VANISHING OF BRAUER CLASSES ON K3 SURFACES UNDER SPECIALIZATION

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ABSTRACT. Given a Brauer class on a K3 surface defined over a number field, we prove that there exists infinitely many specializations where the Brauer class vanishes, under certain technical hypotheses, answering a question of Frei–Hassett–Várilly-Alvarado.

CONTENTS

1. INTRODUCTION

Let X be a K3 surface over a number field K and let $\alpha \in \mathrm{Br}(X)$ be a Brauer class on X. Let $\mathscr{X} \to \mathscr{S}$ be a smooth projective model, where $\mathscr{S} \hookrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is an open subset of the spectrum of the ring of integers $\mathcal{O}_K$.

For a prime $\mathfrak{P}$ of $\mathscr{S}$ where α is unramified, we have a specialization of α in the Brauer group of the reduction $\mathscr{X}_{\mathfrak{P}}$ that we denote by:

$$\alpha_{\mathfrak{P}} \in \mathrm{Br}(\mathscr{X}_{\mathfrak{P}}) \ .$$

In [FHVA22], Frei, Hassett and Várilly-Alvarado ask about what can be said about the locus:

$$\mathcal{S}(X, \alpha) = \{\mathfrak{P} \in \mathscr{S} \,|\, \alpha_{\mathfrak{P}} = 0\} \ .$$

Let $\sigma : K \hookrightarrow \mathbb{C}$ be a complex embedding and let $T(X_\sigma(\mathbb{C}))$ be the transcendental lattice of $X_\sigma(\mathbb{C})$. Let N be the product of primes where X has bad reduction. In this article, we prove the following result.

Theorem 1.1. *Assume that the rank of $T(X_\sigma(\mathbb{C}))$ is different from $2, 4$, and that the torsion order of α in $\mathrm{Br}(X_{\overline{K}})$ is coprime to the discriminant of $T(X_\sigma(\mathbb{C}))$ and N. Then the set $\mathcal{S}(X, \alpha)$ is infinite.*

1.1. **Prior work and applications.** The question of triviality of Brauer classes on smooth projective surfaces under specialization has been raised in [FHVA22]. The authors proved *loc. cit.* that on a K3 surface, a Brauer class becomes trivial for a positive density of primes, when the following assumptions are satisfied: the endomorphism field E of the transcendental lattice of $X_\sigma(\mathbb{C})$ is totally real and $\dim_E(T(X_\sigma(\mathbb{C})))$ is odd. If these assumptions are not satisfied, then Charles [Cha14] proved that the set in Theorem 1.1 has density zero, up to a finite extension of K. Our result hence gives a fairly general answer to this question with no assumptions on the Hodge structure of X, see [FHVA22, Remark 1.4]. The technical conditions appearing in the theorem are artifacts of the proof and we explain their appearance in the strategy of the proof below.

Theorem 1.1 has several applications to rationality problems of cubic fourfolds and derived equivalences of twisted K3 surfaces which have been developed in [FHVA22]. Theorem 1.1 admits also a natural formulation over the complex numbers and in this case it follows from the results of [Voi02, §17.3], which can be furthermore sharpened into an equidistribution type statement in the spirit of [Tay20, TT23]. We simply formulate the statement here.

Theorem 1.2. *Let $\mathscr{X} \to S$ be a non-isotrivial smooth projective family of K3 surfaces over a complex quasi-projective algebraic variety S and let α be a Brauer class on the generic fiber $\mathscr{X}_\eta$. Then the locus in S where the specialization of the Brauer class α vanishes is analytically dense and equidistributed with respect to the metric given by the Chern form of the Hodge bundle.*

1.2. **Strategy of the proof.** The proof of Theorem 1.1 relies on Arakelov intersection theory on integral models of toroidal compactifications of GSpin Shimura varieties. We interpret the locus $\mathcal{S}(X, \alpha)$ as an intersection locus of $\mathscr{S}$ with a family of special divisors in a Shimura variety with level structure at r, the geometric torsion order of α. Following a method initiated by Charles in [Cha18] and generalized in [SSTT22], see also [MST22a, ST20, MST22b], we control the intersection numbers of $\mathscr{S}$ with a sequence of special divisors indexed by integers m at archimedean and non-archimedean places and compare the order of growths. If there were only finitely many primes where α vanishes, then this means that the intersection is supported at finitely many primes independent of m. As m grows, we get a contradiction by comparing the order of growths of the local and global estimates. The assumption on the rank is used at this level, as the results of [SSTT22]

are valid only when the rank of the transcendental lattice is at least 5, and the case of rank 3 follows from [FHVA22].

The main new difficulty that we have to overcome in this paper is that the Shimura variety has a level structure at r and hence the results of [HP20] do not apply directly. We have thus to construct suitable integral models which are ad hoc for our purposes, then we construct Borcherds products to derive the global estimate on the intersection number. This, in particular, explains the technical condition appearing in the statement of Theorem 1.1. We also have to construct the special divisors above r, in a way compatible with the special divisors already constructed outside of r by [HP20]. We then show that [HP20, Theorem A] extends over r for the special divisors and for some well chosen Borcherds products. We do not prove that the resulting generating series is a modular form, a result not needed for our purposes and which requires a deeper understanding of the integral models with full level structure at r.

1.3. **Organization of the paper.** In §2, we explain how to reduce Theorem 1.1 to an intersection theoretic statement in GSpin Shimura varieties. In §3, we introduce GSpin Shimura varieties, their integral models, special divisors and toroidal compactifications. We construct suitable integral models with level structure at r and use them to write down the local and global estimates needed from Arakelov theory. In §4 we prove the global estimate on the intersection number using a well-chosen Borcherds products. In §5 we estimate the archimedean contributions, and in §6 we estimate the non-archimedean contributions.

2. K3 surfaces and Brauer classes: some reductions

We prove in this section how to deduce Theorem 1.1 from Theorem 2.2. Then we explain the connection to special quasi-endomorphisms on Kuga-Satake abelian varieties. For background on Brauer classes on K3 surfaces, we refer to [Huy16, Chapter 18] and [FHVA22, §4].

2.1. **Background results.** Let X be a K3 surface over a number field K and let $\alpha \in \mathrm{Br}(X)$. Let $\sigma : K \hookrightarrow \mathbb{C}$ be a complex embedding and let (L, Q) be the transcendental lattice of the complex K3 surface $X_\sigma(\mathbb{C})$. Then we have the following maps between the different Brauer groups:

$$\mathrm{Br}(X) \to \mathrm{Br}(X_{\overline{K}}) \simeq \mathrm{Br}(X_\sigma) \subset \mathrm{Br}^{an}(X_\sigma(\mathbb{C})) \ ,$$

where $\mathrm{Br}^{an}(X_\sigma(\mathbb{C}))$ is the analytic Brauer group of $X_\sigma(\mathbb{C})$. By a theorem of Gabber, the Brauer group $\mathrm{Br}(X_{\overline{K}})$ is torsion and is in fact equal to the torsion part of the analytic group $\mathrm{Br}^{an}(X_\sigma(\mathbb{C}))$.

It follows that the image of the class α in $\mathrm{Br}^{an}(X_\sigma(\mathbb{C}))$ is torsion of order $r \geq 1$, which may be different from its torsion order in $\mathrm{Br}(X)$. We refer to r as the *geometric torsion order* of α.

The class α admits B-lifts to $H^2(X_\sigma(\mathbb{C}), \mathbb{Q})$ that we now recall following [Huy16, Page 415]. From the exponential exact sequence, we get the exact sequence

$$0 \to H^2(X_\sigma(\mathbb{C}), \mathbb{Z})/\mathrm{Pic}(X_\sigma(\mathbb{C})) \to H^2(X_\sigma(\mathbb{C}), \mathcal{O}_{X_\sigma(\mathbb{C})}) \to \mathrm{Br}^{an}(X_\sigma(\mathbb{C})) \to 0 \,.$$

Since the torsion part of $\mathrm{Br}^{an}(X_\sigma(\mathbb{C}))$ is equal to $\mathrm{Br}(X_\sigma)$, we get:

$$0 \to H^2(X_\sigma(\mathbb{C}), \mathbb{Z}) + \mathrm{NS}(X_\sigma(\mathbb{C}))_\mathbb{Q} \to H^2(X_\sigma(\mathbb{C}), \mathbb{Q}) \to \mathrm{Br}(X_\sigma) \to 0,$$

yielding an isomorphism

$$\mathrm{Hom}_\mathbb{Z}(L, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Br}(X_\sigma) \,.$$

In particular, the r-torsion sub-groups are isomorphic:

$$\frac{1}{r}L^\vee/L^\vee \simeq \mathrm{Hom}_\mathbb{Z}(L, \frac{1}{r}\mathbb{Z}/\mathbb{Z}) \simeq \mathrm{Br}(X_\sigma)[r].$$

Let $\beta \in \frac{1}{r}L^\vee/L^\vee$ be a preimage of α. To summarize, we have proven the following proposition.

Proposition 2.1. *Let L be the transcendental lattice of $X_\sigma(\mathbb{C})$ and let $\alpha \in \mathrm{Br}(X)$ be a Brauer class of geometric torsion order $r \geq 1$. Then there exists $\beta \in \frac{1}{r}L^\vee/L^\vee$ which corresponds to the image of α in $\mathrm{Br}(X_{\overline{K}})[r]$.*

2.2. **Compatibility with reductions.** We keep the notations from the previous section and let $\mathscr{X} \to \mathscr{S}$ be a smooth projective model of X where $\mathscr{S} \hookrightarrow \mathrm{Spec}(\mathcal{O}_K)$ is a Zariski open subset.

Let $\mathfrak{P}$ be a prime of good reduction for X, i.e., in $\mathscr{S}$, where the Brauer class α is unramified and has torsion r coprime to the residual characteristic of $\mathfrak{P}$. This excludes only finitely many primes.

The Kummer exact sequence yields the following commutative diagram, where the middle vertical arrow is an isomorphism by smooth base change theorem:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{NS}(X_{\overline{K}}) \otimes \mathbb{Z}/r\mathbb{Z} & \longrightarrow & H^2_{\text{ét}}(X_{\overline{K}}, \mu_r) & \longrightarrow & \mathrm{Br}(X_{\overline{K}})[r] & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{NS}(\mathscr{X}_{\overline{\mathfrak{P}}}) \otimes \mathbb{Z}/r\mathbb{Z} & \longrightarrow & H^2_{\text{ét}}(\mathscr{X}_{\overline{\mathfrak{P}}}, \mu_r) & \longrightarrow & \mathrm{Br}(\mathscr{X}_{\overline{\mathfrak{P}}})[r] & \longrightarrow & 0
\end{array}$$

By Artin's comparison theorem [AGV$^+$72, XVI 4], for every prime number ℓ, we have a natural $\mathrm{Gal}(\overline{K}/K)$-module $L_{\mathbb{Z}_\ell}$ in $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1))$. By compatibility of Poincaré pairings in Betti and étale cohomology, the dual lattice of $L_{\mathbb{Z}_\ell}$ is equal to $L^\vee_{\mathbb{Z}_\ell}$. If ℓ does not divide the discriminant of L, then in fact L is self-dual at ℓ and $L^\vee_{\mathbb{Z}_\ell} = L_{\mathbb{Z}_\ell}$.

Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class of geometric torsion order r. By Proposition 2.1, there exists $\beta \in \frac{1}{r}L^\vee/L^\vee$ that lifts α. For every prime number ℓ, we let $\beta_\ell \in \frac{1}{r}L^\vee/L^\vee \otimes \mathbb{Z}_\ell$ denote the ℓ-adic component of β.

