ON THE EQUIDISTRIBUTION OF SOME HODGE LOCI

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Abstract. We prove the equidistribution of the Hodge locus for certain non-isotrivial, polarized variations of Hodge structure of weight 2 with $h^{2,0} = 1$ over complex, quasi-projective curves. Given some norm condition, we also give an asymptotic on the growth of the Hodge locus. In particular, this implies the equidistribution of elliptic fibrations in quasi-polarized, non-isotrivial families of K3 surfaces.

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1. Introduction

Let S be a complex, quasi-projective curve and let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ be an integral, polarized variation of Hodge structure of weight 2 over S with $h^{2,0} = 1$. We assume that the bilinear form associated Q is even of signature $(2, b)$. For $s \in S$, let $\rho(\mathbb{V}_{\mathbb{Z},s})$ be the *Picard number* of $\mathbb{V}_{\mathbb{Z},s}$, that is, the rank of the group of integral $(1,1)$-classes in $\mathbb{V}_{\mathbb{Z},s}$. Let M be the minimum value of the integers $\rho(\mathbb{V}_{\mathbb{Z},s})$ for $s \in S$. It is a classical result of Green [Voi02, Prop. 17.20] and Oguiso [Ogu03] that the Noether-Lefschetz locus

$$\mathrm{NL}(\mathbb{V}_{\mathbb{Z}}) = \{s \in S,\ \rho(\mathbb{V}_{\mathbb{Z},s}) > M\}$$

is a countable dense subset of S, when the variation of Hodge structure is non-trivial. A weaker result obtained in [BKPSB98] assumes the base S to be projective. One might then ask how the set of points where the Picard rank jumps distributes inside S. The goal of this paper is to investigate quantitative statements on the distribution of the Hodge locus.

Say that the polarized variation of Hodge structure $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ is simple if there is no polarized sub-variation of Hodge structure $\{\mathbb{V}'_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}', Q\}$ such that $\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}}$ is non-zero and torsion free. In fact, starting from an arbitrary polarized variation of Hodge structure $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ with $h^{2,0} = 1$, the minimal sub-variation $\{\mathbb{V}'_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}', Q\}$ for which $\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}}$ is torsion free is simple. Its orthogonal with respect to Q is also an integral sub-variation of Hodge structure by Deligne and Schmid's semi-simplicity theorem [Del72, Sch73] and which is purely of type $(1,1)$, thus it is trivial, up to taking a finite étale cover of S.

Say also that $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ is non-trivial if the line bundle $\mathcal{F}^2\mathcal{V}$ is not isotrivial. By a result of Griffiths [Gri74, Chapter II] the first Chern class ω of $\mathcal{F}^2\mathcal{V}$ is positive definite, and the integration with respect to ω defines a finite measure μ on S.

Assume that for each $s \in S$, the lattice $(\mathbb{V}_{\mathbb{Z},s}, Q)$ is isomorphic to an even quadratic lattice $(V, (\,.\,))$ of signature $(2, b)$ with $b \geq 3$ and $Q(x) = \frac{(x.x)}{2}$ for all $x \in V$.

Let $\mathbb{V}_{\mathbb{Z}}^{\vee} \subset \mathbb{V}_{\mathbb{Q}}$ be the dual local system to $\mathbb{V}_{\mathbb{Z}}$ with respect to Q, i.e the fiber $\mathbb{V}_{\mathbb{Z},s}^{\vee}$ at each point $s \in S$ is equal to

$$\{x \in \mathbb{V}_{\mathbb{Q},s},\ \forall y \in \mathbb{V}_{\mathbb{Z},s},\ (x.y) \in \mathbb{Z}\}.$$

The fibers of the local system $\mathbb{V}_{\mathbb{Z}}^{\vee}/\mathbb{V}_{\mathbb{Z}}$ are isomorphic to the finite group $V^{\vee}/V$. For $s \in S$ and $\lambda \in \mathbb{V}_{\mathbb{Z},s}^{\vee}$, there exists $\gamma \in V^{\vee}/V$ such that $\lambda \in \gamma + V$ with the previous identification, and therefore $Q(\lambda) \in Q(\gamma) + \mathbb{Z}$. In general we have $Q(\lambda) \in \cup_{\gamma \in V^{\vee}/V}(Q(\gamma) + \mathbb{Z})$.

If H is a subgroup of $V^\vee/V$ we let $H^\perp$ be the orthogonal of H in $V^\vee/V$ with respect to the reduction of the form $(\,.\,)$ valued in $\mathbb{Q}/\mathbb{Z}$. The main result of this paper is the following theorem.

Theorem 1.1. *Let $\{\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a non-trivial, polarized, simple variation of Hodge structure of weight 2 over a quasi-projective curve S with $h^{2,0} = 1$. Assume that the quadratic lattice $(V, (\;,\;))$ is even and that the local system $\mathbb{V}_\mathbb{Z}^\vee/\mathbb{V}_\mathbb{Z}$ is trivial. Let H be a maximal totally isotropic subgroup of $V^\vee/V$. Let μ be the measure defined by integrating the first Chern class of $\mathcal{F}^2\mathcal{V}$ and let $\gamma \in H^\perp \subset V^\vee/V$. Set $A_\gamma = \{-Q(x+\gamma),\, x \in V\}$. Then*

(i) *For $n \in \mathbb{Q}_{>0}$, the number $N(\gamma, n)$ of points $s \in S$ (counted with multiplicity) for which there exists a $(1,1)$-element x in $\mathbb{V}_{\mathbb{Z},s}^\vee$ of class γ in $\mathbb{V}_{\mathbb{Z},s}^\vee/\mathbb{V}_{\mathbb{Z},s}$ and $(x.x) = -2n$ is equal to zero if $n \notin A_\gamma$, otherwise it satisfies*

$$N(\gamma, n) \sim \mu(S)\frac{(2\pi)^{1+\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}.\prod_p \mu_p(\gamma, n, V)$$

as n tends to infinity along A_γ, where

$$\prod_{p<\infty} \mu_p(\gamma, n, V) \asymp 1.$$

If S is projective, then the error term is $O_\epsilon(n^{\frac{2+b}{4}+\epsilon})$ for every $\epsilon > 0$.

(ii) *The set of such points equidistributes in S with respect to μ.*

We refer to Example 2.3 for the definition of the factors $\mu_p(\gamma, n, V)$. The product $\prod_{p<\infty} \mu_p(\gamma, n, V)$ is called the *singular series.* The Hodge locus has a schematic structure (see [Voi02, Chapter 17]). The multiplicity of a point evoked in Theorem 1.1 is the multiplicity in this schematic sense.

This number can also be seen as the multiplicity of intersection of S with special divisors, the so-called *Heegner divisors*, in the moduli space of Hodge structure of K3 type over V. This moduli space is in fact a Shimura variety of orthogonal type. As a part of their study of the André–Oort conjecture [Yaf07], Clozel and Ullmo proved in [CU05] that the Heegner divisors are equidistributed with respect to the measure induced by the Bergman metric ([Hel64, Chap VIII]). The latter is simply given, up to an absolute constant, by integrating the top power of the first Chern class of the Hodge bundle. What we prove here is the equidistribution of the intersection of the Heegner divisors with any fixed quasi-projective curve as above with respect to the measure given by integration the first Chern class of the Hodge bundle.

Corollary 1.2. *Let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ be a non-trivial, polarized, simple variation of Hodge structure of weight 2 over a quasi-projective curve S and μ the measure on S defined by integrating the first Chern class of $\mathcal{F}^2\mathcal{V}$. Then the set of points $s \in S$ for which there exists a $(1,1)$-class x in $\mathbb{V}_{\mathbb{Z},s}$ such that $(x.x) = -2n$ are equidistributed with respect to μ along positive integers n in the infinite set $\{-Q(x),\, x \in V\}$.*

Indeed, we can always take a finite étale cover $\tilde{S}$ of S for which the pullback of the local system $\mathbb{V}_{\mathbb{Z}}^{\vee}/\mathbb{V}_{\mathbb{Z}}$ is trivial and apply Theorem 1.1 to $\tilde{S}$ which imply the corollary.

In particular, we deduce various equidistribution results for points in some 1-parameter families of complex varieties.

Corollary 1.3. *Let Λ_{K_3} be the K3 lattice and $P \subset \Lambda_{K_3}$ a primitive Lorentzian anisotropic sublattice of rank $\rho \leq 4$. Let $\mathcal{X} \xrightarrow{\pi} S$ be a non-isotrivial family of K3 surfaces with generic Picard group equal to P over a quasi-projective curve S and let $\{R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}, \mathcal{F}^{\bullet}\mathcal{H}, Q\}$ be the induced variation of Hodge structure on S. Let μ be the measure induced by integrating the first Chern class of $\mathcal{F}^2\mathcal{H}$. Fix $H \subset P^{\vee}/P$ a maximal isotropic group, $\gamma \in H^{\perp}$ and let $A_{\gamma} = \{Q(x+\gamma),\, x \in P\}$.*

(i) *For $n \in \mathbb{Q}_{>0}$, the number $N(\gamma, n)$ of points $s \in S$ (counted with multiplicity) for which $\mathcal{X}_s$ admits a parabolic line bundle of type (γ, n) is zero if $n \notin A_{\gamma}$, otherwise it satisfies*

$$N(\gamma,n) \sim \mu(S) \frac{(2\pi)^{\frac{22-\rho}{2}}.n^{10-\frac{\rho}{2}}}{\sqrt{|P^{\vee}/P|}\Gamma(\frac{22-\rho}{2})}. \prod_{p<\infty} \mu_p(n,\gamma,V)$$

as n tends to infinity in A_{γ}, and where $V = P^{\perp}$.

If S is projective, then the error term is $O_{\epsilon}(n^{\frac{2+b}{4}+\epsilon})$ for every $\epsilon > 0$.

(ii) *The previous set is equidistributed in S with respect to μ.*

(iii) *If $P^{\vee}/P$ has no non-trivial isotropic subgroup, then the set of points $s \in S$ (counted with multiplicity) for which $\mathcal{X}_s$ admits an elliptic fibration of norm less than n is equidistributed with respect to μ as n tends to infinity.*

For the definition of a parabolic line bundle of type (γ, n) and the norm of an elliptic fibration, we refer to Definition4.4. If the Lorentzian sublattice P is generated by a single element, the corollary says that the number of elliptic surfaces of norm less than n^2 (or volume less than n) in a generic family of quasi-polarized K3 surfaces "grows like" n^{20}. In the case of twistor families of K3 surfaces, an analogous result was shown by Simion Filip in [Fil16] and an improvement of the error term was given by Bergeron and Matheus in [BM17]. The main term there grows also like n^{20}, and Filip works with the full K3 lattice Λ_{K3}. His method is different from ours, although it was the starting point of this paper. Notice also the analogy between the coefficient of the main term in our case and in Filip's case. Indeed, due to the Siegel mass formula (see [ERS91]), the product $\prod_{p<\infty}\mu_p(n,\gamma,P)$ can be expressed as a sum of volumes of some homogeneous spaces (compare to Filip's formula 3.1.6 in [Fil16]). There is also a generalization of the previous corollary which concerns families of hyperkähler manifolds over a quasi-projective curve which we discuss of the section 4.3.

There are several arithmetic statements which shed light on the arithmetic analogues of the above results, the curve S being replaced by an open subset of the spectrum of the ring of integers of a number field. A result by Charles [Cha18] shows that the set of primes where the reduction of two elliptic curves defined over a number field are geometrically isogenous is infinite. More recently, Shankar and Tang [ST17] proved by using similar techniques that, given a simple abelian surface defined over a number field and which has real multiplication, there are infinitely many places where its reduction is not absolutely simple.

