RATIONAL CURVES ON ELLIPTIC K3 SURFACES

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Abstract. We prove that any non-isotrivial elliptic K3 surface over an algebraically closed field k of arbitrary characteristic contains infinitely many rational curves. In the case when $\mathrm{char}(k) \neq 2, 3$, we prove this result for any elliptic K3 surface. When the characteristic of k is zero, this result is due to the work of Bogomolov-Tschinkel and Hassett.

1. Introduction

Let X be a K3 surface over an algebraically closed field k. In [BT00, Corollary 3.28], Bogomolov and Tschinkel prove that when the characteristic of k is zero and X admits a non-isotrivial elliptic fibration then X contains infinitely many rational curves. Later, Hassett in [Has03, Section 9] handled the general case of arbitrary elliptic complex K3 surfaces. In this note, we extend the above results to the case where k has positive characteristic.

Theorem 1.1. *Let X be an elliptic K3 surface over an algebraically closed field k. Then X contains infinitely many rational curves in the following cases:*
(1) *X admits a non-isotrivial elliptic fibration;*
(2) $\mathrm{char}(k) \neq 2, 3$.

In characteristic zero, this is the content of [BT00, Corollary 3.28] and [Has03, Section 9]. When k has positive characteristic, the main ingredients in case (1) are a result on the image of ℓ-adic monodromy representations associated to non-isotrivial 1-dimensional families of elliptic curves, see Proposition 2.5. The proof is inspired from [BT00], though we simplify some arguments presented there. The proof in case (2) follows the arguments of Hassett in [Has03, Section 9]. This note is split into two parts. In the first section, some background on elliptic K3 surfaces is recalled. The main result is proved in the second section.

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2. Background on elliptic K3 surfaces

Let k be an algebraically closed field of positive characteristic and $\mathbb{P}^1_k$ the projective line over k. We recall some facts about elliptic K3 surfaces. For a more comprehensive introduction, see [Huy16, Chapter 11].

An elliptic K3 surface is a K3 surface X which admits a surjective morphism $X \xrightarrow{\pi} \mathbb{P}^1_k$ whose generic fiber is a smooth integral curve of genus 1. If moreover the morphism π admits a section, then X is said to be a Jacobian elliptic K3 surface. The fibration is said to be non-isotrivial if not all the smooth fibers are isomorphic. For Jacobian elliptic K3 surfaces, the latter condition is equivalent to the fact that the j-invariant of the generic fiber is not in k.

2.1. **Tate-Shafarevich group.** Let $X \xrightarrow{\pi} \mathbb{P}^1_k$ be an elliptic K3 surface. For every integer $d \geq 0$, one can associate to X an elliptic K3 surface $J^d(X)$ as follows. If η denotes the generic point of $\mathbb{P}^1_k$, then the generic fiber X_η over $k(\eta)$ is a smooth integral curve of genus 1. Then one can associate to it a smooth curve of genus 1, $Jac^d(X_\eta)$, which coarsly represents the étale sheafification of the functor

$$\mathrm{Pic}^d : (\mathrm{Sch}/k(\eta))^\circ \to (\mathrm{Sets}),\ S \mapsto \mathrm{Pic}^d(X_\eta \times S)/\sim .$$

Then $J^d(X) \to \mathbb{P}^1_k$ is defined as the unique relatively minimal smooth model of $Jac^d(X_\eta)$. For $d = 0$, we denote it simply $J(X)$ and it is a Jacobian elliptic K3 surface, see[Huy16, Chap.11, Section 4.1] or [CD89, Thm. 5.3.1] for more details. For every smooth fiber X_t, $t \in \mathbb{P}^1_k$, the fiber $J(X)_t$ is isomorphic to the Jacobian elliptic curve associated to X_t. Let $J(X)^{sm} \subset J(X)$ be the open set of π-smooth points, viewed as a smooth group scheme over $\mathbb{P}^1_k$. Then the open π-smooth locus $X^{sm} \to \mathbb{P}^1_k$ is a $J(X)^{sm}$-torsor over $\mathbb{P}^1_k$. Hence for an arbitrary Jacobian elliptic K3 surface $Y \to \mathbb{P}^1_k$, define the *Tate-Shafarevich group* $\text{Ш}(Y)$ as the set of isomorphism classes of Y^{sm}-torsors over $\mathbb{P}^1_k$. The group structure on $\text{Ш}(Y)$ depends on the choice of the section, however the isomorphism class does not.