If ℓ is coprime to r, then $\beta_\ell = 0$ and if ℓ is coprime to the discriminant of L, then

$$\beta_\ell \in \frac{1}{r}L^\vee/L^\vee \otimes \mathbb{Z}_\ell \simeq \frac{1}{r}L/L \otimes \mathbb{Z}_\ell \simeq L_{\mathbb{Z}_\ell}/rL_{\mathbb{Z}_\ell} \ .$$

Theorem 1.1 is then a consequence of the following statement.

Theorem 2.2. *Assume that r is coprime to the discriminant of L. Then there exist infinitely many prime ideals $\mathfrak{P}$ such that there exists $\lambda \in \mathrm{Pic}(\mathscr{X}_{\overline{\mathfrak{P}}})$ which satisfies the following: for every prime ℓ coprime to $\mathfrak{P}$, the image of λ under the isomorphism*

$$H^2_{\text{ét}}(\mathscr{X}_{\overline{\mathfrak{P}}}, \mathbb{Z}_\ell(1)) \simeq H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)),$$

lies in $L_{\mathbb{Z}_\ell}$ and the residue class of λ in $L_{\mathbb{Z}_\ell}/rL_{\mathbb{Z}_\ell}$ is equal to β_ℓ.

2.3. **Proof of Theorem 1.1.** Assuming Theorem 2.2, we will prove in this section Theorem 1.1. Let $\mathfrak{P}$ be a prime ideal given by Theorem 2.2, and where α is unramified. Let r be the geometric torsion order of α, which is coprime to p, the residual characteristic of $\mathfrak{P}$. We have then the following diagram, where the middle vertical arrow is an isomorphism by proper and smooth base change theorem:

$$\begin{array}{ccccccc}
\oplus_{\ell|r}L_{\mathbb{Z}_\ell} \subset \oplus_{\ell|r}H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{Br}(X_{\overline{K}})[r] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
(\lambda)_{\ell|r} \in \oplus_{\ell|r}H^2_{\text{ét}}(\mathscr{X}_{\overline{\mathfrak{P}}}, \mathbb{Z}_\ell(1)) & \longrightarrow & \mathrm{Br}(\mathscr{X}_{\overline{\mathfrak{P}}})[r] & \longrightarrow & 0,
\end{array}$$

By construction, the image of λ in $\mathrm{Br}(X_{\overline{K}})[r]$ is equal to α. By commutativity of the diagram, this implies that the image of λ in $\mathrm{Br}(\mathscr{X}_{\overline{\mathfrak{P}}})[r]$ is equal to the specialization $\alpha_{\mathfrak{P}}$ of α. Since $\lambda \in \mathrm{NS}(\mathscr{X}_{\overline{\mathfrak{P}}})$, we can conclude that $\alpha_{\mathfrak{P}} = 0$ in $\mathrm{Br}(\mathscr{X}_{\overline{\mathfrak{P}}})[r]$. Finally, we use the following lemma, which is taken from [FHVA22, Lemma 4.4], to conclude.

Lemma 2.3. *We have $\alpha_{\mathfrak{P}} = 0$ in $\mathrm{Br}(\mathscr{X}_{\mathfrak{P}})$.*

2.4. **Special endomorphisms on Kuga-Satake abelian varieties.** We explain in this section our strategy for proving Theorem 2.2. By [MP15, Theorem 3] (and [KMP16, IIK21] when the characteristic is equal to 2), up to extending the number field K, we can associate to X an abelian variety A defined over K, the *Kuga-Satake* abelian variety such that for any prime $\mathfrak{P}$ of good reduction for X and A, the $\mathbb{Z}_\ell$ and crystalline realizations of the primitive cohomology of $\mathscr{X}_{\mathfrak{P}}$ embed in those of $\mathrm{End}(\mathcal{A}_{\mathfrak{P}})$.

Let $\mathfrak{P}$ be a place where both X and A have good reduction. For every $\beta \in L/rL$, we will define a subgroup

$$V_\beta(\mathcal{A}_{\overline{\mathfrak{P}}}) \subset \mathrm{End}(\mathcal{A}_{\overline{\mathfrak{P}}}),$$

of special endomorphisms of $\mathcal{A}_{\overline{\mathfrak{P}}}$, see Sections 3.1.2 and 3.2.1.

Proposition 2.4. *Let $r \geq 1$ and let $\mathfrak{P}$ be a prime of good reduction of residual characteristic coprime to r. Then there exists $\beta \in L/rL$ such that $V_\beta(\mathcal{A}_{\overline{\mathfrak{P}}})$ is different from zero if and only if there exists $\lambda \in \mathrm{Pic}(\mathscr{X}_{\overline{\mathfrak{P}}})$ that satisfies the conditions of Theorem 2.2.*

Proof. Let $\widetilde{\beta} \in \frac{1}{r}L/L$ be a lift of the class α as given by Proposition 2.1 and let $\beta \in L/rL$ its image after multiplication by r.

From the properties of the Kuga-Satake abelian variety, we have an inclusion:

$$V_\beta(\mathcal{A}_{\overline{\mathfrak{P}}}) \hookrightarrow \mathrm{Pic}(\mathscr{X}_{\overline{\mathfrak{P}}}).$$

Then for any non-zero endomorphism $f \in V_\beta(\mathcal{A}_{\overline{\mathfrak{P}}})$, the class $\lambda = f$ gives the desired result. Indeed, by definition of special endomorphisms, for any prime ℓ, we have an ℓ-adic realization $f \in L_{\mathbb{Z}_\ell}$ which by definition has residue equal to β in L/rL, hence it satisfies Theorem 2.2. $\qquad\square$

We conclude that Theorem 2.2 is implied by the following statement which we will prove in Section 3.5.

Theorem 2.5. *For $\beta \in L/rL$ as in the proof above, there exists infinitely many primes $\mathfrak{P}$ coprime to r such that A has good reduction at $\mathfrak{P}$ and $V_\beta(\mathcal{A}_{\overline{\mathfrak{P}}}) \neq \{0\}$.*

3. GSpin Shimura varieties: integral models and Arakelov intersection theory

We introduce in this section GSpin Shimura varieties, their integral models and their toroidal compactifications. Our main references are [AGHMP18, MP16, HP20, Per19] to which we refer for more details.

3.1. **GSpin Shimura varieties over** $\mathbb{Q}$. Let (L, Q) be a quadratic even lattice of signature $(n, 2)$, $n \geq 1$ and denote the bilinear form associated to (L, Q) by:

$$(x \cdot y) = Q(x + y) - Q(x) - Q(y), \forall x, y \in L .$$

We can construct a Shimura datum associated to (L, Q) as follows: let $G = \mathrm{GSpin}(L_{\mathbb{Q}})$ be the reductive algebraic group over $\mathbb{Q}$ of spinor similitudes and consider the Hermitian symmetric domain

$$\mathcal{D} = \{\omega \in \mathbb{P}(L_{\mathbb{C}}), (\omega \cdot \omega) = 0, (\omega \cdot \overline{\omega}) < 0\}.$$

Then $(G, \mathcal{D})$ is a Hodge type Shimura datum with reflex field equal to $\mathbb{Q}$. For any choice of a compact open subgroup $K \subset G(\mathbb{A}_f)$, we get a Shimura variety defined over $\mathbb{Q}$ whose set of complex points is

$$M(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K,$$

and whose canonical model M is a smooth Deligne-Mumford stack over $\mathbb{Q}$.

The choice of the lattice (L, Q) specifies a particular open subgroup of $G(\mathbb{A}_f)$ defined as $K = C(L \otimes \widehat{\mathbb{Z}}) \cap G(\mathbb{A}_f)$, where $C(L \otimes \widehat{\mathbb{Z}})$ is the

$\widehat{\mathbb{Z}}$-Clifford algebra of $(L \otimes \widehat{\mathbb{Z}}, Q)$. The group K is the largest compact-open subgroup of $G(\mathbb{A}_f)$ that stabilizes $L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and acts trivially on $L^\vee/L$ where $L^\vee$ is the dual lattice of L defined as:

$$L^\vee = \{x \in L_{\mathbb{Q}} | \forall y \in L, (x \cdot y) \in \mathbb{Z}\}.$$

As mentioned above, the Shimura variety M is of Hodge type and carries a family of Kuga-Satake abelian varieties $A \xrightarrow{\pi} M$ whose relative cohomology can be understood in terms of algebraic representations of G as follows. By construction, G has an algebraic action by left multiplication on $C(V)$ where $V = L \otimes_{\mathbb{Z}} \mathbb{Q}$, and $C(V)$ is the Clifford algebra of (V, Q). There is also an action of G on V via an algebraic group morphism $G \to \mathrm{SO}(V)$. Letting $H = C(V)$, then we have an inclusion $V \hookrightarrow \mathrm{End}_{\mathbb{Q}}(H)$ given by left multiplication and it is in fact a G-equivariant map. This yields filtered vector bundles with integrable connection on M, denoted $(\mathbb{V}_{dR}, F^\bullet \mathbb{V}_{dR})$ and $(\mathbb{H}_{dR}, F^\bullet \mathbb{H}_{dR})$ related by a morphism of flat filtered vector bundles

$$\mathbb{V}_{dR} \hookrightarrow \mathbb{H}_{dR} .$$

The vector bundle $\mathbb{V}_{dR}$ is endowed with a bilinear form

$$(\cdot) : \mathbb{V}_{dR} \times \mathbb{V}_{dR} \to \mathcal{O}_M,$$

for which the line bundle $\omega = F^1 \mathbb{V}_{dR}$ is isotropic and $F^0 \mathbb{V}_{dR} = (F^1 \mathbb{V}_{dR})^\perp$. Moreover, we have a canonical isomorphism of filtered vector bundles:

$$\mathbb{H}_{dR} \simeq \underline{\mathrm{Hom}}(R^1 \pi_* \Omega^\bullet_{A/M}, \mathcal{O}_M),$$

see [AGHMP18, §4.1] for more details.

The constructions above are functorial in the following way: for any inclusion $(L_1, Q) \subseteq (L_2, Q)$ of quadratic lattices, then the previous discussion produces Shimura varieties M_1 and M_2 over $\mathbb{Q}$ which admit Kuga-Satake abelian schemes $A_i \to M_i$ and filtered vector bundles with integrable connections $(\mathbb{V}^i_{dR}, F^\bullet \mathbb{V}^i_{dR})$ and $(\mathbb{H}^i_{dR}, F^\bullet \mathbb{H}^i_{dR})$, for $i = 1, 2$. We have a finite morphism $\eta : M_1 \to M_2$, which is étale if L_1 has finite index in L_2. We also have morphism of Kuga-Satake abelian schemes

$$\begin{array}{ccc} A_1 & \longrightarrow & \eta^* A_2 \\ & \searrow \quad \swarrow & \\ & M_1 & \end{array}$$

which is an isogeny in the finite index case, of degree a power of $|L_2/L_1|$. Finally, we have canonical isomorphisms of filtered vector bundles with integrable connections:

$$\eta^* \mathbb{V}^2_{dR} \simeq \mathbb{V}^1_{dR}, \quad \text{and} \quad \eta^* \mathbb{H}^2_{dR} \simeq \mathbb{H}^1_{dR} .$$

3.1.1. *Integral models and their compactifications.* We recall in this section the construction of integral models of GSpin Shimura varieties following [HP20, §6], [AGHMP18, §§4.2, 4.3], and their toroidal compactifications following [HP20, Per19].

Let p be a prime number. The lattice L is said to be *maximal* at p if $L \otimes \mathbb{Z}_p$ is a maximal lattice of $L \otimes \mathbb{Q}_p$ over which the quadratic form is $\mathbb{Z}_p$-valued. In particular, if the lattice L is self-dual at p, then L is maximal at p. We say that L is maximal if it is maximal at all primes.