1.1. **Outline of the proof.** Let us now sketch the proof of Theorem 1.1. Let D_V be the period domain associated to the quadratic lattice (V,Q), namely the complex analytic variety defined by

$$D_V=\{x\in\mathbb{P}(V_{\mathbb{C}}),\,(x.x)=0,\,(x.\overline{x})>0\}.$$

Let Γ_V be the stable orthogonal group of V defined by

$$\Gamma_V:=\mathrm{Ker}\left(\mathrm{O}(V)\to\mathrm{O}(V^\vee/V)\right),$$

where

$$V^\vee:=\{x\in V_{\mathbb{Q}},\,\forall y\in V,\,(x.y)\in\mathbb{Z}\}$$

denotes as before the dual lattice of V. By Baily and Borel [BB66], the complex analytic quotient $\Gamma_V\backslash D_V$ can be endowed with a natural structure of quasi-projective variety called an *orthogonal modular variety*. It is the structure that we consider in the whole text. Let $\mathcal{L}$ denote the Hodge bundle on $\Gamma_V\backslash D_V$ and let ω be its first Chern class. Recall that $\mathcal{L}$ is an ample line bundle [BB66].

For $\gamma\in V^\vee/V$ and $n\in -Q(\gamma)+\mathbb{Z}$ with $n>0$, let $\mathcal{Z}(\gamma,-n)$ denote the associated Heegner divisor in $\Gamma_V\backslash D_V$ which parametrizes Hodge structures on V for which there exists a rational Hodge class $\lambda\in\gamma+V$ with $(\lambda.\lambda)=-2n$ (see Section 2.2.2 for the precise definition). Let $\{S,\mathbb{V}_{\mathbb{Z}},\mathcal{F}^\bullet\mathcal{V},Q)\}$ be given as in Theorem 1.1. Since the local system $\mathbb{V}_{\mathbb{Z}}^\vee/\mathbb{V}_{\mathbb{Z}}$ is trivial, we have a corresponding holomorphic period map

$$\rho:S\to\Gamma_V\backslash D_V.$$

This map is in fact algebraic by Borel [Bor72][1]. The pullback of the Hodge bundle $\mathcal{L}$ along ρ is equal to $\mathcal{F}^2\mathcal{V}$. The idea of the proof is to

[1]In fact, in [Bor72] the theorem is stated for smooth quotients but see [Huy16, Remark 4.2] for how one can reduce to this case.

obtain a global estimate of the cardinality (with multiplicity) of the set

$$\{s \in S, \exists\, \lambda \in \gamma + \mathbb{V}_{\mathbb{Z},s},\, (\lambda.\lambda) = -2n\}.$$

To this end, following ideas of Maulik [Mau14], we use linear dependence relations between Heegner divisors to get an upper bound. These relations follow from Borcherds' construction in [Bor99] of a modular form on the Picard group of the orthogonal modular variety $\Gamma_V \backslash D_V$. Then we extend those relations to a suitable toroidal compactification of $\Gamma_V \backslash D_V$. It is at this level that we need a restriction on γ being in $H^{\perp}$ for a maximal isotropic subgroup H of $V^{\vee}/V$, since for arbitrary γ, we don't know how to control the intersection of S with the boundary divisor of the given toroidal compactification.

To obtain a lower bound, we construct a suitable fibration over every small enough simply connected open subset $\Delta \subset S$. Then following ideas of Green (see [Voi02, Chap.17]), we obtain a map to the homogeneous space $A_0 = \{x \in V_{\mathbb{R}}, Q(x) = -1\}$. This map turns out to be, outside a Lebesgue negligible analytic subset, a local diffeomorphism. We use then a result of equidistribution of integral points on A_0 proven by Eskin–Oh in a more general context in [EO06, Th.1.2]. The proof of the latter relies on results from ergodic theory, namely the ergodicity of unipotent flows, which is also an important ingredient in the proof of the main result of [CU05].

1.2. **Outline of the paper.** In section 2 we recall the construction of the Borcherds' modular form and its implications on the linear dependence relations between Heegner divisors following ideas of Maulik in [Mau14]. We then explain how to extend those relations to the toroidal compactification of $\Gamma_V \backslash D_V$ determined by the perfect cone decomposition following the work of Peterson in [Pet15]. This will allow us, under some mild assumptions, to give global estimates on the growth of the Hodge locus in a curve. We conjecture that these estimates still hold without those assumptions. In section 3 we construct a fibration in spheres over the small open subsets of S which, combined with equidistribution results of Eskin and Oh [EO06], allow to deduce a lower estimate on the cardinality of Hodge locus. In section 4 we explain how one can reduce to the case where the group $V^{\vee}/V$ has no non-trivial isotropic subgroup and then prove the result in this case. The end of the section is devoted to prove corollary 1.3.

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1.4. **Notations.** If $\epsilon > 0$ $f, g_\epsilon, h : \mathbb{N} \to \mathbb{R}$ are real functions and g_ϵ does not vanish, then:

(1) $f = O_\epsilon(g)$ if there exists an integer $n_\epsilon \in \mathbb{N}$, a positive constant $C_\epsilon > 0$ such that
$$\forall n \geq n_\epsilon,\ |f(n)| \leq C_\epsilon |g_\epsilon(n)|.$$
(2) $f \asymp h$ if and $f = O(h)$ and $h = O(f)$.

2. The Weil representation and modular forms

2.1. **General setting.** We recall in this section some results about Weil representations and certain vector-valued modular forms associated to quadratic lattices. Our main references are [Bor98] and [Bor99].

Let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$: the elements of this group consist of pairs (M, ϕ), where
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$
and ϕ is a holomorphic function on the Poincaré upper half-plane $\mathbb{H}$ such that $\phi(\tau)^2 = c\tau + d$, $\tau \in \mathbb{H}$. The group structure is defined by
$$(M_1, \phi_1).(M_2, \phi_2) = (M_1 M_2, \tau \mapsto \phi_1(M_2.\tau)\phi_2(\tau)),$$
for $(M_1, \phi_1), (M_2, \phi_2) \in \mathrm{Mp}_2(\mathbb{R})$, where $M_2.\tau$ stands for the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given by fractional linear transformations.

The map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ given by $(M, \phi) \mapsto M$ is a double cover of $\mathrm{SL}_2(\mathbb{R})$. Denote by $\mathrm{Mp}_2(\mathbb{Z})$ the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under this map. It is well known (see [Ser77, P.78]) that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the elements:
$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau \mapsto \sqrt{\tau}\right).$$
Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a finite-dimensional complex representation of $\mathrm{Mp}_2(\mathbb{Z})$ that factors through a finite quotient of $\mathrm{Mp}_2(\mathbb{Z})$ and let $k \in \frac{1}{2}\mathbb{Z}$. The group $\mathrm{Mp}_2(\mathbb{Z})$ has a right action on the space of functions $f : \mathbb{H} \to V$ given by
$$(1) \qquad \left(f.(M, \phi)_k\right)(\tau) = \phi(\tau)^{-2k} \rho(M, \phi)^{-1} f(M.\tau).$$

Fix an eigenbasis $(v_\gamma)_{\gamma \in I}$ of V with respect to the action of T. A holomorphic function $f : \mathbb{H} \to V$ which is invariant under the action of T has a Fourier expansion

$$f(\tau)=\sum_{\gamma\in I}\sum_{n\in\mathbb{Q}}c(\gamma,n)e^{2i\pi n\tau}v_\gamma. \tag{2}$$

For $(\gamma,n)\in I\times\mathbb{Q}$, the coefficient $c(\gamma,n)$ is non-zero only if $e^{2i\pi n}$ is the eigenvalue of T acting on v_γ. The function f is said to be holomorphic at infinity if $c(\gamma,n)=0$ for all $n<0$ and $\gamma\in I$.

Definition 2.1. *A holomorphic function $f:\mathbb{H}\to V$ is a* ***modular form of weight*** k ***and type*** ρ*, if it satisfies the following conditions:*

(i) *f is invariant under the action (1) of* $\mathrm{Mp}_2(\mathbb{Z})$.
(ii) *f is holomorphic at infinity.*

Moreover, if $c(\gamma,0)=0$ *for all $\gamma\in I$ in the formula (2), we say that f is a cusp form.*

Let $M_k(\rho)$ denote the $\mathbb{C}$-vector space of modular forms of weight k and type ρ, and let $S_k(\rho)$ be the subspace of cusp forms. Both $M_k(\rho)$ and $S_k(\rho)$ are finite-dimensional vector spaces over $\mathbb{C}$ (see [Bor99, Section 2]).

Let (V,Q) be an even lattice of signature (b^+,b^-) with the underlying non-degenerate symmetric bilinear form denoted by $(\,.\,)$ and such that $Q(x)=\frac{(x.x)}{2}$ for $x\in V$. Let $V^\vee$ be the dual lattice of V. We can associate to the quadratic lattice (V,Q) a representation ρ_V of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ whose underlying vector space is $\mathbb{C}[V^\vee/V]$. For this, it is enough to specify the action of S and T on a basis $(v_\gamma)_{\gamma\in V^\vee/V}$ of $\mathbb{C}[V^\vee/V]$ as follows :

$$\begin{aligned}\rho_V(T)v_\gamma&=e^{2i\pi Q(\gamma)}v_\gamma,\\ \rho_V(S)v_\gamma&=\frac{i^{\frac{b^--b^+}{2}}}{\sqrt{|V^\vee/V|}}\sum_{\delta\in V^\vee/V}e^{-2i\pi(\gamma,\delta)}v_\delta,\end{aligned} \tag{3}$$

where $\gamma\in V^\vee/V$. We denote by ρ_V^* the dual representation of ρ_V.

Remark 2.2. By a result of McGraw [McG03, Prop. 5.6], the complex vector space $\mathrm{M}_k(\rho_V^*)$ has a rational structure $\mathrm{M}_k(\rho_V^*)_\mathbb{Q}$ given by modular forms with rational coefficients, and similarly for $S_k(\rho_V^*)$.

We present an example of a modular form which will be crucial for our later study.

Example 2.3. Assume V has signature $(2,b)$ where $b\geq 3$. There is an Eisenstein series E_V in $\mathrm{M}_k(\rho_V^*)$ whose Fourier expansion is given by (see [BK01, Prop.4]):

$$E_V(\tau)=\sum_{\gamma\in V^\vee/V}\sum_{n\in-Q(\gamma)+\mathbb{Z},n\geq 0}c(\gamma,n)q^nv_\gamma,$$

where $q = e^{2i\pi\tau}$, $\tau \in \mathbb{H}$, and the coefficients $c(\gamma, n)$ are given by:

$$\begin{cases} c(0,0) &= 2 \\ c(\gamma,n) &= -\frac{2^{2+\frac{b}{2}}.\pi^{1+\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}.\prod_p \mu_p(\gamma,n,V) \quad \text{for all}\, n>0, \end{cases}$$

where the product is ranging over all primes p. The factors $\mu_p(\gamma, n, V)$ are defined as follows. For $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$ such that n is positive and a a positive integer, let

$$N(\gamma, n, L, a) = |\{\alpha \in L/aL,\, Q(\alpha+\gamma) + n \equiv 0 \pmod{a}\}|.$$

For a prime p, Siegel proves in [Sie35, Hilfssatz 13] that for s sufficiently large, the value of $p^{-(1+b)s}N(\gamma, n, L, p^s)$ is independent of s and we define

$$\mu_p(\gamma, n, V) := \lim_{s\to\infty} p^{-(1+b)s} N(\gamma, n, V, p^s).$$

The infinite product $\prod_p \mu_p(\gamma, n, V)$ converges as long as every factor is different from zero. Since Q is indefinite of rank greater than 5, this is equivalent by Hasse-Minkowski theorem ([Ser77, p.41]) to the equation $Q(\alpha) + n = 0$ having a solution α in $\gamma + V$. In this situation, we say that n satisfy *local congruence conditions* and by [BD08, Proposition 2] (see also [Iwa97, Section 11.5]), we have

$$\prod_p \mu_p(\gamma, n, V) \asymp 1.$$

Hence the estimate

$$c(\gamma, n) \asymp n^{\frac{b}{2}}.$$

The coefficients $c(\gamma, n)$ are rational numbers by [BK01, Proposition 14] so that $E_V \in \mathrm{M}_k(\rho_V^*)_{\mathbb{Q}}$.