Proposition 2.1 (Chap.11, Section 5.2, 5.5(i), 5.6 [Huy16])**.** *Let $X \to \mathbb{P}^1_k$ be a Jacobian elliptic K3 surface. The Tate-Shafarevich group $\text{Ш}(X)$ is isomorphic to the Brauer group $\mathrm{Br}(X)$ of X and we have an injective map*

$$\text{Ш}(X) \hookrightarrow WC(X_\eta),$$

where $WC(X_\eta)$ is the Weil-Châtelet group of the generic fiber of $X \to \mathbb{P}^1_k$.

Recall that the Brauer group of X is defined as the étale cohomology group $H^2(X, \mathbb{G}_m)$ and recall also that for an elliptic curve E over a field K, the Weil-Châtelet group, denoted $WC(E)$, is defined as the set of isomorphism classes of torsors under E over K, see [Huy16, Chapter 11, Section 5.1].

For every positive integer d and for every smooth fiber X_t, $t \in \mathbb{P}^1_k$, $J^d(X)_t$ is isomorphic to $\mathrm{Pic}^d(X_t)$. Moreover, one has an isomorphism

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & J^1(X) \arrow[dl, "\pi_1"] \\ & \mathbb{P}^1_k & \end{tikzcd}$$

and $J(J^d(X)) \simeq J(X)$. In addition, the class $[J^d(X)]$ of $J^d(X)$ in $\mathrm{Br}(J(X))$ is equal to $d[X]$.

For every integers d, d', we have natural rational maps of algebraic varieties

$$\begin{tikzcd} J^d(X) \times_{\mathbb{P}^1_k} J^{d'}(X) \arrow[rr, dashed] \arrow[dr] & & J^{d+d'}(X) \arrow[dl] \\ & \mathbb{P}^1_k & \end{tikzcd}$$

For a positive integer ℓ, the diagonal embedding

$$J^1(X) \to \underbrace{J^1(X) \times_{\mathbb{P}^1_k} \cdots \times_{\mathbb{P}^1_k} J^1(X)}_{\ell\,\text{times}}$$

composed with the rational map above defines a rational map η_ℓ which fits into the following commutative diagram

$$\begin{tikzcd} J^1(X) \arrow[rr, dashed, "\eta_\ell"] \arrow[dr, "\pi"'] & & J^\ell(X) \arrow[dl, "\pi_\ell"] \\ & \mathbb{P}^1_k & \end{tikzcd}$$

The map η_ℓ is defined over the smooth locus of π.

2.2. **Rational curves.** Let X be a K3 surface over k. A rational curve on X is an integral closed subscheme C of dimension 1 and of geometric genus 0. Recall the following existence result, attributed to Bogomolov and Mumford, with a refinement of Li and Liedtke ([LL12, Theorem 2.1]).

Proposition 2.2 (Bogomolov-Mumford)**.** *Let L be a non-trivial effective line bundle on a K3 surface X over k. Then L is linearly equivalent to a sum of effective rational curves.*

2.3. **Relative effective Cartier divisors.**

Definition 2.3. *Let $X \to \mathbb{P}^1_k$ be an elliptic K3 surface. A relative effective Cartier divisor on $X/\mathbb{P}^1_k$ is a closed subscheme $\mathcal{M}$ on X such that $\mathcal{M} \to \mathbb{P}^1_k$ is finite flat. If moreover $\mathcal{M}$ is irreducible, it is called a multisection.*

Given an elliptic K3 surface X and a multisection $\mathcal{M}$ on X, the map $\mathcal{M} \to \mathbb{P}^1_k$ is finite flat and its degree is by definition the degree of $\mathcal{M}$.

