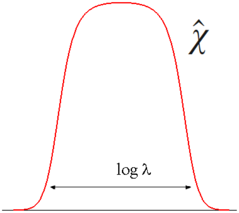
Bérard's Remainder Estimate

Case where M has no conjugate points. Bérard proved a $\log\lambda$ improvement on counting function remainder. This implies a better L^∞ estimate.

$$\|u\|_{L^\infty} \lesssim \frac{\lambda^{\frac{n-1}{2}}}{(\log\lambda)^{1/2}} \|u\|_{L^2}$$

This is achieved by shrinking the spectral window by a factor of $\log\lambda$.





Means that we need to run propagator for $\log\lambda$ time.

Spectral Window of $1/\log \lambda$

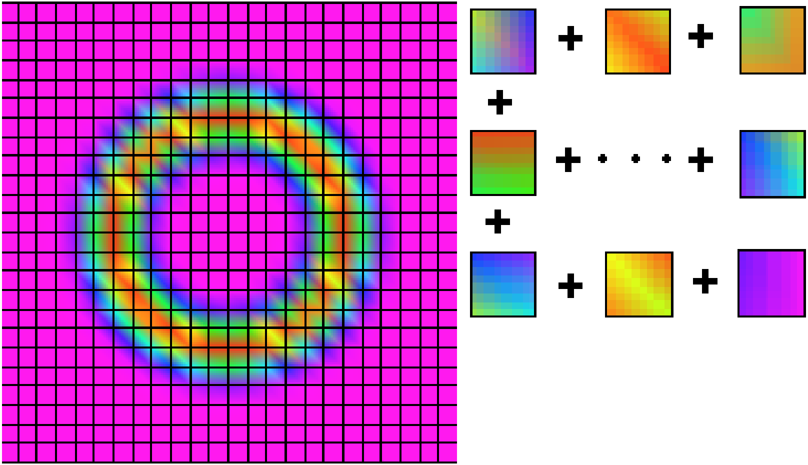
We need to evaluate

$$\int_{t<\log \lambda} e^{it\sqrt{-\Delta}} e^{it\lambda} dt$$

Cannot achieve this on any manifold but for manifolds without conjugate point we can use the universal cover. If M has no conjugate points its universal cover $\widetilde{M}$ is a manifold with infinite injectivity radius. Therefore we can find a solution for

$$\begin{cases} (i\partial_t + \sqrt{-\Delta}_{\tilde{M}})U(t) = 0 \\ U(0) = \delta_y \end{cases}$$

for all time on $\widetilde{M}$



Expression for Propagator Kernel

$e^{it\sqrt{-\Delta}}$ has kernel

$$e(x, y, t) = \sum_{g \in \Gamma} \tilde{e}(x, gy, t)$$

where Γ is the group of automorphisms of the covering $\pi : \widetilde{M} \to M$ and the fundamental solution of

$$\begin{cases} (i\partial_t + \sqrt{-\Delta}_{\tilde{M}})U(t) = 0 \\ U(0) = \delta_y \end{cases}$$

is given by

$$U(t)u = \int_{\tilde{M}} \tilde{e}(x, y, t)u(y)dy$$

The case of constant negative curvature

We will reduce to the simple case where M is two dimensional and has constant negative curvature, therefore $\widetilde{M}$ is the hyperbolic plane.
We study

$$\chi_\lambda = \chi((\sqrt{\Delta} - \lambda)A)$$

where $A = A(\lambda)$ controls the size of the spectral window.
Therefore

$$\chi_\lambda = \int_\epsilon^{2\epsilon} e^{itA\sqrt{-\Delta}} e^{-itA\lambda} \hat{\chi}(t) dt$$

So

$$\chi_\lambda \chi_\lambda^\star = \int_\epsilon^{2\epsilon} \int_\epsilon^{2\epsilon} e^{iA(t-s)\sqrt{-\Delta}} e^{-iA(t-s)\lambda} \hat{\chi}(t) \hat{\chi}(s) dt ds$$