Let Ω be the finite set of primes $p \in \mathbb{Z}$ at which the lattice $L_{\mathbb{Z}_p}$ is not maximal. Then by [HP20, §6], there is a normal and flat integral model $\mathcal{M} \to \mathbb{Z}[\Omega^{-1}]$ with generic fiber M, which is a Deligne-Mumford stack and which enjoys the following properties:

(1) The Kuga-Satake abelian scheme extends to an abelian scheme $\mathcal{A} \to \mathcal{M}$.
(2) The line bundle $\omega = F^1 \mathbb{V}_{dR}$ extends to a line bundle $\boldsymbol{\omega}$ on $\mathcal{M}$.
(3) $\mathcal{M}$ is smooth at a prime p if the lattice (L, Q) is almost self-dual and regular if p is odd, and p^2 does not divide the discriminant of L.

To explain the last condition, we say that L is almost self-dual at p if either p is odd and L is self-dual at p or $p = 2$ and $v_2(|L^\vee/L|) \leq 1$, where v_2 is the 2-adic valuation.

3.1.2. *Special divisors.* By [AGHMP18, §4.5], for every scheme $S \to \mathcal{M}$, there is a functorial subspace

$$V(\mathcal{A}_S) \subset \operatorname{End}(\mathcal{A}_S)_{\mathbb{Q}}$$

of *special quasi-endomorphisms*, the construction of which will be recalled in Section 3.2. The space $V(\mathcal{A}_S)$ is endowed with a positive definite quadratic form Q such that $x \circ x = Q(x) \cdot \operatorname{Id}_{\mathcal{A}_S}$ for $x \in V(\mathcal{A}_S)$. One in fact can define for every $\beta \in L^\vee/L$, a subset

$$V_\beta(\mathcal{A}) \subset V(\mathcal{A}_S)$$

of special quasi-endomorphisms whose different cohomological realizations are prescribed by β, see [AGHMP18, P. 447]. We have then the following result which is [AGHMP18, Proposition 4.5.8].

Proposition 3.1. *For every $\beta \in L^\vee/L$, $m \in Q(\beta) + \mathbb{Z}$, there is a finite, unramified and relatively representable $\mathcal{M}$-stack whose functor of points assigns to every scheme $\mathcal{S} \to \mathcal{M}$ the set*

$$\mathcal{Z}(\beta, m)(\mathcal{S}) = \{x \in V_\beta(\mathcal{A}_S) \,|\, Q(x) = m\}$$

By [HM22, Proposition 2.4.3], $\mathcal{Z}(\beta, m)$ is a generalized Cartier divisor in the sense of [HM22, Definition 2.4.1] and can also be seen as Cartier divisor on $\mathcal{M}$ by [HM22, Remark 2.4.2]. We will henceforth refer to it as *special divisor.*

We can give an explicit description of the set of complex points of the special divisors as follows: in $\mathcal{M}(\mathbb{C})$, a point $s \in \mathcal{M}(\mathbb{C})$ can be lifted to a pair

$$(h, g) \in \mathcal{D} \times G(\mathbb{A}_f),$$

and the group of special quasi-endomorphisms of $\mathcal{A}_s$ is canonically identified with

$$\{x \in L_{\mathbb{Q}} | (x \cdot h) = 0\} .$$

Then the special divisors are given, for every $\beta \in L^\vee/L$ and m, by the following double quotient

$$\mathcal{Z}(\beta, m)(\mathbb{C}) = G(\mathbb{Q}) \backslash \bigcup_{\substack{\lambda \in g.(\beta + \widehat{L}) \\ Q(\lambda) = m}} \{(h, g) \in \mathcal{D} \times G(\mathbb{A}_f), (h.\lambda) = 0\}/K .$$

3.2. **GSpin Shimura varieties with level structure.** We fix a maximal quadratic lattice (L, Q) for the rest of the paper and let $r \geq 1$. Consider the inclusion of quadratic lattices

$$(rL, Q) \subset (L, Q).$$

The discussion from the previous section applies to both lattices (L, Q) and (rL, Q) yielding normal flat integral models

$$\mathcal{M} \to \operatorname{Spec}(\mathbb{Z}) , \quad \text{and} \quad \mathcal{M}_r \to \operatorname{Spec}(\mathbb{Z}[\Omega^{-1}])$$

of M and M_r. Here Ω is the set of primes where rL is not maximal, i.e., the prime divisors of r. We have thus an abelian scheme $\mathcal{A}_r \to \mathcal{M}_r$, and a Hodge line bundle $\boldsymbol{\omega}_r$. We also have a finite étale map $\eta : M_r \to M$ which extends to a finite map over $\mathbb{Z}[\Omega^{-1}]$ by [HP20, Proposition 6.6.1] that we still denote by

$$\eta : \mathcal{M}_r \to \mathcal{M}_{\mathbb{Z}[\Omega^{-1}]} ,$$

and such that $\eta^* \boldsymbol{\omega} \simeq \boldsymbol{\omega}_r$.

The Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{M}$ pulls back to an abelian scheme $\eta^* \mathcal{A}$ on $\mathcal{M}_r$ with an isogeny

$$\begin{array}{ccc} \mathcal{A}_r & \longrightarrow & \eta^* \mathcal{A} \\ & \searrow \quad \swarrow & \\ & \mathcal{M}_r & \end{array}$$

which extends the isogeny over the generic fibers.

The following lemma is an easy consequence of the construction of the module of special quasi-endomorphisms, see also [HP20, Proposition 6.6.2, 6.6.3] which refers to [AGHMP17, Proposition 2.6.4] for the proof. To simplify notations, we will drop the index r in the notation of special divisors in $\mathcal{M}_r$, as it will be clear from their coset in which space they live.

Lemma 3.2. *For every $\beta \in L^\vee/L$, $m \in Q(\beta) + \mathbb{Z}$, we have an equality of Cartier divisors:*

$$\eta^* \mathcal{Z}(\beta, m) = \bigsqcup_{\substack{\gamma \in L^\vee/rL \\ \overline{\gamma} = \beta}} \mathcal{Z}(\gamma, m) \ .$$

Definition 3.3. *Let $\widetilde{\mathcal{M}}_r$ be the normalization of $\mathcal{M}$ in $\mathcal{M}_r$. This **a** normal flat integral model over $\mathbb{Z}$ of M_r extending $\mathcal{M}_r \to \mathrm{Spec}(\mathbb{Z}[\Omega^{-1}])$.*

It follows from the definition that we have the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Spec}(\mathbb{Z}) & \longleftarrow & \widetilde{\mathcal{M}}_r & \longrightarrow & \mathcal{M}_r \\
\| & & \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\
\mathrm{Spec}(\mathbb{Z}) & \longleftarrow & \mathcal{M} & \longrightarrow & \mathcal{M}_{\mathbb{Z}[\Omega^{-1}]} \ .
\end{array}
$$

The Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{M}$ pulls back to an abelian scheme $\eta^* \mathcal{A}$ on $\widetilde{\mathcal{M}}_r$, and the line bundle $\boldsymbol{\omega}$ pulls-back to a line bundle $\eta^* \boldsymbol{\omega}$ on $\widetilde{\mathcal{M}}_r$ which extends $\boldsymbol{\omega}_r$. By abuse of notations, we still denote $\boldsymbol{\omega}_r$ this extension.

Our goal in the next section is to extend the Cartier divisors $\mathcal{Z}(\beta, m) \to \mathcal{M}_r$ to $\widetilde{\mathcal{M}}_r$ such that the extension has good moduli interpretation and Lemma 3.2 still holds. We will work at each prime in Ω then glue the constructions.

3.2.1. *Almost self-dual case.* Let p be a prime number dividing r, hence $p \in \Omega$. We make the additional assumption that the lattice L is **almost self-dual at** p as this will be satisfied in our applications. Then the level K_2 at p is hyperspecial and the Shimura variety $\mathcal{M}_{(p)}$ is the smooth canonical model over $\mathbb{Z}_{(p)}$ constructed in [Kis10, MP16, KMP16]. Let $\pi : \mathcal{A} \to \mathcal{M}$ be the Kuga-Satake abelian scheme. For $\ell \neq p$, we have an inclusion of étale sheaf of $\mathbb{Z}_\ell$-modules

$$\mathbb{V}_\ell \subset \mathrm{End}_{\mathbb{Z}_\ell}(\mathbb{H}_\ell)$$

where

$$\mathbb{H}_\ell = H^1_{\text{ét}}(\mathcal{A}/\mathcal{M}_{(p)}, \mathbb{Z}_\ell) \ .$$

We also have an inclusion of filtered vector bundles with integrable connections:

$$\mathbb{V}_{dR} \subset \mathrm{End}(\mathbb{H}_{dR}) \ ,$$

where

$$\mathbb{H}_{dR} = H^1_{dR}(\mathcal{A}/\mathcal{M}_{(p)}) \ ,$$

and a crystal of modules over the formal completion of $\mathcal{M}_{(p)}$ along the special fiber:

$$\mathbb{V}_{\mathrm{crys}} \subset \mathrm{End}(\mathbb{H}_{\mathrm{crys}}),$$

where $\mathbb{H}_{\mathrm{crys}} = R^1 \pi_* \mathcal{O}_{\mathcal{A}_{2,\mathbb{F}_p}}^{\mathrm{crys}}$. Moreover, the formal completion of the de Rham vector bundle $\mathbb{V}_{dR}$ with its integrable connection is isomorphic to $\mathbb{V}_{\mathrm{crys}}$.

We recall now the construction of special divisors in $\mathcal{M}_{(p)}$. For any scheme S over $\mathbb{Z}_{(p)}$, the module of special quasi-endomorphisms $V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}}$ is by definition the set of quasi-endomorphisms $x \in \mathrm{End}_{\mathbb{Z}_{(p)}}(\mathcal{A}_S)$ such that:

- the de Rham realization x_{dR} lies in $\mathbb{V}_{dR|_S}$ and
- the ℓ-adic realization x_ℓ lies in $\mathbb{V}_{\ell|_S} \otimes \mathbb{Q}_\ell$ and
- the p-adic realization x_p over the generic fiber $S_\mathbb{Q}$ lies in $\mathbb{V}_{p|_{S_\mathbb{Q}}} \otimes \mathbb{Q}_p$, and
- its crystalline realization x_{crys} lies in $\mathbb{V}_{crys|_{S_{\mathbb{F}_p}}}$.

Let $\beta \in L^\vee/L$ and let $\beta_\ell \in L^\vee/L \otimes \mathbb{Z}_\ell$ be its ℓ-adic component for every prime ℓ. For $\ell \neq p$, the local system $\mathbb{V}_{\mathbb{Z}_\ell}^\vee/\mathbb{V}_{\mathbb{Z}_\ell}$ is trivial on $\mathcal{M}_{(p)}$ and isomorphic to $\underline{L^\vee/L \otimes \mathbb{Z}_\ell}$. Thus we have a well defined subsheaf

$$\beta_\ell + \mathbb{V}_\ell \subseteq \mathbb{V}_\ell^\vee .$$

We define:

$$V_\beta(\mathcal{A}_S) = \{x \in V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}}, \forall \ell \neq p, x_\ell \in \beta_\ell + \mathbb{V}_{\ell|_S}, x_p \in \beta_p + \mathbb{V}_{p|_{S_\mathbb{Q}}}\} .$$

Via the morphism

$$\eta : \widetilde{\mathcal{M}}_{r,(p)} \to \mathcal{M}_{(p)} ,$$

all the above data pulls-back to $\widetilde{\mathcal{M}}_{r,(p)}$: we have hence ℓ-adic sheaves $\eta^* \mathbb{V}_\ell$, a de Rham vector bundle $\eta^* \mathbb{V}_{dR}$ and a crystal $\eta^* \mathbb{V}_{\mathrm{crys}}$.