2.2. **Borcherds' modular form.** In this section we introduce the Heegner divisors and state a modularity result of their generating series. This will allow us later to control the growth of their intersection with a curve S supporting a variation of Hodge structure. As before, let (V, Q) be an even quadratic lattice and assume henceforth that it has signature $(2, b)$ with $b \geq 3$. Let $\mathrm{O}(V)$ be the orthogonal group of V and Γ_V the subgroup of elements acting trivially on $V^\vee/V$. We refer to [Mau14] and [Huy16, Chapter 6] for more details on this section.

2.2.1. *The period domain.* Let D_V be the *period domain* associated to V, that is the complex analytic variety

$$D_V := \{w \in \mathbb{P}(V_{\mathbb{C}}),\, (w, w) = 0,\, (w, \overline{w}) > 0\}.$$

Let D_V^+ be one of the two connected components of D_V. Let G be the connected component of the identity of the real Lie group $\mathrm{O}(V_{\mathbb{R}})$, where $V_{\mathbb{R}} := V \otimes_{\mathbb{Z}} \mathbb{R}$ is endowed with the real extension of Q. The action of discrete subgroup $\Gamma_V^+ := \Gamma_V \cap G$ on D_V^+ is proper and totally discontinuous and the quotient $\Gamma_V^+\backslash D_V^+$ has the structure of a quasi-projective variety with orbifold singularities by [BB66].

There is another realization of D_V as a Grassmanian. Let $\mathrm{Gr}(2, V_{\mathbb{R}})$ be the Grassmanian of planes of $V_{\mathbb{R}}$ and let $\mathrm{Gr}^+(2, V_{\mathbb{R}})$ be the open subset of positive definite planes. We have a natural split covering of degree 2

$$\begin{aligned} D_V &\longrightarrow \mathrm{Gr}^+(2, V_{\mathbb{R}}) \\ \omega = X + iY &\mapsto P = < X, Y >, \end{aligned}$$

where P is the oriented plane generated by X and Y. The restriction of the map above to D_V^+ is a diffeomorphism. Both of the previous descriptions of D_V^+ will be interchangeably used henceforth.

2.2.2. *Heegner divisors.* In this section, the main result from [Bor99] is used to produce linear dependence relations between certain special divisors that will be defined hereafter.

For any vector $v \in V_{\mathbb{R}}$ such that $Q(v) < 0$, let $v^{\perp}$ be the set of planes in D_V^+ orthogonal to v. Let $\gamma \in V^{\vee}/V$ and $n \in Q(\gamma) + \mathbb{Z}$ with $n < 0$. The union of hyperplanes

$$\bigcup_{v \in \gamma + V,\, Q(v) = n} v^{\perp}$$

is locally finite, invariant under the action of Γ_V^+, and defines an algebraic divisor on $\Gamma_V^+ \backslash D_V^+$ given as

$$\mathcal{Z}(\gamma, n) := \Gamma_V^+ \backslash \left(\bigcup_{v \in \gamma + V,\, Q(v) = n} v^{\perp} \right).$$

In terms of Hodge structures, $\mathcal{Z}(\gamma, n)$ parametrizes Hodge structures on V for which there exists a rational Hodge class λ in $\gamma + V$ with $Q(\lambda) = n$.

The restriction of the tautological line bundle $\mathcal{O}(-1)$ to $D_V^+ \subset \mathbb{P}(V_{\mathbb{C}})$ admits a natural Γ_V^+-equivariant action and defines an algebraic line bundle $\mathcal{L} := \Gamma_V^+ \backslash \mathcal{O}(-1)$ on $\Gamma_V^+ \backslash D_V^+$ called the *Hodge bundle.* We define $\mathcal{Z}(0, 0)$ to be a divisor whose class is equal to the dual of the Hodge bundle.

Finally, we set $\mathcal{Z}(\gamma, n) = 0$ if $n > 0$ or if $n = 0$ and $\gamma \neq 0$. The $\mathcal{Z}(\gamma, n)$ are the *Heegner divisors.* They are Cartier divisors on $\Gamma_V^+ \backslash D_V^+$, and we denote by $[\mathcal{Z}(\gamma, n)]$ their associated class in $\mathrm{Pic}(\Gamma_V^+ \backslash D_V^+)$.

Consider the formal power series

$$\Phi_V(q) = \sum_{\substack{\gamma \in V^{\vee}/V \\ n \in -Q(\gamma) + \mathbb{Z}}} [\mathcal{Z}(\gamma, -n)] q^n v_{\gamma} \in \mathrm{Pic}(\Gamma_V^+ \backslash D_V^+)[[q^{\frac{1}{2d}}]] \otimes \mathbb{C}[V^{\vee}/V].$$

Here d is the order of $V^{\vee}/V$.

The following result is due to the work of Borcherds ([Bor99]), combined with the refinement of McGraw (see remark 2.2):

Theorem 2.4. $\Phi_V(q) \in \mathrm{Pic}(\Gamma_V^+\backslash D_V^+) \otimes \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)_{\mathbb{Q}}$.

We will follow ideas of Maulik in [Mau14, Section 3] with some changes in order to translate the previous theorem in terms of linear dependence relations between the Heegner divisors. This will be achieved by writing Φ_V as a sum of a multiple of an Eisenstein series and a cusp from, then using standard bounds on the growth of coefficients of cusp forms.

By [Bru02, p.27], for each γ in a set of representatives of the quotient of $V^\vee/V$ by the involution $x \mapsto -x$, there exists an Eisenstein series E_γ such that the following decomposition holds

$$\mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)_{\mathbb{Q}} = \bigoplus_\gamma \mathbb{C}.E_\gamma \oplus \mathrm{S}_{1+\frac{b}{2}}(\rho_V^*)_{\mathbb{Q}}$$

where $E_0 = E_V$ is the Eisenstein series from Example 2.3. Since the only $(\gamma, 0)$-coefficient of Φ_V which is non-zero is the one corresponding to $\gamma = 0$, there exists a finite set $\mathcal{I}$, a family $(\mathcal{Z}(\gamma_i, n_i))_{\in\mathcal{I}}$ of Heegner divisors and a family $(g_i)_{i\in\mathcal{I}}$ of cusp forms such that

$$\Phi_V = \frac{1}{2}[\mathcal{Z}(0,0)] \otimes E_V + \sum_{i\in\mathcal{I}} [\mathcal{Z}(\gamma_i, n_i)] \otimes g_i$$

For $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$, by identifying the (γ, n)-coefficient in the above expression we get

$$[\mathcal{Z}(\gamma, -n)] = \frac{1}{2}c(\gamma, n)[\mathcal{Z}(0,0)] + \sum_{i\in\mathcal{I}} g_i(\gamma, n)[\mathcal{Z}(\gamma_i, n_i)]. \tag{4}$$

Notice that all the coefficients in (4) are rational numbers. For a cusp form f, the trivial bounds on the order of growth of its coefficients (see [Sar90, Prop. 1.5.5]) say that

$$|a_{\gamma,n}(f)| \le C_{\epsilon,f}\, n^{\frac{2+b}{4}+\epsilon},$$

for all $\epsilon > 0$, some constant $C_{\epsilon,f} > 0$, and for all $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n \ge 0$.

Hence, we can find a constant $C_\epsilon > 0$ such that for all $i \in \mathcal{I}$, n and γ as before, we have

$$|g_i(\gamma, n)| \le C_\epsilon n^{\frac{2+b}{4}+\epsilon}.$$

Taking into account relation (4) and the expression in Example 2.3, we get:

Proposition 2.5. *For every $\epsilon > 0$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$, the following estimate holds in* $\mathrm{Pic}(\Gamma_V\backslash D_V)_{\mathbb{Q}}$

$$[\mathcal{Z}(\gamma, -n)] = -\frac{(2\pi)^{1+\frac{b}{2}} n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\,\Gamma(1+\frac{b}{2})} \prod_p \mu_p(\gamma, n, V)[\mathcal{Z}(0,0)] + O_\epsilon(n^{\frac{2+b}{4}+\epsilon}).$$

The above proposition is a quantitative version of Lemma 3.7 in [Mau14].

2.3. **Extension to a toroidal compactification.** The goal in this section is to extend the estimate in Proposition 2.5 to a well chosen toroidal compactification of $\Gamma_V\backslash D_V$. This will allow us to control their growth in cohomology and the growth of their intersection with any curve as in Theorem 1.1. We first start by recalling the construction of the Baily-Borel compactification of $\Gamma_V\backslash D_V$. For a short summary in the case of orthogonal modular varieties, see [GHS13] which we follow closely, or [BJ06, Part III] for the general case.

2.3.1. *Baily-Borel compactification.* There is a "minimal" compactification of $\Gamma_V\backslash D_V$ constructed by Baily and Borel in [BB66] and which proceeds by adding rational boundary components and then showing that the resulting space is a projective algebraic variety.

The rational boundary components correspond precisely to maximal rational parabolic subgroups of G, which in turn are the stabilizers of totally isotropic subspaces of $V_{\mathbb{Q}}$. Since Q has signature $(2, b)$, such spaces have dimension 1 or 2. Hence, we obtain the following description:

$$(\Gamma_V^+\backslash D_V^+)^{BB} = \Gamma_V^+\backslash D_V^+ \sqcup \bigsqcup_{\Pi} X_\Pi \sqcup \bigsqcup_{\ell} Q_\ell.$$

where ℓ and Π run through representatives of the finitely many Γ_V^+-orbits of isotropic lines and isotropic planes in $V_{\mathbb{Q}}$. Each X_Π is a modular curve, and Q_ℓ is a point. They are also known as 1-*cusps* and 0-*cusps* respectively.

2.3.2. *Extension of the relations between Heegner divisors.* The boundary of the Baily-Borel can be singular and the Zariski closure of the Heegner divisors may not be Cartier. To solve this problem, we extend the relation (4) to a well-chosen toroidal compactification of $\Gamma_V\backslash D_V$. We work with the toroidal compactification considered in [Pet15, Section 5.2] and which is given by the perfect cone decomposition. We denote it by $\overline{\Gamma_V\backslash D_V}^{tor}$. Above each cusp determined by an isotropic subspace I of V, the boundary divisors are determined by the one dimensional rays in the $\mathrm{Stab}(I)$-invariant decomposition of the positive cone of $I^\perp/I$ and in this situation they lie in its boundary. Hence above every 1-cusp F there is only one irreducible Cartier boundary divisor Δ_F and there are no other boundary divisors. Also the closure $\overline{\mathcal{Z}(\gamma, n)}$ of a Heegner divisor $\mathcal{Z}(\gamma, n)$ is Cartier for all $\gamma \in V^\vee/V$ and $n \in Q(\gamma) + \mathbb{Z}$. For more details, see [Pet15, 5.2.4]. The rest of the section is devoted to bound the coefficients of the boundary divisors in some particular cases. We start first by recalling Peterson's results in our context, especially Theorem 5.3.3 in [Pet15].

