ORBIFOLDS

July 16, 2010

Outline

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1. Definition and Examples of Orbifolds
2. Orbifolds as quotients of manifolds by Lie group actions
3. Stratification of orbifolds

What is an orbifold?

Definition

An n-dimensional orbifold $\mathcal{O}$ is a second countable Hausdorff topological space together with a maximal n-dimensional orbifold atlas.

Definition

An orbifold atlas is a collection of mutually compatible orbifold charts whose images cover $\mathcal{O}$.

So what's an orbifold chart?

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Orbifold charts

Definition

An orbifold chart or uniformizing system $(\widetilde{U}, \Gamma_U, \pi_U)$ on a connected open set U is given by

$$\widetilde{U} \twoheadrightarrow \Gamma_U \backslash \widetilde{U} \xrightarrow{\sim} U, \qquad \pi_U : \widetilde{U} \to U$$

where

- $\widetilde{U}$ is a connected open subset of $\mathbf{R}^n$
- Γ_U is a finite group acting on $\widetilde{U}$ by diffeomorphisms
- $\pi_U : \widetilde{U} \to U$ is a continuous map inducing a homeomorphism from the orbit space $\Gamma_U \backslash \widetilde{U}$ onto U.

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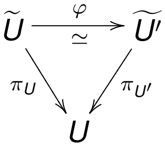
Isomorphic charts

Definition

An isomorphism between two charts $(\widetilde{U}, \Gamma_U, \pi_U)$ and $(\widetilde{U}', \Gamma_{U'}, \pi_{U'})$ on the same open set U is a diffeomorphism $\phi : \widetilde{U} \to \widetilde{U'}$ such that $\pi_{U'} \circ \phi = \phi \circ \pi_U$.



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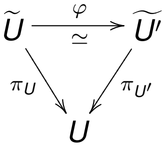

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Every automorphism (ϕ, Γ_U) of a chart $(\widetilde{U}, \Gamma_U, \pi_U)$ is inner.
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Injections of charts

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An injection $(\widetilde{V}, \Gamma_V, \pi_V) \to (\widetilde{U}, \Gamma_U, \pi_U)$ of charts on $V \subseteq U$ is an open embedding $\varphi : \widetilde{V} \to \widetilde{U}$ such that the diagram

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\ \varphi\ } & \widetilde{U} \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi_U} \\ V & \hookrightarrow & U \end{array}$$

commutes.

Given such an injection, there exists a monomorphism $\lambda : \Gamma_V \to \Gamma_U$ such that $\varphi \circ \gamma = \lambda(\gamma) \circ \varphi$ for all $\gamma \in \Gamma_U$.

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Induced charts

Proposition

Given a chart $(\widetilde{U}, \Gamma_U, \pi_U)$ on an open set U and given a connected open subset $W \subset U$, there exists a unique up to isomorphism chart $(\widetilde{W}, \Gamma_W, \pi_W)$ on W that injects into $(\widetilde{U}, \Gamma_U, \pi_U)$. This is the chart induced on W by $(\widetilde{U}, \Gamma_U, \pi_U)$.

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Compatibility of charts

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Two charts $(\widetilde{U}, \Gamma_U, \pi_U)$ and $(\widetilde{U}', \Gamma_{U'}, \pi_{U'})$ on open subsets U and U' are said to be compatible if for every $x \in U \cap U'$, there exists a neighborhood

$$x \in W \subset U \cap U'$$

such that $(\widetilde{U}, \Gamma_U, \pi_U)$ and $(\widetilde{U}', \Gamma_{U'}, \pi_{U'})$ induce isomorphic charts on W.

Caution: There's a subtlety here.

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Good versus Evil

An orbifold is good if it is of the form

$$\mathcal{O} = \Gamma \backslash M$$

where Γ is a discrete group acting properly discontinuously on M.
(Γ does *not* have to be finite.)

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Singular points

Definition

A point $x \in \mathcal{O}$ is said to be singular if in a chart $(\widetilde{U}, \Gamma_U, \pi_U)$ on a neighborhood U of x, the isotropy group

$$Iso(\widetilde{x}) = \{\gamma \in \Gamma_U : \gamma(\widetilde{x}) = \widetilde{x}\}$$

is non-trivial.
The group $Iso(\widetilde{x})$ is the abstract isotropy group of x.

Exercise

The isotropy group of x is well-defined (up to isomorphism) independently of the choice of chart.