For any $\mathbb{Z}_{(p)}$-scheme $S \to \widetilde{\mathcal{M}}_{r,(p)}$, we define the group of special quasi-endomorphisms:

$$V(\eta^* \mathcal{A}_S)_{\mathbb{Z}_{(p)}} \subset \mathrm{End}(\eta^* \mathcal{A}_S)_{\mathbb{Z}_{(p)}}$$

as the quasi-endomorphisms $f \in \mathrm{End}(\eta^* \mathcal{A}_S)_{\mathbb{Z}_{(p)}}$ whose étale, de Rham and crystalline realizations lies in the subsheaves $\eta^* \mathbb{V}_{\ell|_S} \otimes \mathbb{Q}_\ell$, $\eta^* \mathbb{V}_{dR|_S}$, $\eta^* \mathbb{V}_{crys|_S}$. This is simply the pull-back of $V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}}$.

For $\ell \neq p$, notice that the étale local system $\frac{1}{r} \eta^* \mathbb{V}_\ell^\vee / r \cdot \eta^* \mathbb{V}_\ell$ is trivial on $\widetilde{\mathcal{M}}_{r,(p)}$ and isomorphic to $\frac{1}{r} L^\vee / rL \otimes \mathbb{Z}_\ell$. Hence, given $\beta \in L/rL$ and β_ℓ its ℓ-adic component, we have a well defined subsheaf

$$\beta_\ell + r \cdot \eta^* \mathbb{V}_\ell .$$

We define then

$$V_{\beta_\ell}(\eta^* \mathcal{A}_S) = \{x \in V(\eta^* \mathcal{A}_S)_{\mathbb{Z}_{(p)}} | x_\ell \in \beta_\ell + r \cdot \eta^* \mathbb{V}_\ell\},$$

and

$$V_{\beta_p}(\eta^* \mathcal{A}_S) = \{x \in V(\eta^* \mathcal{A}_S)_{\mathbb{Z}_{(p)}} | x_p \in \beta_p + r \cdot \eta^* \mathbb{V}_{p,S_\mathbb{Q}}, \text{ and } x_{\mathrm{crys}} \in \eta^* \mathbb{V}_{crys,S_{\mathbb{F}_p}}\}.$$

Finally, we define

$$V_\beta(\eta^* \mathcal{A}_S) = \cap_\ell V_{\beta_\ell}(\eta^* \mathcal{A}_S) \cap V_{\beta_p}(\eta^* \mathcal{A}_S) .$$

We define now a functor on $\mathbb{Z}_{(p)}$-schemes as follows:

$$S \mapsto \mathcal{Z}(\beta, m)(S) = \{f \in V_\beta(\eta^* \mathcal{A}_S), f \circ f = m \cdot \mathrm{Id}_{\eta^* \mathcal{A}_S}\} .$$

Proposition 3.4. *Let $\beta \in L/rL$. Then the above functor is representable by a finite unramified $\widetilde{\mathcal{M}}_{r,(p)}$-stack which coincides over $\mathbb{Q}$ with $\mathcal{Z}(\beta, m)_\mathbb{Q}$.*

Proof. The proof is similar to [AGHMP17, Proposition 2.7.2]. $\square$

In particular, it results from the previous proposition that the divisors $\mathcal{Z}(\beta, m)$ glues as a finite unramified $\widetilde{\mathcal{M}}_r$-stack over $\mathbb{Z}$ and étale locally it is a Cartier divisor on $\widetilde{\mathcal{M}}_r$. Moreover, we have an equality of Cartier divisors:

$$\eta^* \mathcal{Z}(m) = \cup_{\beta \in L/rL} \mathcal{Z}(\beta, m), \tag{3.2.1}$$

valid over $\mathbb{Z}$, and which extends Lemma 3.2.

Proposition 3.5. *Let $\beta \in L/rL$, $m \in Q(\beta) + r\mathbb{Z}$. Then the Cartier divisor $\mathcal{Z}(\beta, m) \to \widetilde{\mathcal{M}}_r$ is flat over $\mathbb{Z}_{(p)}$.*

Proof. We have the relation $\eta^* \mathcal{Z}(m) = \cup_{\gamma \in L/rL} \mathcal{Z}(\gamma, m)$, $\mathcal{Z}(m)$ is flat over $\mathbb{Z}_{(p)}$ by the same argument as in [MP16, Prop 5.21], hence has no vertical components. Since η is a finite map, we conclude that none of the $\mathcal{Z}(\gamma, m)$ has vertical components and by the lemma below applied to the complete local ring at a point, they are flat over $\mathbb{Z}_{(p)}$. $\square$

Lemma 3.6. *Let R be a normal, local, flat $\mathbb{Z}_{(p)}$-algebra and let a be a non-zero divisor. Then all the associated primes of a have height 1. In particular, if $\mathrm{div}(a) \subset \mathrm{Spec}(R)$ has no vertical components of $\mathrm{Spec}(R \otimes \mathbb{F}_p)$, then $\mathrm{div}(a)$ is flat over $\mathbb{Z}_{(p)}$.*

Proof. This lemma is similar to [HP20, Lemma 7.2.4] when R is Cohen-Macaulay but since we only assume normality, we give a detailed proof. By Serre's normality criterion, for every ideal $\mathfrak{P}$ of height ≥ 2, $R_\mathfrak{P}$ has depth at least 2 and hence $\mathfrak{P}$ cannot be associated to a, as otherwise the depth of $R_\mathfrak{P}/aR_\mathfrak{P}$ would be 0, which is not possible as

$$\mathrm{depth}(R_\mathfrak{P}/aR_\mathfrak{P}) = \mathrm{depth}(R_\mathfrak{P}) - 1 \geq 1 ,$$

by [Sta23, Lemma 10.72.7.]. For the second part, to prove that $\mathrm{div}(a)$ is flat, it is enough to prove that it has no p-torsion. By assumption, a is not contained in any minimal prime over p, which are the same as the associated primes by the above. Hence a is not a zero divisor in R/pR, which is equivalent to p not being a zero divisor in R/aR, since R is local and normal. $\square$

3.3. **Arithmetic Chow groups.** We introduce in this section Arakelov Chow groups following [GS90] and [BBK07]. For more details on this section, we also refer to [SSTT22, §3.1] and [Tay22, §3].

Let $(rL, Q) \subset (L, Q)$ be an inclusion of quadratic lattices of signature $(n, 2)$ as before, in particular L is maximal with discriminant coprime to r. Let $\widetilde{\mathcal{M}}_r$, $\mathcal{M}$ be the normal integral models over $\mathbb{Z}$ of the GSpin Shimura varieties associated to (rL, Q) and (L, Q) constructed in the previous section.

Let Σ be a rational polyhedral K_r-admissible cone decomposition. By the main theorem of [Per19, Theorem 1], $\widetilde{\mathcal{M}}_r$ has a toroidal compactifications $\widetilde{\mathcal{M}}_r^\Sigma$ which is proper, normal and flat over $\mathbb{Z}$. Over $\mathbb{C}$, it is compatible with the toroidal compactification of its complex fiber as constructed in [AMRT10, Chapter III]. Let $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{M}}_r^\Sigma, \mathcal{D}_{pre})_\mathbb{Q}$ be the first arithmetic Chow group of prelog forms as defined in [BBK07, Definition 1.15].

For any toroidal stratum representative (Ξ, σ) of type III where σ is a ray, let $\mathcal{B}^{\Xi,\sigma}$ be the corresponding boundary divisor of $\widetilde{\mathcal{M}}_r^\Sigma$ and for Υ a toroidal stratum representative of type II, let $\mathcal{B}^\Upsilon$ be the corresponding boundary divisor of type II. Then by [Per19, Theorem 1], both $\mathcal{B}^{\Xi,\sigma}$ and $\mathcal{B}^\Upsilon$ are relative Cartier divisors over $\mathbb{Z}$, hence flat over $\mathbb{Z}$.

Let $\beta \in L/rL$ and $m \in \mathbb{Z}$. We have defined in the previous section a special divisor

$$\mathcal{Z}(\beta, m) \to \widetilde{\mathcal{M}}_r^\Sigma,$$

and following [BZ21], see also [EGT23, Theorem 1.2], we define a corrected divisor in $\widetilde{\mathcal{M}}_r^\Sigma$:

$$\mathcal{Z}^{tor}(\beta, m) = \mathcal{Z}(\beta, m) + \sum_\Upsilon \mu_\Upsilon(\beta, m)\mathcal{B}^\Upsilon + \sum_{(\Xi,\sigma)} \mu_{\Xi,\sigma}(\beta, m)\mathcal{B}^{\Xi,\sigma},$$
(3.3.1)

where the coefficients $\mu_\Upsilon(\beta, m)$ and $\mu_{\Xi,\sigma}(\beta, m)$ are defined in [Tay22, Eqs (4.5.1), (4.6.1)].

Following [Bru02, BZ21], the divisors $\mathcal{Z}^{tor}(\beta, m)$ can be endowed with a Green function $\Phi_{\beta,m}$ such that the pair:

$$\widehat{\mathcal{Z}}(\beta, m) = (\mathcal{Z}^{tor}(\beta, m), \Phi_{\beta,m})$$

is an element of $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{M}}_r^\Sigma, \mathcal{D}_{pre})_\mathbb{Q}$.

The Hodge line bundle $\boldsymbol{\omega}_r$ has a canonical Hermitian metric with prelog singularities, the *Petersson metric*, see [HP20, Equation (4.2.3)] for a definition. Hence it defines an element

$$\widehat{\boldsymbol{\omega}}_r \in \widehat{\mathrm{CH}}^1(\widetilde{\mathcal{M}}_r^\Sigma, \mathcal{D}_{pre}).$$

3.3.1. *Arithmetic height and main estimates.* Let K be a number field and let
$$\rho : \mathscr{S} = \mathrm{Spec}(\mathcal{O}_K) \to \widetilde{\mathcal{M}}_r^\Sigma$$
be an $\mathcal{O}_K$-point. Then the height $h_{\widehat{\mathcal{Z}}(\beta,m)}(\mathscr{S})$ of $\mathscr{S}$ with respect to $\widehat{\mathcal{Z}}(\beta,m)$ is defined as the image of $\widehat{\mathcal{Z}}(\beta,m)$ under the composition:

$$\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{M}}_r^\Sigma, \mathcal{D}_{pre}) \xrightarrow{\rho^*} \widehat{\mathrm{CH}}^1(\mathscr{S}) \xrightarrow{\widehat{\deg}} \mathbb{R} .$$

It is given by, see [SSTT22, Equation (3.1)]:

$$h_{\widehat{\mathcal{Z}}(\beta,m)}(\mathscr{S}) = \sum_{\mathfrak{P} \subset \mathcal{O}_K} (\mathcal{Z}^{tor}(\beta,m).\mathscr{S})_\mathfrak{P} \log|\mathcal{O}_K/\mathfrak{P}| + \sum_{x \in \mathscr{S}(\mathbb{C})} \Phi_{\beta,m}(x) ,$$

where for a prime $\mathfrak{P} \subset \mathcal{O}_K$:

$$(\mathcal{Z}^{tor}(\beta,m).\mathscr{S})_\mathfrak{P} = \sum_{v \in \left(\mathscr{S} \times_{\widetilde{\mathcal{M}}_r^\Sigma} \mathcal{Z}^{tor}(\beta,m)\right)(\overline{\mathbb{F}}_\mathfrak{P})} \mathrm{length}\left(\mathcal{O}_{\mathcal{Z}^{tor}(\beta,m) \times_{\widetilde{\mathcal{M}}_r^\Sigma} \mathscr{S}, v}\right) ,$$

and $\mathbb{F}_\mathfrak{P}$ is the residual field of $\mathfrak{P}$.

3.4. **Main estimates.** Let $\beta \in L/rL$ and let $m \in \mathbb{Z}$ represented by $\beta + rL$.

Let $c(\beta,m)$ be the (β,m)-th Fourier coefficient of Eisentein series E_{rL} as in [BK03, Prop. 3.1, (3.3)], see also [SSTT22, §3.3]. For $n \geq 3$, we have $|c(\beta,m)| = -c(\beta,m) \ll_\epsilon m^{\frac{n}{2}}$ along the integers m representable by $\beta + rL$.