Let X_0 be a smooth fiber of $X \to \mathbb{P}^1_k$ over a point $0 \in \mathbb{P}^1_k$. Then we have a map given by the intersection product

$$\mathrm{Pic}(X) \xrightarrow{(X_0,\ )} \mathbb{Z}.$$

It sends any multisection to its degree. The image of the above map is a non-zero subgroup of $\mathbb{Z}$, of finite index. Denote by d_X its index. It is called the degree of the elliptic fibration $X \to \mathbb{P}^1_k$. Remark that an elliptic fibration is Jacobian if and only if its degree is equal to one.

Lemma 2.4. *Let $X \to \mathbb{P}^1_k$ be an elliptic K3 surface.*

(1) *The order of $[X]$ in $\mathrm{Br}(J(X))$ is equal to d_X.*
(2) *There exists a multisection of degree $d_{\mathcal{M}} = d_X$ which is a rational curve.*
(3) *There exists at least one multisection $\mathcal{M}$ such that $d_{\mathcal{M}} = d_X$ and which is moreover generically étale over $\mathbb{P}^1_k$.*

Proof. For (2), let $\mathcal{M}$ be a multisection of degree d_X. By Proposition 2.2, $\mathcal{M}$ is linearly equivalent to a sum of rational curves $\sum_i C_i$. Then there exists a unique curve C_i which is horizontal and all the others are vertical. Then C_i satisfies the desired properties.

For (1), notice that X_η is a torsor under the elliptic curve $J(X)_\eta$ and that d_X is the index of X_η, i.e is the greatest common divisor of the degrees of residue fields of closed points of X_η (see [Lic68, 1]). Since the order of X_η in $WC(J(X)_\eta)$ is equal to its index by [Lic68, Theorem 1], it implies that the order of $[X]$ is exactly d_X. By [Lic68, Section 5, Theorem 4][1], it is also equal to the minimal degree of residue fields of separable closed points. Hence there exists a closed separable point

[1]More precisely, see the proof given there.

in X_η of degree d_X. Taking its closure yields a separable multisection. This proves (3). $\square$

2.4. **Monodromy.** Let $X \xrightarrow{\pi} \mathbb{P}^1_k$ be an elliptic K3 surface. Let U be the largest Zariski open subset of $\mathbb{P}^1_k$ over which the map π is smooth. Thus $X_U \to U$ is a torsor under the smooth group scheme $J(X)_U \to U$. For $b \in U$ a closed point and m prime to $p := \mathrm{char}(k)$, the étale fundamental group $\pi_1^{\text{ét}}(U, b)$ of U acts on the group of m-torsion points in $J(X)_b$ and defines a group morphism

$$\rho : \pi_1^{\text{ét}}(U, b) \to \mathrm{Aut}\left(\varprojlim_{\gcd(m,p)=1} J(X)_b[m] \right) = \prod_{\gcd(\ell,p)=1} \mathrm{Aut}(\mathrm{T}_\ell J(X)_b).$$

This action preserves the Weil paring and factors as follows:

$$\rho : \pi_1^{\text{ét}}(U, b) \to \prod_{\ell \wedge p=1} \mathrm{SL}(\mathrm{T}_\ell J(X)_b).$$

For every prime ℓ, we denote by ρ_{ℓ^∞} the representation of $\pi_1^{\text{ét}}(U, b)$ on the Tate module $\mathrm{T}_\ell J(X_b)$ and denote by ρ_ℓ its reduction modulo ℓ. Then ρ_{ℓ^∞} is simply the projection on the ℓ-factor in the previous map. The monodromy group Γ is the image of $\pi_1^{\text{ét}}(U, b)$ under ρ. The next result on the image of the monodromy group will be crucial in the proof of Theorem 1.1.