We have

$$e(x, y, At) = \sum_{g \in \Gamma} \tilde{e}(x, gy, At)$$

$$= \sum_{g \in \Gamma} \int_0^\infty e^{i\theta(d(x,gy)-tA)} \theta^{1/2} a(x, gy, tA, \theta) d\theta$$

- Away from diagonal $x = gy$ the principal symbol of $a(x, gy, tA, \theta)$ is $(\sinh(d(x, gy)))^{-1/2}$.
- Only significant contributions when $d(x, gy) = At$ so sum is finite
- If $(t - s)$ is bounded away from zero can directly substitute this expression for the kernel of $e^{i(t-s)A\sqrt{-\Delta}}$.

Small $t-s$

$$(\chi_\lambda \chi_\lambda^\star)_1 = \int_\epsilon^{2\epsilon}\int_\epsilon^{2\epsilon} e^{iA(t-s)\sqrt{-\Delta}} e^{-iA(t-s)\lambda}\hat{\chi}(t)\hat{\chi}(s)\zeta(A(t-s))dtds$$

for ζ cut off function supported on $[-2\epsilon, 2\epsilon]$ and $\zeta = 1$ on $[\epsilon, \epsilon]$.

$$(\chi_\lambda \chi_\lambda^\star)_1 u = \int_M K_1(x,y)u(y)dy$$

$$K_1(x,y) = \int_{\mathbb{R}^2}\int_M e(x,z,At)e(z,y,As)e^{-iA(t-s)\lambda}\hat{\chi}(t)\hat{\chi}(s)u(y)dzdsdt$$

$$= \sum_{g,g'\in\Gamma}\int e^{iA(\theta(d(x,gz)-t)-\eta(d(g'z,y)-s))}e^{iA(t-s)\lambda}\theta^{1/2}\eta^{1/2}d\Lambda$$

where

$$d\Lambda = \frac{b(t,s)\zeta(A(t-s))d\eta d\theta dzdsdt}{(\sinh(d(x,gz)))^{1/2}(\sinh(d(g'z,y)))^{1/2}}$$

Scaling $\theta \to \lambda\theta$ and $\eta \to \lambda\eta$ combined with stationary phase in $(t, \theta), (s, \eta)$ gives

$$K_1(x, y) = \frac{\lambda}{A^2} \sum_{g, g' \in \Gamma} \int_M e^{i\lambda(d(x, gz) - d(g'z, y))} d\Lambda$$

$$d\Lambda = \frac{\tilde{a}(x, y, z) dz}{(\sinh(d(x, gz)))^{-1/2} (\sinh(d(g'z, y)))^{-1/2}}$$

and the restriction

$$d(g'z, y) \in [d(x, gz) - \epsilon, d(x, gz) + \epsilon]$$

Turn one sum into an integral over H^2

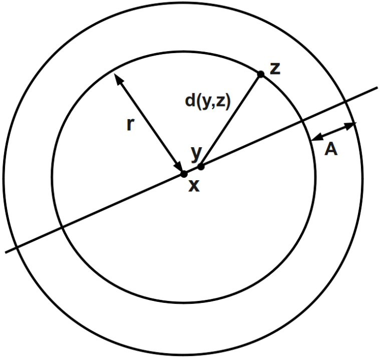
$$K_1(x, y) = \frac{\lambda}{A^2} \sum_{g \in \Gamma} \int_{H^2} e^{i\lambda(d(x, z) - d(z, gy))} d\tilde{\Lambda}$$

but for $g \neq \text{Id}$ zero contribution. So

$$K_1(x,y) = \frac{\lambda}{A^2} \int_{H^2} e^{i\lambda(d(x,z)-d(z,y))} d\tilde{\Lambda}$$

Stationary Phase

- Phase is stationary (in angular variables) when z is on the geodesic to y from x.
- Nondegeneracy depends on the distance between x and y.
- Pick up one factor of A from radial integral.