Our first main result is the following global height bound.

Proposition 3.7. *As $m \to \infty$ and represented by $\beta + rL$, we have*
$$\sum_{\mathfrak{P} \subset \mathcal{O}_K} (\mathcal{Z}(\beta,m).\mathscr{S})_\mathfrak{P} \log|\mathcal{O}_K/\mathfrak{P}| + \sum_{x \in \mathscr{S}(\mathbb{C})} \Phi_{\beta,m}(x) = O(c(\beta,m)) .$$

Recall from [SSTT22, §6] that for a subset $S \subset \mathbb{N}$, the logarithmic asymptotic density is defined as:
$$\limsup_{X \to \infty} \frac{|\{s \in S, X \leq s < 2X\}|}{\log X} .$$

The second main results are estimates in average of multiplicities at archimedean and non-archimedean places.

Proposition 3.8. *For every $x \in \mathscr{S}(\mathbb{C})$, there is a decomposition:*
$$\Phi_{\beta,m}(x) = c(\beta,m)\log(m) + A(\beta,m) + o\left(c(\beta,m)\log(m)\right) .$$

Moreover, there exists a subset $S_{\mathrm{bad}} \subset \mathbb{Z}_{>0}$ of logarithmic asymptotic density 0 such that
$$\lim_{\substack{m \to \infty \\ m \notin S_{\mathrm{bad}}}} \frac{A(\beta,m)}{m^{\frac{b}{2}}\log m} = 0.$$

Next, we have the estimate at the non-archimedean places. Let N be the product of primes where ρ intersects the boundary of $\widetilde{\mathcal{M}}_r^\Sigma$.

Proposition 3.9. *Given $D, X \in \mathbb{Z}_{>0}$, D coprime to N, let $S_{D,X}$ denote the set*

$$\{m \in \mathbb{Z}_{>0} \mid X \le m < 2X, \sqrt{\frac{m}{D}} \in \mathbb{Z}, (m, N) = 1\}.$$

For a fixed prime $\mathfrak{P}$ and a fixed D, we have

$$\sum_{m \in S_{D,X}} (\mathscr{S}.\mathscr{Z}(\beta, m))_{\mathfrak{P}} = o(X^{\frac{b+1}{2}} \log X).$$

3.5. **Proof of Theorem 2.5.** Assuming Propositions 3.7 to 3.9 from the previous section, we prove here Theorem 2.5 which proves Theorem 2.2 and hence Theorem 1.1.

Let K a number field and fix an embedding $\sigma : K \hookrightarrow \mathbb{C}$. Let X be a K3 surface over K, and let $(T(X_\sigma(\mathbb{C})), Q)$ be the transcendental lattice of $X_\sigma(\mathbb{C})$. Let $\alpha \in \mathrm{Br}(X)$ be a Brauer class of geometric torsion order r, assumed to be coprime to the discriminant of $T(X_\sigma(\mathbb{C}))$. Let $\beta \in T(X_\sigma(\mathbb{C}))/rT(X_\sigma(\mathbb{C}))$ be a lift (multiplied by r) given by Proposition 2.1 and let A be the Kuga-Satake abelian variety associated to X.

Let $T(X_\sigma(\mathbb{C})) \subset L$ be a maximal lattice containing $T(X_\sigma(\mathbb{C}))$. Then we can see $\beta \in L/rL$. We are now in the setup of the previous sections: let $\widetilde{\mathcal{M}}_r^\Sigma$ be the toroidal compactification of the integral model of the Shimura variety associated to (rL, Q) constructed in the previous sections. The Kuga-Satake abelian variety defines a K-point in $\widetilde{\mathcal{M}}_r^\Sigma$ which extends to a morphism:

$$\rho : \mathscr{S} \to \widetilde{\mathcal{M}}_r^\Sigma,$$

where $\mathscr{S} = \mathrm{Spec}(\mathcal{O}_K)$.

Assume by contradiction that the conclusion of Theorem 2.5 does not hold. Then there exists finitely many primes $\mathfrak{P}_1, \ldots, \mathfrak{P}_r$ such that for every $m \in \mathbb{Z}$, the support of the intersection of $\mathscr{S}$ and $\mathscr{Z}(\beta, m)$ is contained in $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_r\}$.

By Proposition 3.8, there exists a subset $S_{\mathrm{bad}} \subset \mathbb{Z}_{>0}$ of logarithmic asymptotic density zero such that outside S_{bad} we have:

$$\sum_{x \in \mathscr{S}(\mathbb{C})} \Phi_{\beta,m}(x) \asymp c(\beta, m) \log(m) + o(c(\beta, m) \log(m)) \asymp -|c(\beta, m)| \log(m).$$

Let $S_{D,X}^{good} = \{m \in S_{D,X}, m \notin S_{bad}, (m, N) = 1\}$, then one can easily check that $|S_{D,X}^{good}| \asymp X^{\frac{1}{2}}$. By choosing m representable by $\beta + rL$ which is guaranteed by Lemma 5.3, we have $|c(\beta, m)| \gg X^{\frac{n}{2}}$ for $m \in S_{D,X}^{good}$. Hence we get

$$\sum_{m \in S_{D,X}^{good}} \sum_{x \in \mathscr{S}(\mathbb{C})} \Phi_{\beta,m}(x) \gg X^{\frac{n+1}{2}} \log X. \qquad (3.5.1)$$

On the other hand, by Proposition 3.9, we get by summing over the finitely many places where either $\mathscr{S}$ intersects a $\mathcal{Z}(\beta, m)$ or which are of bad reduction:

$$\sum_{m \in S_{D,X}^{\text{good}}} (\mathscr{S}.\mathcal{Z}(\beta, m))_{\mathfrak{P}} \log |\mathcal{O}_K/\mathfrak{P}| = o(X^{\frac{n+1}{2}} \log X). \qquad (3.5.2)$$

The combination of Equation (3.5.1) and Equation (3.5.2) contradicts Proposition 3.7. This proves the desired result.

The rest of the paper is devoted to proving the main estimates in Section 3.4.

4. GLOBAL ESTIMATE

We prove in this section Proposition 3.7. Our method is inspired from [HP20] and relies on Fourier-Jacobi expansions of Borcherds products at the cusps of GSpin Shimura varieties.

4.1. **Background results.** Let (L, Q) be a quadratic lattice of signature $(n, 2)$ and let $\mathcal{M}$ be the normal integral model over $\mathbb{Z}[\Omega^{-1}]$ of the GSpin Shimura variety associated to (L, Q) constructed in [HP20]. Let $f \in M_{1-\frac{n}{2}}^!(\rho_L)$ be a weakly holomorphic modular form of weight $1 - \frac{n}{2}$ with respect to the conjugate Weil representation $\overline{\rho}_L$. We assume that the principal part of f has integral coefficients and denote it by:

$$\sum_{\beta \in L^\vee/L} \sum_{\substack{m \in -Q(\beta)+\mathbb{Z} \\ m < 0}} c(\beta, m) q^m, \; c(\beta, m) \in \mathbb{Z} \; .$$

By the main theorem of [HP20, Theorem A], there exists a Borcherds products $\psi(f)$ associated to f which defines, after multiplying f by a suitable integer, a rational section of $\boldsymbol{\omega}^{\frac{c(0,0)}{2}}$ over $\mathbb{Q}$ and its divisor in $\mathcal{M}$ is equal to:

$$\text{div}(\psi(f)) = \sum_{(\beta, m)} c(\beta, -m) \mathcal{Z}(\beta, m).$$

Let Σ an admissible polyhedral cone decomposition and let $\mathcal{Z}^{tor}(\beta, m)$ be the completed divisor as defined in Equation (3.3.1). Then by [Tay22, Proof of Thm 3.1], the divisor of the Borcherds products on $\mathcal{M}^\Sigma$ is equal to:

$$\text{div}(\psi(f)) = \sum_{(\beta, m)} c(\beta, -m) \mathcal{Z}^{tor}(\beta, m) \; .$$

In fact, the above relation can be upgraded into an equality by [HP20, Equation (1.2.2)] in $\widehat{\text{CH}}^1(\mathcal{M}^\Sigma, \mathcal{D}_{pre})_\mathbb{Q}$:

$$\widehat{\text{div}}(\psi(f)) = \sum_{(\beta, m)} c(\beta, -m) \widehat{\mathcal{Z}}^{tor}(\beta, m) \; .$$

On the other hand,

$$\widehat{\mathrm{div}}(\psi(f)) = \frac{c(0,0)}{2}\widehat{\boldsymbol{\omega}} \ .$$

Hence we get the following equality:

$$\frac{c(0,0)}{2}\widehat{\boldsymbol{\omega}} = \sum_{(\beta,m)} c(\beta,m)\widehat{\mathcal{Z}}^{tor}(\beta,m) \ .$$

4.2. **Expansions at the cusp.** We assume now that we are given two quadratic lattices $(rL, Q) \subset (L, Q)$ where L is a maximal lattice, almost self-dual at r. Let Σ be a K_r-admissible polyhedral cone decomposition and let $\widetilde{\mathcal{M}}_r^\Sigma$ be the toroidal compactification of the integral model of the GSpin Shimura variety associated to (rL, Q) constructed in Section 3.2. It is normal, proper and flat over $\mathbb{Z}$.

We recall in this section the theory of integral q-expansions at the cusps following [HP20, §§5,8]. We assume that (rL, Q) is isotropic and let (Ξ, σ) be a toroidal stratum representative such that σ is top dimensional.

We will first describe the Fourier-Jacobi expansion over $\mathbb{C}$ and then over $\mathbb{Z}_{(p)}$ where p is a prime number. Associated to the cusp label representative Ξ, there is an admissible parabolic subgroup $P_\Xi \subset G$, a connected component $\mathcal{D}^\circ$ of $\mathcal{D}$ and an element $h \in G(\mathbb{A}_f)$. Such cusp label representative determines a mixed Shimura datum $(Q_\Xi, \mathcal{D}_\Xi)$, see [HP20, §4.4]. The unipotent radical W_Ξ and its center U_Ξ are both equal and are described at the level of $\mathbb{Q}$-points by:

$$U_\Xi(\mathbb{Q}) \simeq K_\mathbb{Q} \otimes I_\mathbb{Q}$$

where $I_\mathbb{Q}$ is the $\mathbb{Q}$-isotropic line determined by the cusp label representative Ξ, $I = rL \cap I_\mathbb{Q}$, and $K = I^\perp/I$. Define the $\mathbb{Z}$-lattice $\Gamma_\Xi = K_\Xi \cap U_\Xi(\mathbb{Q})$ and the torus:

$$T_\Xi = \Gamma_\Xi(-1) \otimes \mathbb{G}_m.$$

The level K_r determines a mixed Shimura variety M_Ξ associated to the mixed Shimura datum $(Q_\Xi, \mathcal{D}_\Xi)$. Let $K_{\Xi 0}$ be the compact open subgroup of $Q_\Xi(\mathbb{A}_f)$ determined as in the end of page 220 of [HP20]. It defines another mixed Shimura variety $M_{\Xi 0}$ over $\mathbb{Q}$ associated to the same datum $(Q_\Xi, \mathcal{D}_\Xi)$ and an étale morphism of Deligne–Mumford stacks $M_{\Xi 0} \to M_\Xi$.