Let I be an isotropic primitive plane of V, F the associated 1-cusp. The isomorphism class of the definite lattice $I^\perp/I$ depends only on the cusp F. We denote it by K_F and let Θ_F be the associated theta function, i.e the function defined by

$$\Theta_F(\tau) = \sum_{\gamma \in K_F^\vee/K_F} \sum_{x \in \gamma + K_F} q^{-Q(x)} v_\gamma,\ q = e^{2i\pi\tau},\ \tau \in \mathbb{H},$$

where $(v_\gamma)_{\gamma \in K_F^\vee/K_F}$ is the standard basis of $\mathbb{C}[K_F^\vee/K_F]$.

Let $I^\# = I_\mathbb{Q} \cap V^\vee$. Following [Bri83, 4.1], I is said to be *strongly primitive* if $I^\# = I$. The cardinality N_F of the finite group $H_I = I^\#/I$ depends only on F and is called the imprimitivity of F. Let $H_I^\perp := \{x \in L^\vee/L,\, \forall y \in H_I,\, (x,y) = 0\}$.

Proposition 2.6. *Let $I \subset V$ a primitive isotropic plane. Then*

(i) *$H_I^\perp/H_I \simeq K_F^\vee/K_F$ as quadratic finite modules.*

(ii) *$|V^\vee/V| = |K_F^\vee/K_F|.N_F^2$.*

Proof. Assertion (i) follows from Lemma page 77 in [Bri83]. For (ii), notice that $H_I^\perp \simeq \{\ell \in \mathrm{Hom}(V^\vee/V, \mathbb{Q}/\mathbb{Z}),\ \ell_{/H_I} = 0\}$ and that the cardinality of the latter is equal to $\frac{|V^\vee/V|}{N_F}$. □

Let $p : H_I^\perp \to K_F^\vee/K_F$ be the composite of the projection $H_I \to H_I^\perp/H_I$ followed by the isomorphism (i) from the last proposition. By construction, it is a morphism of quadratic finite modules. We have an induced map $p^* : \mathbb{C}[K_F^\vee/K_F] \to \mathbb{C}[V^\vee/V]$ which maps an element v_γ, $\gamma \in K_F^\vee/K_F$, to

$$p^* v_\gamma = \sum_{\substack{\delta \in H_I^\perp \\ p(\delta) = \gamma}} v_\delta.$$

Using (ii) from the previous proposition, it is straightforward that p^* commutes with the action of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ given by the Weil representation as in Section 2.1 Equation (3). Hence, for any $k \in \frac{1}{2}\mathbb{Z}$, we have a map

$$p^* : \mathrm{M}_k(\rho^*_{K_F})_\mathbb{Q} \to \mathrm{M}_k(\rho^*_V)_\mathbb{Q}.$$

For $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$, let

$$a(\gamma, n, F) = \frac{N_F}{24}(E_2.p^*(\Theta_F))(\gamma, n),$$

where $E_2(\tau) = 1 - 24\sum_{n \geq 1} \sigma_1(n) q^n$, $q = e^{2i\pi\tau}$, $\tau \in \mathbb{H}$, is the weight 2 Eisenstein series.

The following result is an application of Theorem 5.3.3 in [Pet15] to formula (4)

Proposition 2.7. *Let $\gamma \in V^\vee/V$, $n \in -Q(\gamma)+\mathbb{Z}$. Then we have the following linear equivalence relations in* $\mathrm{Pic}(\overline{\Gamma_V\backslash D_V}^{tor})$

(5)
$$\begin{aligned}[\overline{\mathcal{Z}(\gamma,-n)}] = &\frac{c(\gamma,n)}{2}[\overline{\mathcal{Z}(0,0)}] + \sum_{F\in S_1} u(\gamma,n,F)\Delta_F \\ &+ \sum_{i\in\mathcal{I}} g_i(\gamma,n)[\overline{\mathcal{Z}(\gamma_i,n_i)}] + \sum_{i\in\mathcal{I}}\sum_{F\in S_1} g_i(\gamma,n)a(\gamma_i,n_i,F)\Delta_F,\end{aligned}$$

where
$$u(\gamma,n,F) = \frac{c(\gamma,n)}{2}a(0,0,F) - a(\gamma,n,F),$$
and the coefficients $c(\gamma,n)$ are defined in 2.3.

Taking into account the estimates preceding Proposition 2.5, we get

Proposition 2.8. *For every $\epsilon > 0$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ with $n>0$, we have:*
$$\begin{aligned}[\mathcal{Z}(\gamma,-n)] = &-\frac{(2\pi)^{1+\frac{b}{2}}n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\prod_p \mu_p(\gamma,n,V)[\mathcal{Z}(0,0)] \\ &+ \sum_{F\in S_1} u(\gamma,n,F)\Delta_F + O_\epsilon(n^{\frac{2+b}{4}+\epsilon}),\end{aligned}$$
in $\mathrm{Pic}(\overline{\Gamma_V\backslash D_V}^{tor})_{\mathbb{Q}}$.

Remark 2.9. The term $u(\gamma,n,F)$ can a priori be as large as $c(\gamma,n)$. However, when F is strongly primitive, Lemma 2.11 shows that $c(\gamma,n)$ cancels because of the term $a(\gamma,n,F)$, hence giving a sharper control on the growth of $u(\gamma,n,F)$.

2.4. **Some consequences.** We turn now to the consequences of the previous proposition on the distribution of Hodge loci in 1-dimensional variation of Hodge structure. Let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a simple, non trivial, polarized variation of Hodge structure over a complex quasi-projective curve S such that the local system $\mathbb{V}_{\mathbb{Z}}^\vee/\mathbb{V}_{\mathbb{Z}}$ is trivial. Let $\rho : S \to \Gamma_V\backslash D_V$ be the corresponding period map. Let $\overline{S}$ be a smooth compactification of S such that the following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{\rho} & \Gamma_V\backslash D_V \\ \downarrow & & \downarrow \\ \overline{S} & \xrightarrow{\overline{\rho}} & \overline{\Gamma_V\backslash D_V}^{tor} \end{array}$$

Let $\gamma \in V^\vee/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ such that $n>0$. Since the variation is assumed to be simple, we can express the degree of the divisor $\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}$ on $\overline{S}$ as follows:

$$\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}) = \sum_{s\in\overline{S}} \mathrm{ord}_s(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}),$$

where $\mathrm{ord}_s(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)})$ is the multiplicity of the intersection of $\overline{S}$ with $\overline{\mathcal{Z}(\gamma,-n)}$ at a point $s\in\overline{S}$.

Notice that $\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(0,0)}) = -\mu(S)$, where μ is the finite measure on S given by integration of the first Chern class of $\mathcal{F}^2\mathcal{V}$. By Proposition 2.8 we have:

Corollary 2.10. *For every $\epsilon>0$, $\gamma\in V^\vee/V$ and $n\in -Q(\gamma)+\mathbb{Z}$ with $n>0$, we have:*

$$\begin{aligned}\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}) =& \frac{(2\pi)^{1+\frac{b}{2}}n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\prod_p \mu_p(\gamma,n,V)\mu(S)\\ &+\sum_{F\in S_1} u(\gamma,n,F)\deg_{\overline{S}}(\overline{\rho}^*\Delta_F) + O_\epsilon(n^{\frac{2+b}{4}+\epsilon}).\end{aligned}$$

Assume now that $\deg_{\overline{S}}(\overline{\rho}^*\Delta_F)=0$ if F corresponds to a totally isotropic plane which is not strongly primitive. The following lemma gives a control on the coefficient $u(\gamma,n,F)$ when F is associated to a strongly primitive totally isotropic plane.

Lemma 2.11. *Let $\gamma\in V^\vee/V$, I an isotropic, strongly primitive plane of V, F the associated 1-cusp and $K_F=I^\perp/I$. Then for all $\epsilon>0$ we have the following estimate:*

$$|u(\gamma,n,F)| \ll n^{\frac{b}{2}-1+\epsilon}$$

Proof. Let $\mathrm{M}_k^{\leq s}(\rho_V^*)$ be the vector space of vector-valued quasi-modular from of weight k and depth less than s (see [IRR14, Definition 1] and [MR05, Section 17.1] for definitions and properties of quasi-modular forms). Let D be the derivation operator $q\frac{d}{dq}$. Then we have the following structure theorem

$$\mathrm{M}^{\leq 1}_{1+\frac{b}{2}}(\rho_V^*) = \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)\oplus D(\mathrm{M}_{\frac{b}{2}-1}(\rho_V^*)).$$

For a proof, we refer to [MR05, Section 17.1] where it is proven for scalar quasi-modular forms, but the reader may notice that the proof generalizes easily to vector-valued quasi-modular forms.

The product $E_2.p^*(\Theta_F)$ is an element of $\mathrm{M}^{\leq 1}_{1+\frac{b}{2}}(\rho_V^*)$, hence we can write

$$E_2.p^*(\Theta_F) = \sum_i \alpha_i E_L^i + g + D(\tilde{g}), \tag{6}$$

where g is a cusp form of weight $1+\frac{b}{2}$, $(E_L^i)_i$ is a basis of Eisenstein series of $\mathrm{M}_{\frac{b}{2}+1}(\rho_V^*)$ with $E_L^0=E_L$ and $\tilde{g}\in\mathrm{M}_{\frac{b}{2}-1}(\rho_V^*)$. By comparing the constant coefficients, we get $\alpha_0=\frac{1}{2}$ and $\alpha_i=0$ for $i\neq 0$, since I is strongly primitive. Hence for $\gamma \in V^\vee/V$, $n \in -Q(\gamma)+\mathbb{Z}$ with $n \geq 0$, we have

$$(E_2.p^*(\Theta_F))(\gamma, n) = \frac{c(\gamma,n)}{2} + g(\gamma,n) + n\tilde{g}(\gamma,n).$$

Since $\tilde{g}$ is a modular form of weight $\frac{b}{2}-1$, we have $\tilde{g}(\gamma,n) \ll_\epsilon n^{\frac{b}{2}-2+\epsilon}$ for all $\epsilon > 0$. Also g is a cusp form and by (see [Sar90, Prop. 1.5.5]) $|a_{\gamma,n}(f)| \leq C_{\epsilon,f} n^{\frac{2+b}{4}+\epsilon}$. Combining these estimates we get the desired result. □

Remark 2.12. If I is not strongly primitive, then for $\gamma \notin H_I^\perp$, we have $u(\gamma, n, F) = c(\gamma, n)$, so the estimate in Lemma 2.11 fails. Even for $\gamma \in H_I^\perp$, all the Einsenstein series E_δ for $\delta \in H_I$ appear in the decomposition (6) with non-zero coefficients, so again Lemma 2.11 fails.

In view of the previous lemma, Corollary 2.10 rewrites

Corollary 2.13. *If $\overline{S}$ only meets the boundary of $\overline{\Gamma_V\backslash D_V}^{tor}$ in divisors Δ_F corresponding to strongly primitive totally isotropic planes, then for every $\epsilon > 0$ we have*

$$\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}) = \mu(S)\frac{(2\pi)^{1+\frac{b}{2}}n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\prod_p \mu_p(\gamma,n,V) + O_\epsilon(n^{u+\epsilon}),$$

for $\gamma \in V^\vee/V$, $n \in -Q(\gamma)+\mathbb{Z}$ with $n > 0$ satisfying local congruence conditions of Example 2.3 and $u = \max(\frac{b}{2}-1, \frac{2+b}{4})$. If S is projective, then we can choose $u = \frac{2+b}{4}$

Remark 2.14. In the case where the discriminant of (V,Q) is square free, all the primitive isotropic planes are strongly primitive by Proposition 2.6(ii), so the estimate 2.11 holds for all the coefficients $u(\gamma,n,F)$ for $\gamma \in V^\vee/V$, $n \in -Q(\gamma)+V$. The condition on the curve S in 2.13 is then automatically satisfied. Notice that here the control on the error term is sharper than the one in Theorem 1.1. This is because we don't know how to bound the intersection of $\overline{S}$ and $\overline{\mathcal{Z}(\gamma,-n)}$ at the boundary points, see Remark 2.12. However we conjecture that $|S \cap \mathcal{Z}(\gamma,-n)|_{mult}$ grows as the main term in the corollary.

