The spectral results on $N(j)$

Definition. Two linear maps $f, f' : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are called isospectral if for each $Z \in \mathbb{R}^n$, the maps $L_z f, L_z f' : \mathbb{R}^n \rightarrow \mathbb{R}^n$ have the same eigenvalues (with multiplicities) in $\mathbb{C}$.

Proposition [GGSWV]
Let $f, f' : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be isospectral, and let $L$ be a cocompact lattice in $\mathbb{R}^n$ associated to the closed Riemannian manifolds $(N(j)$ and $(N(j')$ are isospectral if $f$ and $f'$ are obviously isospectral because the eigenvalues for both of them are $\pm \iota Z$, each with multiplicity $\dim v/2$).

9. The special family $N^{ab} := N(I^{a,b})$

Notation. $\mathbb{H} := \mathbb{R} \langle 1, i, j, k \rangle$ denote the algebra of quaternions with the usual multiplication, endowed with the inner product for which $(1, i, j, k)$ is an orthonormal basis.

Definition. For $a, b \in \mathbb{N}_0$ with $a + b > 0$ define

\[ v := \mathbb{H} \langle i, a, b, \ldots, \iota \rangle \mathbb{H} (\text{orthogonal sum}), \]

\[ i, j, k := \mathbb{H} \langle 1, i, j, k \rangle, \text{the space of pure quaternions,} \]

\[ L := \mathbb{H} \langle i, j, k \rangle, \text{the associated Riemannian manifold.} \]

Definition. $j \rightarrow \mathbb{R}^n$ is of Heisenberg type if $B_{j} = -Z \cdot \mathbb{H}d_{j}$ for all $Z \in \mathbb{R}^n$.

Remark. If $j, j' \rightarrow \mathbb{R}^n$ are both of Heisenberg type $\Rightarrow f$ and $f'$ are isospectral.

We denote the resulting Riemannian manifolds by $N^{ab} := N(I^{a,b})$, resp. $\tilde{N}^{ab} := \tilde{N}(I^{a,b})$.

10. Szabó's isospectral pairs $N^{a,b,0}$ and $N^{a,0}$

Remark. For all pairs $(a, b) \in \mathbb{N}_0$ with fixed sum $a + b = (\dim v)/4 > 0$ the associated Riemannian manifolds $N^{a,b,0}$ are of Heisenberg type and thus mutually isospectral.

Proposition [Sz]
For every $a \in \mathbb{N}$ the manifolds $N^{0,a}$ and $N^{a,0}$ are homogeneous.

Remark [Sz]: $N^{ab,0}$ is not locally homogeneous if both $a$ and $b$ are nonzero.

One can not hear the local homogeneity property of a closed Riemannian manifold.

11. Weak local symmetry of $N^{a,0}$

Weakly symmetric spaces were introduced by A. Selberg in 1956.

A Riemannian manifold $M$ is called weakly symmetric if each $p \in M$ and each nontrivial geodesic starting in $p$ there exists an isometry $f$ of $M$ which fixes $p$ and reverses $\gamma$ (equivalently: $df_{p}(x)(0) = -\gamma(0)$).

This is not Selberg's original definition, but was Z.I. Szabó's definition of what he called ray symmetry in 1993.

Now, for any given point $p \in N^{a,0}$ and any given tangent vector at $p$, we find an isometry $f$ of $N^{a,0}$ which fixes $p$ and whose differential maps the given tangent vector to its negative. In particular, since $N^{a,0}$ is homogeneous we only consider the case $p = \exp_{0}((1, 0, \ldots, 0), 0)$. Therefore, we conclude

Theorem:
For any $a \in \mathbb{N}$ the Riemannian manifold $N^{a,0}$ is weakly symmetric.

Since $N^{a,0}$ and $N^{0,b}$ are locally isometric, the manifold $N^{a,b}$ is weakly locally symmetric.

Theorem: The symmetry $f$ in the proof of the Theorem will in general not descend to the quotient manifold $N^{a,0}$. So we cannot conclude weak symmetry of $N^{a,0}$ but only weak local symmetry.

12. Failure of the type $\mathcal{A}$ condition for $N^{a,b}$ with $a, b > 0$

Notation:
(i) Let $j, k \rightarrow \mathbb{R}^n$ be any linear map (not necessarily one of our images $f^{a,b}$). Inner products $\langle \cdot, \cdot \rangle$ and norms $\|\cdot\|$, will refer to the metric $g(j)$ on $N(j)$. We denote the Levi-Civita connection and the Ricci tensor of $N(j)$ by $\nabla$ and $\Delta$, resp.

(iv) We identify vectors in $T_{p}G(j) = L_{p}g(j)$ with their preimage in $g(j)$. Correspondingly, we will decompose $Y \in T_{p}G(j)$ as $Y = Y^{\perp} + Y^{\parallel}$ with $Y^{\perp} = v, Y^{\parallel} = c$.

Lema: Let $j$ be of Heisenberg type and let $p = \exp((x, z) \in N(j))$, where $x, z \in v, |x| = 1, |z| \leq 3$. Then for all $Y, Y_{2}, Y_{3} \in T_{p}M(j)$ we have

(i) $\Delta_{p_{r}}(Y, Y_{2}) = -\frac{1}{4} \cdot d_{v} \cdot \frac{1}{2} (Y^{\perp}, Y_{2}^{\perp}) + \frac{1}{2} (Y^{\parallel}, Y^{\parallel}, Y_{2}^{\parallel}, Y_{2}^{\parallel}) + \frac{1}{2} \cdot (d_{v} - \frac{1}{2} \cdot d_{3}(Y_{2}^{\perp}, Y_{2}^{\perp}) + (d_{v} - \frac{1}{2} \cdot d_{3}(Y^{\perp}, Y^{\perp}) + (d_{v} - \frac{1}{2} \cdot d_{3})(Y^{\parallel}, Y^{\parallel}, Y_{2}^{\parallel}, Y_{2}^{\parallel})$,

(ii) $(\nabla_{Y_{2}}Y_{1}, Y_{1}) = \langle [Y_{1}, Y_{2}], Y_{1} \rangle$.

Theorem:
For $a, b > 0$ the Riemannian manifolds $N^{a,b}$ are not of Type $\mathcal{A}$.

Since the type $\mathcal{A}$ condition is a local condition and since $N^{a,b}$ and $N^{0,b}$ are locally isometric, we conclude that $N^{a,b}$ are not of Type $\mathcal{A}$. 