Arrive at

$$K_1(x,y) = \frac{\lambda}{A} \frac{e^{i\lambda d(x,y)} a(x,y)}{(1+\lambda|x-y|)^{-1/2}}$$

This is true for all A including $A = 1$ which is the Sogge case. Therefore

$$\|(\chi_\lambda \chi_\lambda^\star)_1 u\|_{L^p} \lesssim \frac{\lambda^{2\delta(n,p)}}{A} \|u\|_{L^{p'}}$$

Can make this estimate very small by increasing A however we still need to address the terms given by $(t-s)$ large. This term will limit how large we make A.

Use Hadamard parametrix for large $t - s$

For $|t-s| > \epsilon$ we assume that $t > s$ and use

$$e^{iAt\sqrt{-\Delta}} e^{-iAs\sqrt{-\Delta}} = e^{iA(t-s)\sqrt{-\Delta}}$$

and write

$$(\chi_\lambda \chi_\lambda^\star)_2 = \int e^{iA(t-s)\sqrt{-\Delta}} e^{i(t-s)\lambda} \hat{\chi}(t)\hat{\chi}(s)(1-\zeta(A(t-s)))dtds$$

$$= \frac{1}{A^2} \int_{t,s<\epsilon A} e^{i(t-s)\sqrt{-\Delta}} e^{i(t-s)\lambda} b(t,s)(1-\zeta(((t-s)))dtds$$

$$= \frac{1}{A^2} \sum_{g\in\Gamma} \int_{t,s<\epsilon A} \int_M \int_0^\infty e^{i(\theta(d(x,gy)-(t-s))-(t-s)\lambda)} \theta^{1/2} d\Lambda$$

$$d\Lambda = \frac{\tilde{a}(x,y,\theta,t,s)d\theta dydtds}{(\sinh(d(x,gy))^{-1/2}}$$

After the usual scaling $\theta \to \lambda\theta$ and stationary phase in (t, θ) we obtain

$$(\chi_\lambda \chi_\lambda^\star)_2 u = \int K_2(x, y) u(y) dy$$

$$K_2(x, y) = \frac{\lambda^{1/2}}{A} \sum_{g \in \Gamma} (\sinh(d(x, gy))^{-1/2} e^{i\lambda d(x, gy)}$$

where

$$\epsilon \leq d(x, gy) \leq \epsilon A$$

Because of the exponential growth there are $e^{\epsilon R}$ such terms at distance $R + O(1)$ from x. Therefore

$$K_2(R, x, y) = \zeta(d(x, gy) - R) K_2(x, y) \Rightarrow |K_2(R, x, y)| \leq \frac{\lambda^{1/2} e^{cR}}{A}$$

If

$$T_\lambda^R u = \int K(R,x,y)u(y)dy$$

then

$$\left\| T_\lambda^R u \right\|_{L^\infty} \lesssim \frac{\lambda^{1/2} e^{cR}}{A} \|u\|_{L^1}$$

As $e^{it\sqrt{-\Delta}}$ is a unitary operator

$$\left\| T_\lambda^R u \right\|_{L^2} \lesssim \frac{1}{A} \|u\|_{L^2}$$

Interpolating

$$\left\| T_\lambda^R u \right\|_{L^p} \lesssim \frac{\lambda^{1/2-1/p} e^{cR(1-2/p)}}{A} \|u\|_{L^{p'}}$$

$$\left\| T_\lambda^R u \right\|_{L^p} \lesssim \frac{\lambda^{2\delta(n,p)-1/2+3/p} e^{cR(1-2/p)}}{A} \|u\|_{L^{p'}}$$

Final Result

Finally let $A = \alpha \log \lambda$

$$\|(\chi_\lambda \chi_\lambda^\star)_2 u\|_{L^p} \lesssim \int_\epsilon^{c \log \lambda} \left\| T_\lambda^R u \right\|_{L^p} dR$$

$$\lesssim \frac{\lambda^{2\delta(n,p)-1/2+3/p+c\alpha}}{\alpha \log \lambda} \|u\|_{L^{p'}}$$

Putting this together with the $A|t-s| \leq \epsilon$ term we obtain (by picking α small enough)

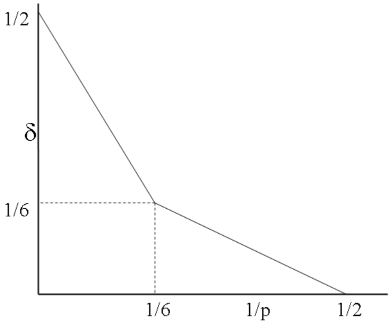
$$\|\chi_\lambda u\|_{L^p} \lesssim C_p f(\lambda, p) \|u\|_{L^2}$$

$$f(\lambda, p) = \frac{\lambda^{\frac{1}{2}-\frac{2}{p}}}{(\log \lambda)^{1/2}}$$

for $6 < p \leq \infty$.