3. Equidistribution in orthogonal modular varieties

The main goal of this section is to prove Proposition 3.8 which gives a lower estimate on the growth of the Hodge locus. The results in this section are independent from those in section 2.

3.1. **Construction of a local map.** Let U be a connected complex manifold and let $\{\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be an integral, polarized variation of Hodge structure of weight 2 over U with $h^{2,0}=1$. Assume that the fiber of $(\mathbb{V}_\mathbb{Z}, Q)$ at a point u_0 (hence at all points of U) is isomorphic to a quadratic even lattice (V,Q) of signature $(2,b)$ as in Section 2.2 and assume also that the local system $\mathbb{V}_{\mathbb{Z}}^{\vee}/\mathbb{V}_{\mathbb{Z}}$ is trivial. It follows that the monodromy representation factors through Γ_V, the stable orthogonal group of (V, Q). Let $\rho : U \to \Gamma_L \backslash D_V$ be the corresponding period map. We will construct in this section a sphere bundle over U that keep track of Hodge classes and a map from the latter to the quadric $A_0 = \{x \in V_{\mathbb{R}},\, Q(x) = -1\}$.

The line bundle $\mathcal{F}^2\mathcal{V}$ is simply the pullback of the Hodge bundle $\mathcal{L}$ via ρ. Let $\mathcal{V}_{\mathbb{R}}$ be the real vector bundle whose sheaf of differentiable sections is equal to $\mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Q}} \mathcal{C}^{\infty}_{\mathbb{R}}$, where $\mathcal{C}^{\infty}_{\mathbb{R}}$ is the sheaf of $\mathcal{C}^{\infty}$ real-valued functions on U. The fiber at a point $u \in U$ of $\mathcal{V}_{\mathbb{R}}$ is isomorphic to $V_{\mathbb{R}}$. This vector bundle contains a sub-vector bundle that we shall note $\mathcal{V}_{\mathbb{R}}^{1,1}$ and whose sheaf of differentiable sections is

$$\mathcal{F}^1\mathcal{V} \otimes \mathcal{C}^{\infty}_{\mathbb{C}} \cap \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{C}^{\infty}_{\mathbb{R}}.$$

Let $\mathcal{V}^{1,1} := \mathcal{F}^1\mathcal{V}/\mathcal{F}^2\mathcal{V}$. Then $\mathcal{V}_{\mathbb{R}}^{1,1}$ is the real part of $\mathcal{V}^{1,1}$, i.e the fiber at each point u is equal to $\mathcal{V}_u^{1,1} \cap V_{\mathbb{R}}$.

Assume that U is simply connected. Parallel transport by the Gauss-Manin connection trivializes the vector bundle $\mathcal{V}_{\mathbb{R}}$, hence it is isomorphic to $U \times V_{\mathbb{R}}$ and this isomorphism preserves the intersection form. Thus one has the commutative diagram

$$\begin{array}{ccc} \mathcal{V}_{\mathbb{R}}^{1,1} & \hookrightarrow & U \times V_{\mathbb{R}} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Projecting forward to $V_{\mathbb{R}}$, we get the parallel transport map:

$$\Xi : \mathcal{V}_{\mathbb{R}}^{1,1} \to V_{\mathbb{R}}.$$

The locus where this map is not submersive were studied in [Voi02, 17.3.4] and goes back to Griffiths and Green. Let us recall the setting and the main result. By Griffiths' transversality, the integrable connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_U$$

induces a $\mathcal{O}_U$-linear map:

$$\overline{\nabla} : \mathcal{F}^1\mathcal{V}/\mathcal{F}^2\mathcal{V} \to \mathcal{F}^0\mathcal{V}/\mathcal{F}^1\mathcal{V} \otimes \Omega^1_U$$

Let $u \in U$, then taking the fibers at u induce a $\mathbb{C}$-linear map

$$\overline{\nabla}_u : \mathcal{V}_u^{1,1} \to \mathcal{V}_u^{0,2} \otimes \Omega^1_{U,u}$$

Then we have the following lemma, due to Green (see Lemma 17.21 from [Voi02]).

Lemma 3.1. *Let $u \in U$, $\lambda \in \mathcal{V}_u^{1,1}$. If the map*

$$\overline{\nabla}_u(\lambda) : T_uU \to \mathcal{V}_u^{0,2}$$

is surjective then Ξ is submersive at (u, λ).

Consider the fibration over U defined by

$$\mathcal{S}_U = \{(u, \lambda),\, u \in U,\, \lambda \in \mathcal{V}_{u,\mathbb{R}}^{1,1},\, Q(\lambda) = -1\} \to U.$$

For every $u \in U$, the restriction of the quadratic form Q to $\mathcal{V}_{u,\mathbb{R}}^{1,1}$ is negative definite, and the fiber $\mathcal{S}_{U,u}$ is thus a $(b-1)$-dimensional sphere. By restriction of Ξ, we get a map:

$$\phi :\, \mathcal{S}_U \to A_0,$$

where $A_0 = \{x \in V_{\mathbb{R}},\, Q(x) = -1\}$.

Lemma 3.2. *Let $u \in U$, $\lambda \in \mathcal{V}_u^{1,1}$ such that $Q(\lambda) = -1$. If the map*

$$\overline{\nabla}_u(\lambda) : T_uU \to \mathcal{V}_u^{0,2}$$

is surjective then ϕ is submersive at (u, λ).

Proof. Let u and λ be as in the statement. The following diagram is commutative

$$\begin{array}{ccccc} T_{(u,\lambda)}\mathcal{S}_U & \longrightarrow & T_{(u,\lambda)}\mathcal{V}^{1,1}_{\mathbb{R}} & \xrightarrow{d_{(u,\lambda)}(Q\circ\Xi)} & \mathbb{R} \\ \downarrow d_{(u,\lambda)}\phi & & \downarrow d_{(u,\lambda)}\Xi & & \downarrow \\ T_\lambda A_0 & \longrightarrow & T_\lambda V_{\mathbb{R}} & \xrightarrow{d_\lambda Q} & \mathbb{R} \end{array}$$

The rows are exact by construction. By Lemma 3.1, $d_{(u,\lambda)}\Xi$ is surjective . Hence the map $d_{(u,\lambda)}\phi$ is surjective which proves the lemma. □

If $\rho(U)$ is not a point, then for $u \in U$ outside the locus where the differential of ρ is identically zero, there exists $\lambda \in \mathcal{S}_{U,u}$ which satisfies the condition of 3.2. Hence, the image $\mathrm{Im}(\phi)$ is open around λ. In particular, the set of points of U for which $\mathcal{V}_u^{1,1}$ contain an extra rational Hodge class x with $Q(x) = -1$ is dense (see [Voi02, Proposition 17.20] and [Ogu03, Theorem 1.1] for a proof without the norm condition imposed by Q).

Lemma 3.2 shows that on order to study the distribution of the Hodge locus in U, one can first study the distribution of points λ in A_0 for which there exists $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{N}$ such that $\sqrt{n}\lambda \in \gamma + V$, since the locus where ϕ is not submersive is a proper real analytic subset of $\mathcal{S}_U$. Hence it is negligible from a measure-theoretic perspective. This will be explained in the following section.

3.2. **Eskin-Oh's equidistribution result.** The study of the distribution of Hodge locus in U amounts via the map ϕ constructed above to the study of radial projections of integral points of $V_{\mathbb{R}}$ on A_0. We will need thus to understand the distribution in A_0 of the set $\{\lambda \in A_0, \sqrt{n}\lambda \in \gamma + V,\}$ for $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + V$ with $n > 0$. This a is a well studied problem and can be dealt with using Hardy-Littlewood's circle method (see [Vau97]). The results we present here follow [EO06] and [Oh04] to which we refer for more details. Recall that $G = \mathrm{O}(V_{\mathbb{R}})^+$ is the connected component of the identity of the real Lie group $\mathrm{O}(V_{\mathbb{R}})$.

Let μ_∞ be the G-invariant measure on A_0 defined in the following way : take W an open subset of $V_{\mathbb{R}}$ and let

$$\mu_\infty(W \cap A_0) = \lim_{\epsilon \to 0} \frac{\mathrm{Leb}\left(\{x \in W, \, |Q(x) + 1| < \epsilon\}\right)}{2\epsilon}.$$

Here Leb is the Lebesgue measure on $V_{\mathbb{R}}$ for which the lattice V is of covolume 1. We can now state the main result of this section which is an application of Theorem 1.2 in [EO06] (see also [Oh04, Section 5]) and the Siegel mass formula [EO06, (1.6)].

Proposition 3.3. *Let Ω be a compact subset of A_0 with zero measure boundary, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$. Then*

$$|\{\lambda \in \gamma + V, \, \frac{1}{\sqrt{n}}\lambda \in \Omega\}| \sim \mu_\infty(\Omega).n^{\frac{b}{2}}.\prod_p \mu_p(\gamma, n, V),$$

as $n \to +\infty$.

Proof. To see how Theorem 1.2 from [EO06] can be applied to our situation, we refer to the proof of Theorem 6.1 in *loc. cit.*. The only difference is that here we don't restrict to fundamental discriminants so we need to check that condition (1.3) in [EO06] holds. In other words, we need to know that for each n_0, there is only finitely many n such that

$$\frac{1}{\sqrt{n}}(\gamma + V) \cap A_0 = \frac{1}{\sqrt{n_0}}(\gamma + V) \cap A_0. \tag{7}$$

For a given n, notice that if (7) holds, then

$$\mathcal{Z}(\gamma, -n) = \mathcal{Z}(\gamma, -n_0),$$

so by Corollary 2.10 this can be true only for finitely many n. ☐

The G-invariant measure μ_∞ can be recovered as integration of a G-invariant volume form on A_0. Indeed, the group G acts transitively on A_0 and the choice of an element ξ in A_0 determines a surjective map

$$\begin{aligned} \pi_\xi : G &\longrightarrow A_0 \\ g &\mapsto g.\xi. \end{aligned} \tag{8}$$

Let H be the stabilizer of ξ. The induced map $G/H \to A_0$ is a diffeomorphism giving A_0 the structure of a symmetric space. Let $\mathfrak{g}_0$ and $\mathfrak{h}_0$ be the Lie algebras of G and H respectively. Then $\mathfrak{g}_0/\mathfrak{h}_0$ is isomorphic to the tangent space of A_0 at ξ via the differential of π_ξ at the identity of G. The space of G-invariant volume forms on A_0 is then identified with $\bigwedge^{b+1}(\mathfrak{g}_0/\mathfrak{h}_0)^\vee$.