The toroidal stratum representative (Ξ, σ) determines partial compactifications $M_\Xi(\sigma)$, $M_{\Xi 0}(\sigma)$ and 0-dimensional boundary component $Z^{\Xi, \sigma}$ of M_r^Σ, $M_\Xi(\sigma)$, and $M_{\Xi 0}(\sigma)$. We denote by $\widehat{M_r^\Sigma}$, resp. $\widehat{M_\Xi}(\sigma)$, $\widehat{M_{\Xi 0}}(\sigma)$, the formal completion of M_r^Σ, resp. $M_\Xi(\sigma)$, $M_{\Xi 0}(\sigma)$ along $Z^{\Xi, \sigma}$. By a theorem of Pink [Pin90, Corollary 7.17, Theorem 12.4], see also [HP20, §2.6] which is our reference, we have an isomorphism

$$\widehat{M_r^\Sigma} \simeq \widehat{M_\Xi}(\sigma) \ .$$

Then by [HP20, Proposition 4.6.2], there exists $K_0 \subset \mathbb{A}_f^\times$ compact open subgroup such that we have the following commutative diagram of formal Deligne–Mumford stacks over $\mathbb{C}$:

$$
\begin{array}{ccc}
\bigsqcup_{a \in \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Xi(\sigma)_{/\mathbb{C}} & \xrightarrow{\simeq} & \widehat{M}_{\Xi,0}(\sigma)/\mathbb{C} \\
\downarrow & & \downarrow \\
\widehat{M}^\Sigma_{r/\mathbb{C}} & \xrightarrow{\simeq} & \widehat{M}_\Xi(\sigma)/\mathbb{C},
\end{array}
\qquad (4.2.1)
$$

such that the vertical arrows are formally étale surjections and

$$
\widehat{T}_\Xi(\sigma) = \mathrm{Spf}\left(\mathbb{Q}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Xi^\vee(1) \\ (\alpha \cdot \sigma) \geq 0}} \right) .
$$

Now given a section ψ of $\boldsymbol{\omega}_r^{\otimes k}$, we get by [HP20, Equation (4.6.10)] a trivialization, *the Fourier-Jacobi expansion* on each copy of $\widehat{T}_\Xi(\sigma)_{/\mathbb{C}}$ indexed by $a \in \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K_0$:

$$
\mathrm{FJ}^{(a)}(\psi) = \sum_{\substack{\alpha \in \Gamma_\Xi^\vee(1) \\ (\alpha \cdot \sigma) \geq 0}} \mathrm{FJ}_\alpha^{(a)}(\psi) \cdot q_\alpha \in \mathbb{C}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Xi^\vee(1) \\ (\alpha \cdot \sigma) \geq 0}} .
$$

Let f be a weakly holomorphic modular from of weight $1 - \frac{n}{2}$ with respect to $\overline{\rho}_{rL}$ and with integral principal part. Let $\psi(f)$ be the associated Howard-Madapusi-Borcherds product, which is a rational section of $\boldsymbol{\omega}_r^{\frac{c(0,0)}{2}}$. Let F be the abelian extension of $\mathbb{Q}$ determined by the reciprocity isomorphism in class field theory:

$$
\mathrm{rec} : \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K_0 \simeq \mathrm{Gal}(F/\mathbb{Q}) .
$$

By [HP20, Proposition 5.4.2], for every $a \in \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K_0$, the Borcherds product $\psi(f)$ has a Fourier-Jacobi expansion given as follows:

$$
\mathrm{FJ}^{(a)}(\psi(f)) = \kappa^{(a)} A^{\mathrm{rec}(a)} q_{\alpha(\rho)} \cdot \mathrm{BP}(f)^{\mathrm{rec}(a)} , \qquad (4.2.2)
$$

where $\kappa^{(a)} \in \mathbb{C}$ is a constant of absolute value 1, and

$$
\mathrm{BP}(f) \in \mathcal{O}_F[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Xi^\vee(1) \\ (\alpha \cdot \sigma) \geq 0}} ,
$$

is the infinite product:

$$
\mathrm{BP}(f) = \prod_{\substack{\lambda \in (I^\perp / I)^\vee \\ (\lambda \cdot \mathscr{W}) > 0}} \prod_{\mu \in hL^\vee / hL} \left(1 - \zeta_\mu \cdot q_{\alpha(\lambda)} \right)^{c(h^{-1}\mu, -Q(\lambda))} .
$$

In the product above, $\mathscr{W}$ is a Weyl chamber as defined in [HP20, Equation (5.3.1)] such the interior of the cone σ is isomorphic to an open subset of $\mathscr{W}$. The number ζ_μ is a root of unity of order dividing $|\frac{1}{r}L^\vee / rL|$.

Finally, the constant A is given as follows, see [HP20, Equation (5.3.6)]: let $I_{\mathbb{Q}}$ be be as before the isotropic line corresponding to the cusp label representative $\Xi = (\mathcal{D}, P, h)$ and let ℓ be a generator of $I \cap h \cdot rL$. Let N be the order ℓ in $(h \cdot rL_{\mathbb{Z}})^{\vee}/(h \cdot rL_{\mathbb{Z}})$. Then

$$A = \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{\frac{2\pi i x}{N}}\right)^{c\left(\frac{xh^{-1}\ell}{N}, 0\right)} .$$

Remark 4.1. *There is a $(2i\pi)^{\frac{c(0,0)}{2}}$ factor in [HP20, Proposition 5.4.2] that disappeared from Equation (4.2.2) and the reason is that Howard-Madapusi already rescaled the original Borcherds constructed by Borcherds in [Bor98], denoted by $\Psi(f)$ in loc. cit., by the factor $(2i\pi)^{\frac{c(0,0)}{2}}$ to obtain $\psi(f)$.*

Lemma 4.2. *For very $a \in \mathbb{Q}_{>0}^{\times}\backslash\mathbb{A}_f^{\times}/K_0$, the constant $\kappa^{(a)}$ is a root of unity.*

Proof. Since $\psi(f)$ is defined over $\mathbb{Q}$, the rationality principle of [HP20, Proposition 4.6.3] stipulates that for any $\tau \in \mathrm{Aut}(\mathbb{C})$:

$$\tau(\mathrm{FJ}_{\alpha}^{(a)}(\psi(f))) = \mathrm{FJ}_{\alpha}^{(aa_{\tau})}(\psi(f)) \ ,$$

where $a_{\tau} \in \mathbb{Q}_{>0}^{\times}\backslash\mathbb{A}_f^{\times}/K_0$ is the unique element with

$$\mathrm{rec}(a_{\tau}) = \tau_{|\mathbb{Q}^{ab}} \ .$$

By identifying the constant term, we get:

$$\tau(\kappa^{(a)}) = \kappa^{(aa_{\tau})} \ .$$

In particular, all the Galois conjugates of $\kappa^{(a)}$ have absolute value 1, hence it is a root of unity. $\square$

4.3. **Integral theory.** Let p be a prime number. We now extend the results from the previous section to $\mathbb{Z}_{(p)}$. We still assume that rL has an isotropic vector, which is always true if $n \geq 3$. Let

$$\widehat{\mathcal{T}}_{\Xi}(\sigma) = \mathrm{Spf}\left(\mathbb{Z}_{(p)}[[q_{\alpha}]]_{\substack{\alpha \in \Gamma_{\Xi}^{\vee}(1) \\ (\alpha \cdot \sigma) \geq 0}}\right) \ ,$$

and let R be the localization of $\mathcal{O}_F$ at a prime $\mathfrak{P} \subset \mathcal{O}_F$ above p.

Proposition 4.3. *There is a unique morphism*

$$\bigsqcup_{a \in \mathbb{Q}_{>0}^{\times}\backslash\mathbb{A}_f^{\times}/K_0} \widehat{\mathcal{T}}_{\Xi}(\sigma)_{/R} \to \widetilde{\widehat{\mathcal{M}}_r^{\Sigma}} \ .$$

of formal Deligne–Mumford stacks which agrees with Equation (4.2.1) by base change to $\mathbb{C}$, and such that for any s in the source with image t, the induced map on étale local rings is faithfully flat.

For primes p where rL is maximal, i.e., those primes who do not divide r, the result above is [HP20, Proposition 8.2.3]. The proof of Proposition 8.2.3 in [HP20] uses Proposition 8.1.1 as the main input. The latter relies on Theorem 4.1.5 from [Per19] and in fact the maximality assumption is not needed in the latter result.

Lemma 4.4. *Assume that $A = 1$. Then the divisor of the Borcherds product $\psi(f)$ in $\widetilde{\mathcal{M}}_{r,(p)}$ is flat over $\mathbb{Z}_{(p)}$.*

Proof. As $\widetilde{\mathcal{M}}_{r,(p)}$ is flat and normal over $\mathbb{Z}_{(p)}$, it is enough to show by Lemma 3.6 that the divisor of $\psi(f)$ in $\widetilde{\mathcal{M}}_{r,(p)}$ does not contain any irreducible component of the special fiber of $\widetilde{\mathcal{M}}_{r,\mathbb{F}_p}$. Since the Fourier-Jacobi expansion of $\psi(f)$ in Equation (4.2.2) is not zero modulo $\mathfrak{P}$, we deduce using the faithful flatness of Proposition 4.3, that the divisor of $\psi(f)$ in every irreducible component that meets the cusp is flat. Hence it is enough to prove that each irreducible component of the special fiber $\widetilde{\mathcal{M}}_{r,\mathbb{F}_p}$ meets the zero cusp of $\mathcal{M}_r^{\Sigma}$.

Recall that we have a finite morphism

$$\eta : \widetilde{\mathcal{M}}_{r,(p)} \to \mathcal{M}_{(p)}$$

and $\mathcal{M}_{(p)}$ is smooth over $\mathbb{Z}_{(p)}$, in particular, every irreducible component of $\mathcal{M}_{\mathbb{F}_p}$ is connected and meets the zero cusp. Since the morphism η is finite, every irreducible component of $\widetilde{\mathcal{M}}_{r,\mathbb{F}_p}$ maps surjectivity to an irreducible component of $\mathcal{M}_{\mathbb{F}_p}$. Hence every irreducible component of $\widetilde{\mathcal{M}}_{r,\mathbb{F}_p}$ meets the 0-cusp of $\widetilde{\mathcal{M}}_{r,\mathbb{F}_p}^{\Sigma}$, which concludes the proof. $\qquad\square$

4.4. **Construction of a flat Borcherds product.** We will construct in this section Borcherds products which satisfy the conditions of Lemma 4.4.

Let $\beta \in \frac{1}{r}L^{\vee}/rL$. For every $m \in Q(\beta) + \mathbb{Z}$, let $a_{\beta,m}$ be the linear form on the space of cusp forms $S_{1+\frac{n}{2}}(\rho_{rL})$ which maps a cusp from g to its $(\beta, m)^{th}$-Fourier coefficient. Then there exists a finite set of indices I such that a_{β,m_i} generates the $\mathbb{Q}$-vector space

$$\mathrm{Span}(a_{\beta,m} \,|\, m \in Q(\beta) + \mathbb{Z}) \subset S_{1+\frac{n}{2}}(\rho_{rL})^{*},$$

where $S_{1+\frac{n}{2}}(\rho_{\rho_{rL}})^{*}$ is the dual of the space of cusp forms.

Let $(g_i)_{i \in I}$ be a dual family [1] of cusp forms to the family $(a_{\beta,m_i})_{i \in I}$ and we can assume that the (g_i) have integral Fourier coefficients by [McG03]. Then for each m, there exists $c_i(\beta, m) \in \mathbb{Q}$ such that we can write

$$a(\beta, m) = \sum_{i \in I} c_i(\beta, m) a(\beta, m_i)$$

---

[1] We don't require $a_{\beta,m_i}(g_i) = 1$, but only $a_{\beta,m_i}(g_j) = 0$ for $j \neq i$.

and $g_i(\beta, m) = c_i(\beta, m)g_i(\beta, m_i)$. Standard estimates on growths of coefficients of cups forms show that:

$$c_i(\beta, m) = O(c(\beta, m)) ,$$

where $c(\beta, m)$ is the (β, m)-coefficient of the Eisenstein series introduced in the paragraph before Proposition 3.7.