Kink Point?

- For $n = 2$, L^6 is the kink point representing change in sharpness regimes
- We get no improvement for $p = 6$, however we have no sharp examples

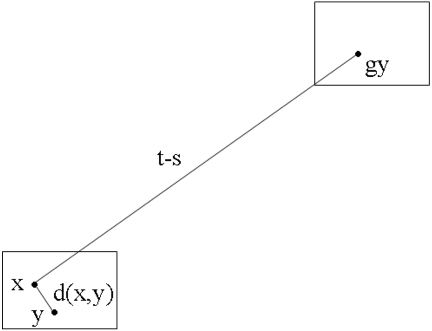


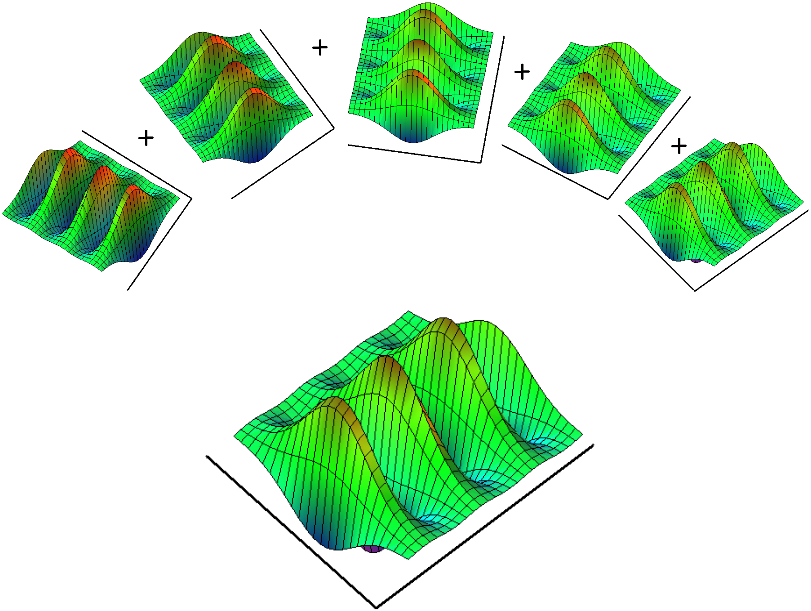
When we interpolate between

$$\left\| T_\lambda^R u \right\|_{L^2} \lesssim \frac{1}{A} \|u\|_{L^2} \quad \text{and} \quad \left\| T_\lambda^R u \right\|_{L^\infty} \lesssim \frac{\lambda^{1/2} e^{cR}}{A} \|u\|_{L^1}$$

we do not take into consideration sharpness regimes.

For the $t - s$ small term we can do this as there is a strong relationship between distance and time.





Wrapping Up

We have the eigenfunction estimates for $p > 6$

$$\|\chi_\lambda u\|_{L^p} \lesssim C_p f(\lambda, p) \|u\|_{L^2}$$

$$f(\lambda, p) = \frac{\lambda^{\frac{1}{2} - \frac{2}{p}}}{(\log \lambda)^{1/2}}$$

- Sharp examples exist for clusters but C_p does not blow up in these examples
- Thought that eigenfunctions estimates are much better, $C_\epsilon \lambda^\epsilon$
- To prove good eigenfunction estimates would need to exploit some cancellation in the sum

$$\sum_{g \in \Gamma} (\sinh(d(x, gy)))^{-1/2} e^{i\lambda d(x,gy)}$$