Let $(e_1, e_2, \xi_1, \ldots, \xi_b)$ be an orthogonal basis of $V_\mathbb{R}$ such that for $i = 1, 2$ and $j = 1, \ldots, b$ we have $Q(e_i) = -Q(\xi_j) = 1$. Let ω_{A_0} be the unique G-invariant volume form on A_0 such that

$$\omega_{A_0,\xi_1} = de_1 \wedge de_2 \wedge d\xi_2 \wedge \cdots \wedge d\xi_b \tag{9}$$

in $\bigwedge^{b+1}(T_{\xi_1}A_0)^\vee$. Let μ_{A_0} be the G-invariant measure on A_0 given by integration of ω_{A_0}. We have then the following proposition:

Proposition 3.4. *For every open subset W of A_0, we have*

$$\mu_\infty(W) = \frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}\mu_{A_0}(W)$$

Proof. It is enough to prove the equality for W open subset of A_0 containing ξ_1. There exists an open subset U^{b+1} in $\mathbb{R}^{b+1}$ containing 0 such that the map

$$\begin{aligned} U^{b+1} &\to A_0 \\ (x_1, x_2, y_2, \ldots, y_b) &\mapsto (x_1, x_2, \sqrt{x_1^2 + x_2^2 - y_2^2 - \cdots - y_b^2 + 1}, y_2, \ldots, y_b). \end{aligned}$$

is a local chart around ξ_1. Let W be its image. For $\epsilon > 0$, the image of the map

$$\begin{aligned} U^{b+1} \times]-\epsilon, \epsilon[&\to A_0 \\ (x_1, x_2, y_2, \ldots, y_b, r) &\mapsto (x_1, x_2, \sqrt{x_1^2 + x_2^2 - y_2^2 - \cdots - y_b^2 + 1 + r}, y_2, \ldots, y_b). \end{aligned}$$

defines a tubular neighbourhood W_ϵ of W in $\mathbb{R}^{b+2}$ and one can check that

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{W_\epsilon} \omega = \frac{1}{2}\mu_{A_0}(W),$$

where $\omega = de_1 \wedge de_2 \wedge d\xi_1 \wedge \cdots \wedge d\xi_b$ and $A_\epsilon = \{x \in V_\mathbb{R},\ |Q(x) + 1| < \epsilon\}$. By change of variable, we have

$$\mathrm{Leb}(\{x \in W_\epsilon, |Q(x) + 1| < \epsilon\}) = \frac{2^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}} \int_{W_\epsilon} \omega.$$

Hence

$$\mu_\infty(W) = \lim_{\epsilon \to 0} \frac{\mathrm{Leb}(\{x \in W, |Q(x)+1| < \epsilon\})}{2\epsilon}$$
$$= \frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}} \mu_{A_0}(W)$$

which proves the lemma. □

Corollary 3.5. *Let Ω be a compact subset of A_0 with zero measure boundary, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$. Then*

$$|\{\lambda \in \gamma + V,\ \frac{1}{\sqrt{n}}\lambda \in \Omega\}| \sim \mu_{A_0}(\Omega).\frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.n^{\frac{b}{2}}.\prod_p \mu_p(\gamma, n, V),$$

as $n \to +\infty$.

3.3. **Quantitative study of the Hodge locus.** The goal of this section is to put together results from the previous sections in order to prove Proposition 3.8 which gives a lower bound on the cardinality of the Hodge locus. Let $\{\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a non-trivial, polarized, simple variation of Hodge structure over a complex quasi-projective curve S and let $\rho : S \to \Gamma_V^+ \backslash D_V^+$ be the associated period map.

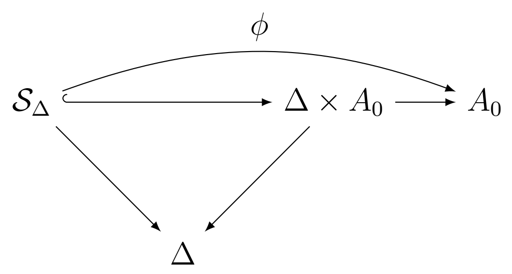
Recall that the Chern class ω of the Hodge bundle $\mathcal{F}^2\mathcal{V}$ defines a volume form on S . For any open subset $\Delta \subset S$, we note $\mu(\Delta) = \int_\Delta \omega$. Let Δ be an open simply connected subset of S. The restriction of ρ to Δ lifts to D_V^+. Let $0 \in \Delta$ be a point in Δ and P_0 the positive definite plane associated to $\rho(0)$. Then P_0 defines a maximal compact subroup $K := \mathrm{SO}(P_0) \times \mathrm{SO}(P_0^\perp)$ of G and a diffeomorphism

$$\pi : G/K \to D_V^+$$
$$g \mapsto g.P_0$$

We constructed in the previous paragraph a map

$$\phi : \mathcal{S}_\Delta \to A_0$$

where $\mathcal{S}_\Delta$ is a sphere bundle over Δ that fits into the following commutative diagram



For any $U \subset S$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ with $n>0$, let

$$|U \cap \mathcal{Z}(\gamma,-n)|_{mult} = \sum_{s \in U \cap \mathcal{Z}(\gamma,-n)} m(s,\gamma,n),$$

where $m(s,\gamma,n) = |\{\lambda \in \mathcal{S}_{\Delta,s},\ \sqrt{n}\lambda \in \gamma + V\}|$.

Lemma 3.6. *Let $s \in S$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ such that $n>0$. Then*

$$m(s,\gamma,n) \leq \mathrm{ord}_s(\rho^*\mathcal{Z}(\gamma,n)).$$

Proof. Let γ, n and s as in the statement of the proposition. Assume that

$$\{\lambda \in \mathcal{S}_{\Delta,s},\ \sqrt{n}\lambda \in \gamma+V\} = \{\lambda_1,\dots,\lambda_k\},$$

where $k = m(s,\gamma,n)$. There exists a finite index congruence subgroup Γ of Γ_V^+ such that the orbits $\Gamma.\lambda_1,\dots,\Gamma.\lambda_k$ are pairwise disjoints. In particular, the divisor

$$\mathcal{Z}' := \Gamma\backslash \left(\bigcup_{\substack{\lambda \in \gamma+V \\ Q(\lambda)=-n}} \lambda^\perp \right) \subset \Gamma\backslash D_V^+$$

has at least k irreducible components. The kernel of the morphism $\pi_1(S) \to \Gamma_V/\Gamma$ defines a finite étale cover $S' \xrightarrow{\iota'} S$ and we have a commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\rho'} & \Gamma\backslash D_V^+ \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\ S & \xrightarrow{\rho} & \Gamma_V^+\backslash D_V^+ \end{array}$$

Remark that $\iota^*\mathcal{Z}(\gamma,n)$ is equal to $\mathcal{Z}'$. Let $s' \in S'$ such that $\iota'(s') = s$. Then $\mathrm{ord}_{s'}(\rho'^*\iota^*\mathcal{Z}(\gamma,n)) \geq k$, since $\mathcal{Z}'$ has at least k irreducible components. By commutativity of the diagram above,

$$\rho'^*\iota^*\mathcal{Z}(\gamma,n) = \iota'^*\rho^*\mathcal{Z}(\gamma,n).$$

Since ι' is étale, we have

$$\mathrm{ord}_{s'}(\iota'^*\rho^*\mathcal{Z}(\gamma,n)) = \mathrm{ord}_s(\rho^*\mathcal{Z}(\gamma,n)),$$

which yields the desired result. □

Remark 3.7. We only have an inequality here because the curve S' may have intersection multiplicity strictly greater than one with a given irreducible component of the Heegner divisor Z'. In fact Theorem 1.1 implies that this does not happen when n is sufficiently large.

Proposition 3.8. *Let $\gamma \in V^\vee/V$. For all $s \in S$, there exists a simply connected open neighborhood $\Delta \subset S$ of s such that*

$$\liminf_n \frac{|\Delta \cap \mathcal{Z}(\gamma, -n)|_{mult}}{n^{\frac{b}{2}} \prod_{p<\infty} \mu_p(\gamma, n, V)} \geq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta),$$

where $n > 0$ ranges over numbers in $-Q(\gamma) + \mathbb{Z}$ represented by $-Q$ in $\gamma + V$.

Proof. The map $\pi : G \to D_V^+ \simeq G/K$ is submersive. Hence there exists U a simply connected open subset of D_V^+ around P_0 such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\sim} & U \times K \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Assume that $\rho(\Delta)$ is contained in U. We have a holomorphic map $\Upsilon : \Delta \to G$ given by the composition

$$\Delta \xrightarrow{\rho} U \to U \times K \xrightarrow{\sim} \pi^{-1}(U)$$
$$t \mapsto \rho(t) \mapsto (\rho(t), 1_K) \mapsto \Upsilon_t$$

Hence, we have a local trivialization of the fibration $\mathcal{S}_\Delta$ given by

$$\begin{array}{ccc} \Delta \times \mathbb{S}^{b-1} & \xrightarrow{\sim} & \mathcal{S}_\Delta \\ (t, \lambda) & \longrightarrow & (t, \Upsilon_t.\lambda) \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

where $\mathbb{S}^{b-1} = \{x \in P_0^\perp,\ Q(x) = -1\}$.

The map

$$\phi : \mathcal{S}_\Delta \to A_0$$

is, by lemma 3.2, submersive at (t, λ) if the map

$$\overline{\nabla}_u(\lambda) : T_u U \to \mathcal{V}_u^{0,2}$$

is surjective, or equivalently not-identically zero, since $T_u U$ is of dimension 1 over $\mathbb{C}$. Let $\mathcal{S}_\Delta^{sing}$ the locus where ϕ is not submersive. Then $\mathcal{S}_\Delta^{sing}$ is a proper real analytic closed subset of $\mathcal{S}_\Delta$ negligible for the Lebesgue measure. Outside $\mathcal{S}_\Delta^{sing}$, ϕ is submersive and in fact a local diffeomorphism by equality of dimensions.

Let ψ be the composite map

$$\psi:\ \Delta\times\mathbb{S}^{b-1}\to\mathcal{S}_\Delta\xrightarrow{\phi}A_0.$$

The pullback $\phi^*\omega_{A_0}$ is a volume form on $\mathcal{S}_\Delta$ and so is $\psi^*\omega_0$ on $\Delta\times\mathbb{S}^{b-1}$.

Let $\epsilon>0$, and let $\mathcal{S}_\Delta^{sing,\epsilon}$ be an open subset containing $\mathcal{S}_\Delta^{sing}$ such that $\int_{\mathcal{S}_\Delta^{sing,\epsilon}}\phi^*\omega_{A_0}\leq\epsilon$. Up to shrinking Δ, we can find a finite open cover W_i of $\mathcal{S}_\Delta\backslash\mathcal{S}_\Delta^{sing,\epsilon}$ such the restriction ϕ_i of ϕ to W_i is a diffeomorphism.

By Corollary 3.5, we get

$$\begin{aligned}|\Delta\cap\mathcal{Z}(\gamma,-n)|_{mult}&\geq\sum_i|\{\lambda\in\mathrm{Im}(\phi_i),\ \sqrt{n}\lambda\in\gamma+V\}|\\&=\sum_i\int_{W_i}\phi^*\omega_{A_0}.\frac{2^{\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.\prod_p\mu_p(\gamma,n,V)+o(n^{\frac{b}{2}})\\&\geq\frac{2^{\frac{b}{2}}.\left(\int_{\mathcal{S}_\Delta}\phi^*\omega_{A_0}-\epsilon\right).n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.\prod_p\mu_p(\gamma,n,V)+o(n^{\frac{b}{2}})\end{aligned}$$

Here we used that $\int_{\mathcal{S}_\Delta}\phi^*\omega_{A_0}=\mu(\Delta).\frac{2.\pi^{1+\frac{b}{2}}}{\Gamma(1+\frac{b}{2})}$, a result we prove in Lemma 3.9 below. Hence we have

$$\liminf_n\frac{|\Delta\cap\mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)}\geq\frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta)-\frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.\epsilon$$

By letting $\epsilon\to 0$, we get the desired result. □

Lemma 3.9. *We have:*

$$\int_{\mathcal{S}_\Delta}\phi^*\omega_{A_0}=\mu(\Delta).\frac{2.\pi^{1+\frac{b}{2}}}{\Gamma(1+\frac{b}{2})}.$$

Proof. The differential of the map $\pi:\ G\to D_L^+$ at the identity of G induces an isomorphism of $\mathfrak{p}_0$ with the tangent space of D_L^+ at P_0. Since ω is a G-invariant 2-form, it corresponds uniquely to an element Ω of $\bigwedge^2\mathfrak{p}_0^\vee$. Fix an orthogonal basis $(e_1,e_2,\xi_1,\dots,\xi_b)$ of $V_\mathbb{R}$ compatible with the decomposition $V_\mathbb{R}=P_0\oplus P_0^\perp$ and such that $Q(e_i)=-Q(\xi_j)=1$ for $i=1,2$ and $j=1,\dots,b$. The Lie algebra $\mathfrak{g}_0$ is then identified with $\mathfrak{so}(2,b)$ and an element $M\in\mathfrak{so}(2,b)$ is written by blocks in the following way

$$\begin{pmatrix}0&\theta&U\\-\theta&0&V\\{}^tU&{}^tV&N\end{pmatrix}\tag{10}$$

where $\theta\in\mathbb{R}$, U and V are $1\times b$-dimensional real matrices, and N is a $b\times b$-dimensional antisymmetric real matrix. For $i,j=1,\dots,b+2$, let $E_{i,j}$ be the matrix whose coefficients are zero except the coefficient

(i,j) which is equal to 1. For $i = 1, \dots, b$, define $U_i = E_{1,2+i} + E_{2+i,1}$ and $V_i = E_{2,2+i} + E_{2+i,2}$. The family $(U_i, V_i)_{i=1,\dots,b}$ is a basis of $\mathfrak{p}_0$.