By [Bru02, Theorem 1.17], there exists a weakly holomorphic modular form $\tilde{f}_m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{rL})$ such that its principal part is equal to

$$(v_\beta + v_{-\beta})q^{-m} - \sum_{i \in I} c_i(\beta, -m)q^{-m_i}(v_\beta + v_{-\beta}) .$$

Let $d = |\frac{1}{r}L^\vee/rL| - 1$ and let

$$C(m) = ((c(\gamma, 0))_{\substack{\gamma \in \frac{1}{r}L^\vee/rL \\ \gamma \neq 0}} \in \mathbb{Q}^d$$

be the vector of constant Fourier coefficients of $\tilde{f}_m$. Then the span of $(C(m))_{m \geq 1}$ is a finite dimensional vector space of $\mathbb{Q}^d$ which admits a basis given by $(C(m_j))_{j \in J}$ for some finite set J.

Finally for any m, there exists coefficients $u_j(m) \in \mathbb{Q}$ such that $u_j(m) = O(c(\beta, m))$ and

$$C(m) = \sum_{j \in J} u_j(m)C(m_j).$$

Define

$$f_m = \tilde{f}_m - \sum_{j \in J} u_j(m)\tilde{f}_{m_j} .$$

Then by construction, all the $(\gamma, 0)^{th}$-Fourier coefficients of f_m vanish, except possibly the $(0, 0)^{th}$-coefficient. Moreover, its principal part is equal to:

$$(v_\beta + v_\beta)q^{-m} - \sum_{i \in I} c_i(\beta, m)(v_\beta + v_{-\beta})q^{-m_i}$$

$$- \sum_{j \in J} u_j(m)\left((v_\beta + v_{-\beta})q^{-m_j} - \sum_{i \in I} c_i(\beta, m_j)(v_\beta + v_{-\beta})q^{-m_i}\right) .$$
$$\tag{4.4.1}$$

The latter can be rewritten as:

$$(v_\beta + v_{-\beta})q^{-m} + \sum_{\ell \in \tilde{I}} z_\ell(m)(v_\beta + v_{-\beta})q^{-m_\ell} ,$$

where $\tilde{I}$ is finite set independent of m, and $z_\ell(m)$ are rational numbers that satisfy:

$$z_\ell(m) = O(c(\beta, m)) .$$

Up to multiplying f_m by an integer, let $\psi(f)$ be the Borcherds product associated to f as in Section 4.1. Then $\psi(f_m)$ is a section of $\boldsymbol{\omega}_r^{\frac{c(0,0)}{2}}$ and we have the following relation in $\widehat{\mathrm{CH}}^1(\mathcal{M}_1^\Sigma, \mathcal{D}_{pre})_{\mathbb{Q}}$

$$\frac{c(0,0)}{2}\widehat{\boldsymbol{\omega}}_r = \widehat{\mathrm{div}}(\psi(f_m)) = \widehat{\mathcal{Z}}^{tor}(\beta, m) + \sum_{\ell \in \tilde{I}} z_\ell(m)\mathcal{Z}^{tor}(\beta, m_\ell). \quad (4.4.2)$$

Assume now that $\beta \in L/rL$ and we will extend the above relation above the primes $p \in \Omega$. Let $\widetilde{\mathcal{M}}_r^\Sigma$ be the proper, flat integral model over $\mathbb{Z}$ extending $\mathcal{M}_r^\Sigma$.

Assume also that rL has an isotropic vector. Then, by choice of the Borcherds product $\psi(f_m)$, the constant A from Equation (4.2.2) is equal to 1, hence by lemma Lemma 4.4, the divisor of the Borcherds $\psi(f_m)$ is flat over $\mathbb{Z}_{(p)}$. By Proposition 3.5, the special divisors $\mathcal{Z}(\beta, m)$ are flat over $\mathbb{Z}_{(p)}$ and the boundary divisors are flat over $\mathbb{Z}_{(p)}$ by [Per19, Theorem 1]. Hence Equation (4.4.2) holds over $\mathbb{Z}_{(p)}$ for all $p \in \Omega$. Hence we conclude.

Proposition 4.5. *Let $\beta \in L/rL$ and assume that rL is isotropic. Then we have in $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{M}}_r^\Sigma, \mathcal{D}_{pre})_{\mathbb{Q}}$:*

$$\frac{c(0,0)}{2}\widehat{\boldsymbol{\omega}}_r = \widehat{\mathrm{div}}(\psi(f_m)) = \widehat{\mathcal{Z}}^{tor}(\beta, m) + \sum_{\ell \in \tilde{I}} z_\ell(m)\mathcal{Z}^{tor}(\beta, m_\ell).$$

4.5. **Summary.** By the previous section, for every $m \in \mathbb{N}$, we have constructed a section $\psi(f_m)$ of $\boldsymbol{\omega}_r^{\frac{c(0,0)}{2}}$ that satisfies the following relation in $\widehat{\mathrm{CH}}^1(\widetilde{\mathcal{M}}_r^\Sigma, \mathcal{D}_{pre})_{\mathbb{Q}}$:

$$\widehat{\mathrm{div}}(\psi(f_m) = \widehat{\mathcal{Z}}^{tor}(\beta, m) + \sum_{\ell \in \tilde{I}} z_\ell(m)\widehat{\mathcal{Z}}^{tor}(\beta, m_\ell).$$

From which it follows that:

$$h_{\widehat{\mathcal{Z}}^{tor}(\beta,m)}(\mathscr{S}) = \frac{c(0,0)}{2}h_{\widehat{\boldsymbol{\omega}}_r}(\mathscr{S}) - \sum_{\ell \in \tilde{I}} z_\ell(m) \cdot h_{\widehat{\mathcal{Z}}^{tor}(\beta,m_i)}(\mathscr{S})$$

It follows that $h_{\widehat{\mathcal{Z}}(\beta,m)}(\mathscr{S}) = O(c(\beta,m))$. Now recall that:

$$\mathcal{Z}^{tor}(\beta, m) = \mathcal{Z}(\beta, m) + \sum_{\Upsilon} \mu_\Upsilon(\beta, m)\mathcal{B}^\Upsilon + \sum_{(\Xi,\sigma)} \mu_\Xi(\beta, m)\mathcal{B}^{\Xi,\Sigma}$$

and from [Tay22, Proposition 4.13], we have the following estimate as $m \to \infty$:

(1) For any type II cusp label representative Υ, we have

$$\mu_\Upsilon(m) \ll_\epsilon m^{\frac{b}{2}-1+\epsilon}.$$

(2) For any type III toroidal stratum representative (Ξ, σ) such that σ is a ray, we have

$$\mu_{\Xi,\sigma}(m) \ll_\epsilon m^{\frac{b-1}{2}+\epsilon}.$$

Hence combining the previous estimates, we get the desired global estimate of Proposition 3.7.

5. Archimedean estimates

Our goal in this section is to prove Proposition 3.8. We follow the approach explained in [SSTT22, §5].

5.1. **Development of the Green function.** The Green function $\Phi_{\beta,m}$ has an explicit expression due to Bruinier [Bru02, §2] and which we recall following [SSTT22, §5].

Let (L, Q) be a quadratic lattice of signature $(n, 2)$. Let $k = 1 + \frac{n}{2}$, $\beta \in L^\vee/L$ and $s > \frac{k}{2}$ a real number. Let:

$$F(s, z) = H\left(s - 1 + \frac{k}{2}, s + 1 - \frac{k}{2}, 2s; z\right),$$

where

$$H(a, b, c; z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is the Gauss hypergeometric function as in [AS64, Chapter 15], and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ for $a, b, c, z \in \mathbb{C}$ and $|z| < 1$.

For $x \in \mathcal{D}$, define:

$$\phi_{\beta,m}(x, s) = 2\frac{\Gamma(s - 1 + \frac{k}{2})}{\Gamma(2s)}$$

$$\sum_{Q(\lambda)=m, \lambda \in \beta+L} \left(\frac{m}{m - Q(\lambda_x)}\right)^{s-1+\frac{k}{2}} F\left(s, \frac{m}{m - Q(\lambda_x)}\right). \quad (5.1.1)$$

Then $\phi_{\beta,m}(x, s)$ admits a meromorphic continuation to the complex plane with a pole at $s = \frac{k}{2}$ with residue $-c(\beta, m)$. We define then:

$$\phi_{\beta,m}(x) = \lim_{s \to \frac{k}{2}} \left(\phi_{\beta,m}(x, s) + \frac{c(\beta, m)}{s - \frac{k}{2}}\right). \quad (5.1.2)$$

Let $s \mapsto C(\beta, m, s)$ be the holomorphic function for $\mathrm{Re}(s) > 1$ defined in [SSTT22, Equation (3.3)], see also [BK03, Equation (3.22)]. Then define:

$$b(\beta, m, s) = -\frac{C\left(\beta, m, s - \frac{k}{2}\right) \cdot \left(s - 1 + \frac{k}{2}\right)}{(2s - 1) \cdot \Gamma\left(s + 1 - \frac{k}{2}\right)}. \quad (5.1.3)$$

By [Bru02, Proposition 2.11], we can write for $x \in \mathcal{D}$:

$$\Phi_{\beta,m}(x) = \phi_{\beta,m)}(x) - b'(\beta, m, \frac{k}{2}).$$

5.2. **Estimate on $b'(\beta, m, \frac{k}{2})$.** We make the following assumptions in this section: L is maximal and $\beta \in L/rL$ has torsion coprime to the discriminant of L. Our goal in this section to prove the following theorem.

Theorem 5.1. *Let $D \geq 1$ be an integer. For $m \to \infty$ representable by $\beta + rL$ and such that $\sqrt{\frac{m}{D}} \in \mathbb{Z}$, we have:*

$$b'(\beta, m, \frac{k}{2}) = |c(\beta, m)| \log(m) + o\left(c(\beta, m) \log(m)\right).$$

Proof. The theorem above has been proved in [SSTT22, Proposition 5.2] under the assumption that L is maximal and $\beta = 0$. We recall the main steps here and make the appropriate modifications.

Taking logarithmic derivatives at $s = \frac{k}{2}$ in Equation (5.1.3) yields:

$$\frac{b'(\beta, m, \frac{k}{2})}{b(\beta, m, \frac{k}{2})} = \frac{C'(\beta, m, 0)}{C(\beta, m, 0)} - \frac{2}{b} - \Gamma'(1) .$$

Let

$$N_{\beta,m}(a) = |\{\lambda \in L/aL | \lambda = \beta \pmod{rL}, Q(\lambda) = m \pmod{a}\}|.$$

Let also d_β denote the order of β in $\frac{1}{r}L^\vee/rL$ and for a prime number p, let:

$$w_p = 1 + 2v_p(2md_\beta) .$$

Define the polynomial $\mathrm{L}_{\beta,m}^{(p)}(t)$:

$$\mathrm{L}_m^{(p)}(t) = N_{\beta,m}(p^{w_p})t^{w_p} + (1 - p^{r-1}t) \sum_{n=0}^{w_p-1} N_{\beta,m}(p^n)t^n \in \mathbb{Z}[t] .$$

For $s \in \mathbb{C}$, define the function $\sigma_{\beta,m}(s)$:

$$\sigma_{\beta,m}(s) = \begin{cases} \displaystyle\prod_{p \backslash 2d_\beta^2 m \det(L)} \frac{\mathrm{L}_{\beta,m}^{(p)}\left(p^{1-\frac{r}{2}-s}\right)}{1-\chi_{D_0}(p)p^{-s}}, & \text{if } r = 2+n \text{ is even,} \\ \displaystyle\prod_{p \backslash 2d_\beta^2 m \det(L)} \frac{1-\chi_{D_0}(p)p^{\frac{1}{2}-s}}{1-p^{1-2s}} \cdot \mathrm{L}_{\beta,m}^{(p)}\left(p^{1-\frac{r}{2}-s}\right), & \text{if } r \text{ is odd.} \end{cases}$$

$$(5.2.1)$$

Here, χ_{D_0} is the quadratic character associated to a fundamental discriminant D_0 of the number field $\mathbb{Q}(\sqrt{D})$ where D is defined by

$$(-1)^{\frac{r}{2}} \det(L), \text{ if } r \text{ is even.}$$

$$2(-1)^{\frac{r+1}{2}} d_\beta^2 m \det(L), \text{ otherwise.}$$

By our choice of m, the fundamental discriminant is independent of m, hence [BK03, Theorem 4.11, (4.73), (4.74)] implies:

$$\frac{C'(\beta, m, 0)}{C(\beta, m, 0)} = \log(m) + \frac{\sigma'_{\beta,m}(k)}{\sigma_{\beta,m}(k)} + O(1).$$