Riemannian orbifolds

Definition

A Riemannian metric g on an orbifold $\mathcal{O}$ is given by specifying a Γ_U-invariant Riemannian metric $g_{\widetilde{U}}$ on each chart $(\widetilde{U}, \Gamma_U, \pi_U)$ subject to the compatibility condition: each injection $\phi : \widetilde{U} \to \widetilde{V}$ of charts is an isometric embedding.

Theorem

Every orbifold admits Riemannian metrics.

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Geodesic charts

Definition

We will say that a chart $(\widetilde{U}, \Gamma_U, \pi_U)$ is a geodesic chart if $\widetilde{U}$ is a convex geodesic ball.

In this case, the entire group Γ_U fixes the center $\widetilde{x}$ of $\widetilde{U}$, so $Iso(x) = \Gamma_U$.

Remark

The orthogonal action of Γ_U on $T_{\widetilde{x}}(\widetilde{U})$ gives a representation of $Iso(x)$ as a subgroup of $O(n)$. Up to conjugacy, this subgroup is unique and is even independent of the choice of Riemannian metric on $\mathcal{O}$.
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Smooth mappings

Definition

Let $\mathcal{O}$ and $\mathcal{O}'$ be orbifolds. A continuous map $f : \mathcal{O} \to \mathcal{O}'$ is said to be a smooth map if for each $p \in \mathcal{O}$, there exist charts $(\widetilde{U}, \Gamma_U, \pi_U)$ and $(\widetilde{V}, \Gamma_V, \pi_V)$ on neighborhoods U of p and V of $f(p)$ with $f(U) \subset V$ and a smooth lift

$$\begin{array}{ccc} \widetilde{U} & \xrightarrow{\widetilde{f}} & \widetilde{V} \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ U & \xrightarrow[f]{} & V \end{array}$$

(The lift $\widetilde{f}$ will necessarily be equivariant with respect to Γ_U and Γ_V.)

In particular, a continuous function $f : \mathcal{O} \to \mathbf{R}$ is smooth if and only if for each chart $(\widetilde{U}, \Gamma_U, \pi_U)$ on $\mathcal{O}$ the map $f \circ \pi_U$ is smooth.

Orbifold covering maps

A smooth map

$$\varrho : \mathcal{O}' \to \mathcal{O}$$

is an orbifold covering map if each $x \in \mathcal{O}$ has a chart

$$x \in U \sim \Gamma_U \backslash \widetilde{U}$$

such that

$$\varrho^{-1}(U) = \sqcup V_i$$

where

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k-sheeted cover For regular points x, $\sharp(\varrho^{-1}(x)) = k$.

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 & & \Big\downarrow \\
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{\scriptstyle \varrho|_{V_i}} \Big\downarrow & & \Big\downarrow \\
U & = & \Gamma \backslash \widetilde{U}
\end{array}
$$

Theorem

Every orbifold $\mathcal{O}$ has a universal cover $\widetilde{O}$. This is a regular cover and $\mathcal{O} = \Gamma \backslash \widetilde{O}$. The orbifold $\mathcal{O}$ is good if and only if $\widetilde{O}$ is a manifold.

Γ is the fundamental group of $\mathcal{O}$.

Remark

One can realize the fundamental group as homotopy classes of loops based at a point. However, the notion of smooth map is inadequate for the notion of homotopy. One needs the concept of orbifold morphisms.

Outline

Theorem

Every orbifold can be realized as the quotient of a manifold by a proper action of a Lie group.

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Strata

Definition

A smooth stratification of a manifold or orbifold $\mathcal{O}$ is a locally finite partition of M into locally closed submanifolds, called the strata, satisfying:
For each stratum N, the closure of N is the union of N with a collection of lower dimensional strata.

The strata of maximal dimension are open in $\mathcal{O}$ and their union has full measure in $\mathcal{O}$.

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Orbit types

Consider a smooth action of a Lie group G on a manifold M.

Definition

We say two points in M have the same G-isotropy type if their isotropy groups are conjugate in G. The set of all points of a given isotropy type is a union of G-orbits.

For $x \in G \backslash M$, the isotropy type of x is defined to be the isotropy type of the associated G-orbit in M.

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Orientability

Definition

A manifold is orientable if it admits a covering by of compatibly oriented charts $(\widetilde{U}, \Gamma_U, \pi_U)$ such that the action of Γ_U on $\widetilde{U}$ is orienation-preserving.

Theorem

If $\mathcal{O}$ is orientable, then each singular stratum has co-dimension at least two in $\mathcal{O}$.