By [CMSP03, 13.1] (see also [GP02, 5.3]), the curvature Θ of the Hodge bundle is given by

$$\Theta(X,Y) = -\lambda([X,Y])$$

for $X, Y \in \mathfrak{p}$ and where λ is the linear form on $\mathfrak{so}(2,b)$ associating to an element M written as in (10) the element $i\theta \in \mathbb{C}$. Notice that λ is the differential of the generator χ of the group of characters of K whose associated automorphic line bundle is the Hodge bundle (see [Zuc81]). A computation shows that

$$\Omega = \frac{i}{2\pi}\Theta = \frac{1}{2\pi}\sum_{i=1}^{b} dU_i \wedge dV_i. \tag{11}$$

Recall that the Killing form B of $\mathfrak{g}_0$ is negative definite on the Lie algebra $\mathfrak{k}_0$ of K and we have thus an orthogonal decomposition (see [Hel64]):

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Let $\xi \in P_0^\perp$ such that $Q(\xi) = -1$. Then $K' = \mathrm{SO}(P_0) \times \mathrm{SO}((\mathbb{R}\xi \oplus P_0)^\perp)$ is a maximal compact subgroup of H and we have similarly an orthogonal decomposition:

$$\mathfrak{h}_0 = \mathfrak{k}'_0 \oplus \mathfrak{p}'_0.$$

Let $\mathfrak{s}^{b-1}$ and $\widetilde{\mathfrak{p}}$ the orthogonal complements of $\mathfrak{k}'_0$ and $\mathfrak{p}'_0$ in $\mathfrak{k}_0$ and $\mathfrak{p}_0$ respectively with respect to the Killing form B of $\mathfrak{g}_0$:

$$\mathfrak{k}_0 = \mathfrak{k}'_0 \oplus \mathfrak{s}^{b-1} \quad , \quad \mathfrak{p}_0 = \mathfrak{p}'_0 \oplus \widetilde{\mathfrak{p}}.$$

The quotient $\mathfrak{g}_0/\mathfrak{h}_0$ can then be identified to $\mathfrak{s}^{b-1} \oplus \widetilde{\mathfrak{p}}$. The space $\mathfrak{s}^{b-1}$ can be identified, via the differential at the identity of $\mathrm{SO}(P_0^\perp)$ of the map π_{ξ_1} introduced in (8), with the tangent space at ξ_1 of the sphere

$$\mathbb{S}^{b-1} := \{x \in P_0^\perp,\ Q(x) = -1\},$$

which explains the notation. Let $\omega_{\mathbb{S}^{b-1}}$ be the unique $\mathrm{SO}(P_0^\perp)$-invariant volume form on $\mathbb{S}^{b-1}$ such that

$$\omega_{\mathbb{S}^{b-1},\xi_1} = d\xi_2 \wedge \cdots \wedge d\xi_b.$$

Then $(\mathbb{S}^{b-1}, \omega_{\mathbb{S}^{b-1}})$ is isometric to a sphere of dimension $b-1$ and radius 1 with its standard volume form, hence

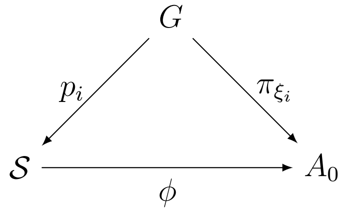
$$\int_{\mathbb{S}^{b-1}} \omega_{\mathbb{S}^{b-1}} = \frac{b.\pi^{\frac{b}{2}}}{\Gamma(1+\frac{b}{2})}. \tag{12}$$

The group G acts transitively on the left on $\mathcal{S} := \mathcal{S}_{D_L^+}$ via $g.(w,\xi) = (g.w, g.\xi)$ for $g \in G$, $w \in D_V^+$ and $\xi \in \mathcal{S}_u$. The map $\phi : \mathcal{S} \to A_0$ is

G-equivariant. For each $i = 1, \ldots, b$, we have thus a surjective map

$$p_i : G \to \mathcal{S}$$
$$g \mapsto (g.P_0, g.\xi_i)$$

that fits into the following commutative diagram



The differential of p_i induces an isomorphism between the tangent space of $\mathcal{S}$ at (P_0, ξ_i) and $\mathfrak{s}^{b-1} \oplus \mathfrak{p}_0$, where $\mathfrak{s}^{b-1}$ is isomorphic to the tangent space of $\mathbb{S}^{b-1}$ at ξ_i. The element $dU_i \wedge dV_i \in \bigwedge^2 \mathfrak{p}_0^\vee$ defines a G-invariant 2-form on $\mathcal{S}$ that we denote by ω_i. Let

$$t_i : K \to \mathbb{S}^{b-1}$$
$$k \to k.\xi_i.$$

The pull back of the form $\omega_{\mathbb{S}^{b-1}}$ along t_i is identified to an element $dY_1^i \wedge \ldots dY_{b-1}^i$ of $\bigwedge^{b-1} \mathfrak{k}_0^\vee$ for an orthogonal family $(Y_1^i, \ldots, Y_{b-1}^i)$ of $\mathfrak{k}_0$. Let $\omega^{(i)}$ be the G-invariant $(b-1)$-from on $\mathcal{S}$ such that $p_i^*\omega^{(i)}$ is equal to $dY_1^i \wedge \ldots dY_{b-1}^i$ in $\bigwedge^{b-1} \mathfrak{g}_0^\vee$

For each $i = 1, \ldots, r$, we have by (9)

$$p_i^*\phi^*\omega_{A_0} = \pi_{\xi_i}^*\omega_{A_0} = dU_i \wedge dV_i \wedge dY_1^i \wedge \ldots dY_{b-1}^i.$$

which is equal to $p_i^*\omega_i \wedge p_i^*\omega^{(i)} = p_i^*(\omega_i \wedge \omega^{(i)})$ by the construction itself. Hence, $\phi^*\omega_{A_0} = \omega_i \wedge \omega^{(i)}$ and summing over i yields

$$\phi^*\omega_{A_0} = \frac{1}{b}\sum_{i=1}^{b} \omega_i \wedge \omega^{(i)}$$

Notice now that the restrictions to $\mathbb{S}^{b-1}$ of the forms $\omega^{(i)}$ are all equal to the form $\omega_{\mathbb{S}_{b-1}}$ and that $\sum_{i=1}^{b} \omega_i = 2\pi\omega$ by (11). Hence

$$\psi^*\omega_{A_0} = \frac{2\pi}{b}\omega \wedge \omega_{\mathbb{S}^{b-1}}.$$

We have in particular

$$\frac{2\pi}{b}\mu(\Delta). \int_{\mathbb{S}^{b-1}} \omega_{\mathbb{S}^{b-1}} = \int_{\mathcal{S}_\Delta} \phi^*\omega_{A_0}.$$

Combined with (12), this yields the desired result. $\square$

4. End of the proof and applications

The goal of the section is to prove Theorem 1.1. We keep the notations from previous sections, i.e $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ is a simple, non trivial, polarized variation of Hodge structure over a quasi-projective curve S such that the local system $\mathbb{V}_{\mathbb{Z}}^{\vee}/\mathbb{V}_{\mathbb{Z}}$ is trivial and $\rho : S \to \Gamma_V \backslash D_V$ is the corresponding period map.

4.1. **First reduction.** Let H be a maximal isotropic subgroup of $V^{\vee}/V$ with respect to Q and let $\overline{\mathbb{V}}_{\mathbb{Z}}$ be the inverse image in $\mathbb{V}_{\mathbb{Z}}$ of $\underline{H}_S$, the trivial local system of fiber H. Then $\{\overline{\mathbb{V}}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ defines a simple, non-trivial, polarized variation of Hodge structure over S. Moreover, the fibers of the local system $\overline{\mathbb{V}}_{\mathbb{Z}}$ are isomorphic to a lattice $\overline{V}$ which has only strongly primitive totally isotropic planes and $\overline{V}^{\vee}/\overline{V} \simeq H^{\perp}/H$.

Proposition 4.1. *If Theorem 1.1 holds for $\{\overline{\mathbb{V}}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ then it holds for $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$.*

Proof. Let Δ be an open simply connected subset of S which satisfies Proposition 3.8 and let $\gamma \in H^{\perp}$ and $n \in -Q(\gamma) + \mathbb{Z}$. Denote by $\overline{\gamma}$ its image in $H^{\perp}/H \simeq \overline{V}^{\vee}/\overline{V}$. Then

$$|\Delta \cap \mathcal{Z}_{\overline{V}}(\overline{\gamma}, -n)|_{mult} = \sum_{t \in H} |\Delta \cap \mathcal{Z}(\gamma + t, -n)|_{mult}$$

where $\mathcal{Z}_{\overline{V}}(\overline{\gamma}, -n)$ is the Heegner divisor associated to the lattice $(\overline{V}, Q)$, to $\overline{\gamma}$ and n. By assumption, Theorem 1.1 holds for $\{\overline{\mathbb{V}}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$:

$$|\Delta \cap \mathcal{Z}_{\overline{V}}(\overline{\gamma}, -n)|_{mult} = -\mu(\Delta)\frac{\overline{c}(\overline{\gamma}, n)}{2} + o(n^{\frac{b}{2}}).$$

The $\overline{c}(\overline{\gamma}, n)$ are the coefficients of the Eisenstein series $E_{\overline{V}}$ constructed out of the lattice $(\overline{V}, Q)$ in a similar fashion to Example 2.3. For $t \in H$, we have by Lemma 3.8

$$|\Delta \cap \mathcal{Z}(\gamma + t, -n)|_{mult} \geq -\mu(\Delta)\frac{c(\gamma + t, n)}{2} + o(n^{\frac{b}{2}})$$

Thus

$$|\Delta \cap \mathcal{Z}(\gamma, -n)|_{mult} \leq \frac{\mu(\Delta)}{2}.\left(-\overline{c}(\overline{\gamma}, n) + \sum_{t \in H \backslash \{0\}} c(\gamma + t, n)\right) + o(n^{\frac{b}{2}})$$

Using Lemma 4.2 below, we have

$$|\Delta \cap \mathcal{Z}(\gamma, -n)|_{mult} \leq -\mu(S)\frac{c(\gamma, n)}{2} + o(n^{\frac{b}{2}})$$