It suffices thus to show that $\frac{\sigma'_{\beta,m}(k)}{\sigma_{\beta,m}(k)} = o(\log(m))$. Taking the logarithmic derivative in (5.2.1) at $s = k$, we get for r even

$$\frac{\sigma'_{\beta,m}(k)}{\sigma_{\beta,m}(k)} = - \sum_{p \backslash 2d_\beta^2 m \det(L)} \left(\frac{p^{1-r} L_{\beta,m}^{(p)'}(p^{1-r})}{L_{\beta,m}^{(p)}(p^{1-r})} + \frac{\chi_{D_0}(p)}{p^k - \chi_{D_0}(p)} \right) \log(p) \,,$$

and for r odd

$$\frac{\sigma'_{\beta,m}(k)}{\sigma_{\beta,m}(k)} = - \sum_{p \backslash 2d_\beta^2 m \det(L)} \left(\frac{p^{1-r} L_{\beta,m}^{(p)'}(p^{1-r})}{L_{\beta,m}^{(p)}(p^{1-r})} - \frac{\chi_{D_0}(p)}{p^{k-\frac{1}{2}} - \chi_{D_0}(p)} + \frac{2}{p^{2k-1} - 1} \right) \log(p)$$

We have $L_{\beta,m}^{(p)}(p^{1-r}) = N_{\beta,m}(p^{w_p}) p^{(1-r)w_p}$ and

$$L_{\beta,m}^{(p)'}(p^{1-r}) = w_p N_{\beta,m}(p^{w_p}) p^{(1-r)(w_p-1)} - \sum_{n=0}^{w_p-1} N_{\beta,m}(p^n) p^{(n-1)(1-r)} \,.$$

Hence

$$\left| \frac{p^{1-r} L_{\beta,m}^{(p)'}(p^{1-r})}{L_{\beta,m}^{(p)}(p^{1-r})} \right| = \left| w_p - \sum_{v=0}^{w_p-1} \frac{N_{\beta,m}(p^v)}{N_{\beta,m}(p^{w_p})} p^{(v-w_p)(1-r)} \right|$$

$$= \left| w_p - \sum_{v=0}^{w_p-1} \frac{\mu_p(\beta, m, v)}{\mu_p(\beta, m, w_p)} \right| \,,$$

where $\mu_p(\beta, m, v) = p^{-n(r-1)} N_{\beta,m}(p^v)$. The proof of [SSTT22, Proposition 5.2] shows then it is enough to prove Lemma 5.2 below. Assuming this lemma, we get:

$$\left| \sum_{p \backslash 2d_\beta^2 m \det(L)} \frac{p^{1-r} L_{\beta,m}^{(p)'}(p^{1-r})}{L_{\beta,m}^{(p)}(p^{1-r})} \cdot \log(p) \right| \leq C \sum_{p \backslash 2d_\beta^2 m \det(L)} \frac{\log(p)}{p}$$

$$= O(\log\log(m))$$

$\square$

Lemma 5.2. *Let $\beta \in L/rL$ be a primitive element, $m \in \mathbb{Z}$ representable by $\beta + rL$, and p a prime number. Then there exists a constant C independent of m and p such that*

$$\left| \omega_p - \sum_{v=0}^{w_p-1} \frac{\mu_p(\beta, m, v)}{\mu_p(\beta, m, w_p)} \right| \leq \frac{C}{p}$$

Proof. For p coprime to r, the result follows from [SSTT22, Proposition 4.1]. Hence we can assume that p divides r. By assumption, L is unimodular at p and β is primitive r-torsion, hence every solution $x \in \beta + r(L/p^k L)$ is good in the sense of [Han04, Definition 3.1]. Let $\delta = 1 + v_p(r)$ Then using that $\beta \neq 0$ modulo p, we get for $v \geq \delta$:

$$N_{\beta,m}(p^v) = p^{(v-\delta)(r-1)} \cdot N_{\beta,m}(p^\delta) = p^{(v-\delta+1)(r-1)},$$

and $\mu_p(\beta, m, v) = \mu_p(\beta, m, \delta) = \frac{1}{p}$. As for $v < \delta$, we have $N_{\beta,m}(p^v) = 1$. Hence:

$$\left| \omega_p - \sum_{v=0}^{w_p-1} \frac{\mu_p(\beta, m, v)}{\mu_p(\beta, m, w_p)} \right| = \frac{1}{\mu_p(\beta, m, \delta)} \left| \sum_{v=0}^{\delta-1} \frac{1}{p} - \frac{1}{p^{v(r-1)}} \right| \ll \frac{1}{p} .$$

$\square$

Lemma 5.3. *Let L be a maximal lattice and let $r \geq 1$ coprime to the discriminant of L. Let $\beta \in L/rL$ be a primitive r torsion element. Then any integer m large enough such that $m = Q(\beta) \pmod{r}$, is representable by $\beta + rL$.*

Proof. There is no local obstruction to finding m and hence the argument of [SSTT22, Corollary 4.7] still applies. $\square$

5.3. **Proof of Proposition 3.8.** We assume in this section that L is maximal lattice, self-dual at primes dividing r. Let $\beta \in L/rL$, $m \in Q(\beta) + r\mathbb{Z}$ and let $\phi_{\beta,m}$ be the Green function defined by Equation (5.1.1).

For $x \in \mathcal{D}$, define

$$A(\beta, m, x) = -2 \sum_{\substack{\sqrt{m}\lambda \in \beta + rL \\ |Q(\lambda_x)| \leq 1, Q(\lambda) = 1}} \log(|Q(\lambda_x)|).$$

Then by [SSTT22, Proposition 5.4], we have:

$$\phi_{\beta,m}(v) = A(\beta, m, x) + O(m^{\frac{b}{2}}) .$$

The reference only proves it for $\beta = 0$ and L maximal, but the same proof applies with minor changes.

Proposition 5.4. *There exists a subset $S_{\mathrm{bad}} \subset \mathbb{Z}_{>0}$ of logarithmic asymptotic density zero such that for every $m \notin S_{\mathrm{bad}}$, we have*

$$A(\beta, m, x) = o(m^{\frac{b}{2}} \log(m)).$$

Proof. Let

$$A(m, x) = -2 \sum_{\substack{\sqrt{m}\lambda \in L \\ |Q(\lambda_x)| \leq 1, Q(\lambda) = 1}} \log(|Q(\lambda_x)|) .$$

Notice that $A(\beta, m, x) \geq 0$ and that

$$\sum_{\beta \in L/rL} A(\beta, m, x) = A(m, x) \ .$$

Hence

$$0 \leq A(\beta, m, x) \leq A(m, x).$$

Now we can use [SSTT22, Theorem 6.1] to bound $A(m, x)$, yielding the desired result. This proves Proposition 3.8. □

6. Non-archimedean estimates

Our goal in this section is to prove the local non-archimedean estimates Proposition 3.9.

Let $\rho : \mathscr{S} \to \widetilde{\mathcal{M}}_r^\Sigma$ be the period map, where $\widetilde{\mathcal{M}}_r^\Sigma$ is the toroidal compactification of the integral model of the GSpin Shimura variety associated to (rL, Q).

6.1. **Good reduction case.** Let $\mathfrak{P}$ be a prime of good reduction. By the moduli interpretation of $\mathcal{Z}(\beta, m)$, see [SSTT22, Lemma 7.2] for a proof, we have:

$$(\mathscr{S}.\mathcal{Z}(\beta, m)) = \sum_{n=1}^\infty |\{x \text{ lifts to order } n, x \in V_\beta(\mathcal{A}_\mathfrak{P}), Q(x) = m\}| \ .$$

In particular,

$$0 \leq (\mathscr{S}.\mathcal{Z}(\beta, m)) \leq (\mathscr{S}.\mathcal{Z}(m)) \ ,$$

where $\mathcal{Z}(m) \to \mathcal{M}$ is the special divisor in $\mathcal{M}$

By [SSTT22, Theorem 7.1], we have the estimate:

$$\sum_{m \in S_{D,X}} (\mathscr{S}.\mathcal{Z}(m)) = o(X^{b/2} \log(X)).$$

Hence, combined with the inequality above, we get:

$$\sum_{m \in S_{D,X}} (\mathscr{S}.\mathcal{Z}(\beta, m)) = o(X^{b/2} \log(X)) \ ,$$

which proves Proposition 3.9]

6.2. **Bad reduction case: type II.** Let $\mathfrak{P}$ be a prime of bad reduction. The toroidal compactification $\widetilde{\mathcal{M}}_r^\Sigma$ has a stratification with two types of boundary components as explained in [Tay22]. We will use the results from that paper to analyze the local intersection multiplicities and we focus now on boundary components of type II.

Let Υ be a toroidal stratum representative of type II , $\mathcal{B}^\Upsilon$ the corresponding boundary component of type II and we assume in this section that that boundary point $\mathscr{S}(\mathbb{F}_\mathfrak{P})$ lies in $\mathcal{B}^\Upsilon(\overline{\mathbb{F}}_p)$.

By Equation (3.2.1), we have

$$0 \leq (\mathscr{S}.\mathcal{Z}(\beta, m))_{\mathfrak{P}} \leq (\mathscr{S}.\mathcal{Z}(m)))_{\mathfrak{P}}. \qquad (6.2.1)$$

Let $D \in \mathbb{Z}_{\geq 1}$. For $X \in \mathbb{Z}_{>0}$, let $S_{D,X}$ denote the set

$$\{m \in \mathbb{Z}_{>0} \mid X \leq m < 2X, \frac{m}{D} \in \mathbb{Z} \cap (\mathbb{Q}^{\times})^2, (m, N) = 1\}.$$

Then by [Tay22], we have

$$\sum_{m \in S_{D,X}} (\mathscr{S}.\mathcal{Z}(m))_{\mathfrak{P}} = o(X^{\frac{b+1}{2}} \log X).$$

Combining Equation (6.2.1) and the previous estimate, we get Proposition 3.9 in the type II case.

6.3. **Bad reduction case: type III.** Let (Ξ, σ) be a toroidal stratum representative of type III where σ is a ray. Let $\mathcal{B}^{\Xi, \sigma}$ be the corresponding boundary component of type III and we assume in this section that the boundary point $\mathscr{S}$ lies in $\mathcal{B}^{\Xi, \sigma}(\overline{F}_p)$.

Similarly, we have Equation (3.2.1)

$$0 \leq (\mathscr{S}.\mathcal{Z}(\beta, m))_{\mathfrak{P}} \leq (\mathscr{S}.\mathcal{Z}(m)) .$$

Let $D \in \mathbb{Z}_{\geq 1}$ coprime to N and $X \in \mathbb{Z}_{>0}$. Let $S_{D,X}$ denote the set

$$\{m \in \mathbb{Z}_{>0} \mid X \leq m < 2X, \frac{m}{D} \in \mathbb{Z} \cap (\mathbb{Q}^{\times})^2, (m, N) = 1\}.$$

Then we have by [Tay22, Proposition 5.4],

$$\sum_{m \in S_{D,X}} (\mathscr{S}.\mathcal{Z}(m))_{\mathfrak{P}} = o(X^{\frac{b+1}{2}} \log X).$$

Combining the two previous estimates concludes the proof of Proposition 3.9 in the type III case.

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