Combined with 3.8, we get the desired result. $\square$

Lemma 4.2. *Let $\gamma \in H^{\perp}$, $n \in -Q(\gamma)+\mathbb{Z}$. Then*
$$\sum_{t\in H} c(\gamma+t,n) = \overline{c}(\overline{\gamma},n) + O_{\epsilon}(n^{\frac{b+2}{4}+\epsilon})$$

Proof. Let $p : H^{\perp} \to H^{\perp}/H \simeq \overline{V}^{\vee}/\overline{V}$. Then p induces two morphisms
$$p_* : \mathbb{C}[V^{\vee}/V] \to \mathbb{C}[\overline{V}^{\vee}/\overline{V}]$$
$$v_{\gamma} \mapsto v_{p(\gamma)} \text{ if } \gamma \in H^{\perp},\ 0 \text{ otherwise.}$$
and
$$p^* : \mathbb{C}[\overline{V}^{\vee}/\overline{V}] \to \mathbb{C}[V^{\vee}/V]$$
$$v_{\delta} \mapsto \sum_{\gamma\in H^{\perp},\, p(\gamma)=\delta} v_{\gamma}$$
which commutes with the Weil representation on both sides. Hence we have two $\mathbb{C}$-linear map: $p_* : \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*) \to \mathrm{M}_{1+\frac{b}{2}}(\rho_{\overline{V}}^*)$ and $p^* : \mathrm{M}_{1+\frac{b}{2}}(\rho_{\overline{V}}^*) \to \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)$. The modular form $p^*E_{\overline{V}} - p^*p_*E_V$ is then a cuspidal form and Lemma 4.2 follows by identifying its coefficients. □

4.2. **An upper bound.** By Theorem 4.1, we may assume that $V^{\vee}/V$ has no non-trivial totally isotropic subgroup in order the prove Theorem 1.1. Hence all the primitive isotropic planes of V are strongly primitive (see the definition preceding Proposition 2.6). Let $\overline{S}$ be a smooth compactification of S which fits in the following commutative diagram
$$\begin{array}{ccc} S & \xrightarrow{\rho} & \Gamma_V\backslash D_V \\ \downarrow & & \downarrow \\ \overline{S} & \xrightarrow{\overline{\rho}} & \overline{\Gamma_V\backslash D_V}^{tor} \end{array}$$

The boundary $\overline{S}\backslash S$ is finite. Let Δ_0 be a finite union of open subsets of $\overline{S}$ around each of those points. Consider Δ an open subset in $S\backslash\Delta_0$ which satisfies lemma 3.8. We can find a finite disjoint family of open subsets $(\Delta_i)_{i\in I}$ included in S which satisfy lemma 3.8 and such that $\mu(S) = \mu(\Delta) + \mu(\Delta_0) + \sum_{i\in I}\mu(\Delta_i)$. For each $i \in I$, we have:
$$\liminf_n \frac{|\Delta_i \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} \geq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^{\vee}/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta_i),$$
for $\gamma \in V^{\vee}/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ satisfying local congruence conditions. Also, by lemma 3.6
$$|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult} \leq \deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}) - \sum_{i\in I}|\Delta_i \cap \mathcal{Z}(\gamma,-n)|_{mult}$$

Hence by Corollary 2.13

$$\limsup_n \frac{|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} \leq \frac{(2\pi)^{1+\frac{b}{2}}\left(\mu(S)-\sum_i \mu(\Delta_i)\right)}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}$$

$$\leq \frac{(2\pi)^{1+\frac{b}{2}}\left(\mu(\Delta)+\mu(\Delta_0)\right)}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}.$$

Since the volume of Δ_0 can be chosen arbitrarily small, we deduce that

$$\limsup_n \frac{|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} \leq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta).$$

Combined with Lemma 3.8, this yields the desired equidistribution result.

4.3. **Elliptic fibrations in families of K3 surfaces.** We now derive some equidistribution results in quasi-polarized families of K3 surfaces. We begin by some background on K3 surfaces. The main references are [Huy16] and [BHPVdV04].

Let X be a K3 surface. The second cohomology group with integer coefficients of X endowed with its intersection form $(.,.)$ is an even unimodular lattice of signature $(3,19)$, hence isomorphic abstractly to the *K3 lattice*

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic lattice and E_8 the unique definite positive even unimodular lattice of rank 8, up to isomorphism. Denote by $Q = \frac{(.,.)}{2}$ the associated quadratic form.

Definition 4.3. *An elliptic K3 surface is a projective K3 surface X together with a surjective morphism $\pi : X \to \mathbb{P}^1$ such the generic fiber is a smooth integral curve of genus one.*

Recall that there is an elliptic fibration on X if, and only if there exists a *parabolic line bundle* on X, i.e a non-trivial line bundle L with $(L.L) = 0$. Indeed, if X admits an elliptic fibration $\pi : X \to \mathbb{P}^1$, then the class of a fiber gives a non-trivial element $e \in \mathrm{Pic}(X)$ such that $(e,e) = 0$. Conversely, let L be a non-trivial line bundle with square zero. Either L or L^{-1} is effective by Riemann-Roch. Assume L is effective. In [Huy16, **8**.2.13], it is shown that up to acting on L by the Weyl group of X, we can assume that L is nef of square zero. Then [**2**.3.10, *loc.cit.*] shows that $\pi_V : X \to \mathbb{P}(\mathrm{H}^0(X,L)^\vee)$ factors through $\mathbb{P}^1$ and induces an elliptic fibration whose fiber class is equal to L.

Let $P \subset \Lambda_{K3}$ be a primitive Lorentzian anistropic sublattice of rank $\rho \leq 4$ and let $V = P^\perp$. Then (V,Q) is an even quadratic lattice of signature $(2, 20-\rho)$ and we have an isomorphism of quadratic finite modules $(V^\vee/V, Q) \simeq (P^\vee/P, -Q)$ (see[Huy16, Prop.**14**.0.2]). Recall that a P-K3 surface is a K3 surface X with a fixed primitive embedding

$P \to \mathrm{Pic}(X)$ such that the image of P contains a quasi-polarization ℓ. If $L \in \mathrm{Pic}(X)$ is of square 0, we can write $L = L_P + L_V$ where $L_P \in P^\vee$ and $L_V \in V^\vee$. Then $(L.L) = (L_P.L_P) + (L_V.L_V)$ and $(L_V.L_V) \leq 0$ since the restriction of the form $(\ ,\ )$ to V is negative definite. Hence $(L_P.L_P) > 0$, unless $L = L_P$, which is excluded since P is assumed to be anisotropic.

Definition 4.4. *Let X be a P-K3 surface, $\gamma \in P^\vee/P$ and $n \in Q(\gamma)+\mathbb{Z}$. A parabolic line bundle L on X is said to be of type (γ, n) if $L_P \in \gamma + P$ and $(L_P.L_P) = 2n$. An elliptic fibration is said to be of type (γ, n) if a line bundle defining the fibration is so. We call n the norm of the elliptic fibration.*

We are now in the setting of Section 2.2 and we follow its notations, namely D_V is the period domain associated to the lattice (V, Q) and $\mathcal{Z}(\gamma, n)$ is the Heegner divisors associated to $\gamma \in V^\vee/V$ and $n \in Q(\gamma)+\mathbb{Z}$.

Proposition 4.5. *Let X be a P-K3 surface, $\gamma \in P^\vee/P$ and $n \in Q(\gamma) + \mathbb{Z}$. Then X admits a parabolic line bundle of type (γ, n) if and only if there exists $t \in \gamma + P$ such that $(t.t) = 2n$ and the period of X lies on the Heegner divisor $\mathcal{Z}(\gamma, -n)$.*

Proof. Let L be a line bundle on X defining an elliptic fibration of type (γ, n). Write $L = L_P + L_V$ as above. Then the element $L_V \in \gamma + V$ satisfies $(L_V.L_V) = -2n$ and take $t = L_P$. Hence, the period of X lies on the Heegner divisor $\mathcal{Z}(\gamma, -n)$. Conversely, if the period of X lies in $\mathcal{Z}(\gamma, -n)$, then there exists $\lambda \in H^{1,1}(X) \cap (\gamma + V)$ such that $(\lambda, \lambda) = -2n$. By assumption, there exists $t \in \gamma + V$ such that $(t.t) = 2n$. Then $L = \lambda + t \in \mathrm{Pic}(X)$ is of square zero and non-trivial. □

Proposition 4.6. *Let X be a P-K3 surface. Then X admits an elliptic fibration of norm less than n if and only if the period of X lies on the union of the Heegner divisors $\mathcal{Z}(\gamma, -s)$ for $\gamma \in P^\vee/P$ and $s \in]0, n]$ represented by Q in $\gamma + P$.*

Proof. The forward direction is clear. For the converse, we can construct a parabolic line bundle L on X of norm less than n in the same way as it was done above. If L is nef, then L defines an elliptic fibration of degree less than n and we are done. Otherwise, there exists a -2-curve C such that $(L.C) < 0$. Then $s_C(L) := L + (L, C).C$ is a parabolic line bundle with positive intersection with C and of norm less than the norm of L. We repeat the process if $s_C(L)$ is not nef. After a finite number of actions by the Weil group, we get a nef line bundle. □

Proof of corollary 1.3. Let $\mathcal{X} \xrightarrow{\pi} S$ be a non-isotrivial family of K3 surfaces with generic Picard group equal to P. The orthogonal to $\underline{P}_{\mathcal{X}}$ in $R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}$ defines a polarized variation of Hodge structure of weight

2 over S with fibers isomorphic to the lattice (V, Q) and to which we can apply Theorem 1.1. Using Proposition 4.5, this proves (i) and (ii). For (iii), let $\Delta \subset S$ an open subset and $\widetilde{N}(n, \Delta)$ the number of $s \in \Delta$ (counted with multiplicity) for which $\mathcal{X}_s$ admits an elliptic fibration of norm less than n. Then by Proposition 4.6 and Theorem 1.1 we have

$$\frac{\widetilde{N}(n,\Delta)}{\widetilde{N}(n,S)} = \frac{\sum_{\gamma\in V^\vee/V}\sum_{s\leq n, s\in Q(\gamma+P)} |\Delta\cap Z(\gamma,-s)|_{mult}}{\sum_{\gamma\in V^\vee/V}\sum_{s\leq n, s\in Q(\gamma+P)} |S\cap Z(\gamma,-s)|_{mult}} \underset{n\to\infty}{\longrightarrow} \frac{\mu(\Delta)}{\mu(S)}$$

□

Remark 4.7. There is an analogous result which concerns families of hyperkähler manifolds and which we state below. See [Huy03] for definitions. Indeed, given a hyperkähler manifold, the *Beauville-Bogomolov-Fujiki* form, defined in [Bea83], endows its second integral Betti cohomology group with a structure of a lattice of signature $(3, b_2 - 3)$.

Corollary 4.8. *Let d be an integer. Let $(\mathcal{X}, \mathcal{L}_{2d}) \to S$ be a non-isotrivial, split quasi-polarized family of hyperkähler manifolds of degree $2d$ over a quasi-projective curve S with generic Picard rank equal to 1 and let $\{R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}, \mathcal{F}^\bullet\mathcal{H}\}$ be the induced variation of Hodge structure over S. Let μ be the measure induced by integrating the first Chern class of $\mathcal{F}^2\mathcal{H}$. Then the set of points $s \in S$ for which $\mathcal{X}_s$ admits a parabolic line bundle L such that $(L.\mathcal{L}_{2d,s}) = 2dn$ becomes equidistributed in S with respect to μ as $n \to +\infty$.*

For the definition of split polarization, we refer to [GHS10, Definition 3.9].

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