Proposition 2.5 ([CH05]). *If the elliptic fibration is not isotrivial, then there exists a constant $c(k)$ depending only on k, such that for every $\ell > c(k)$ the morphism ρ_ℓ is surjective.*

This is the content of [CH05, Theorem 1.1] where the surjectivity is proven for the reduction modulo ℓ, then one uses Lemma 2 in [Ser98, IV-23]. Notice that in [CH05, Theorem 1.1], the base field is supposed to be finite but one can check that the proof given there works for perfect fields, as mentioned in the discussion after Theorem 1.1 in *loc.cit.*

3. Proof of Theorem 1.1

If X has Picard rank $\rho(X)$ at least 20, then the automorphism group of X is infinite and hence X contains infinitely many rational curves, see [Huy16, Chap.13, Remark 1.6] and [BT00, Theorem 4.1]. Hence we assume that $\rho(X) \leq 19$.

The elliptic surface X defines a class in the Tate-Shafarevich group $\text{Ш}(J(X))$ of $J(X)$, which is isomorphic to the Brauer group $\mathrm{Br}(J(X))$ by Proposition 2.1. This class is a sum of two elements $\alpha_p + \alpha$, where α has torsion prime to p and α_p is torsion of order p^a, for some integer a. Here p is the characteristic of k. We will construct infinitely many multisections on X which are rational curves and whose degrees tend to infinity. This will be enough to prove Theorem 1.1. Denote by d_X the degree of X and let ℓ be a prime number with residue $1 \pmod{p^a}$ and such that $\ell > \max(d_X, c(k))$, where $c(k)$ is given by Proposition 2.5. The prime to p torsion part of $\mathrm{Br}(J(X))$ is a divisible group by [Huy16, Chap. 18, Example 1.5]. The Kummer exact sequence and the assumption on the Picard rank ensures furthermore that it is not trivial (see formula (1.8) *loc. cit*). We can thus find an elliptic K3 surface $\pi_\ell : X_\ell \to \mathbb{P}^1$ such that $J(X_\ell) \simeq J(X)$, $\ell[X_\ell, \pi_\ell] = [X, \pi]$ in $\mathrm{Br}(J(X))$ and $d_{X_\ell} = \ell d_X$. Take for instance $\alpha_p + \alpha_\ell$, where α_ℓ is a non-trivial element in $\mathrm{Br}(J(X))$ which satisfies $\ell.\alpha_\ell = \alpha$. Hence $J^\ell(X_\ell) \simeq X$ and we have a rational map defined at the end of section 2.1:

$$\begin{tikzcd} X_\ell \arrow[rr, dashed, "\eta_\ell"] \arrow[dr, "\pi_\ell"'] & & X \arrow[dl, "\pi"] \\ & \mathbb{P}^1_k & \end{tikzcd}$$

By Lemma 2.4, X_ℓ contains a rational multisection $\mathcal{M}_\ell$ of degree $d_{\mathcal{M}_\ell} = d_{X_\ell} = \ell d_X$. If the restriction of η_ℓ to $\mathcal{M}_\ell$ is isomorphic to its images above $\mathbb{P}^1_k$ then $\eta_\ell(\mathcal{M}_\ell)$ is a rational curve on X of degree divisible by ℓ which is the desired result. Otherwise, since the multiplication by ℓ map is étale (by [Gro62, Théorème 2.5]), there exists infinitely many closed points b in the maximal open subset $U \subset \mathbb{P}^1_k$ where π is smooth, $\mathcal{M}_{\ell,U} \to U$ is smooth and two distinct points P_1, P_2 in $X_{\ell,b} \cap \mathcal{M}_\ell$ such that $\ell.(P_1 - P_2) = 0$ in $J(X)_b$. Thus, the point $P_1 - P_2$ is a ℓ-primitive torsion point in $J(X)_b$. Let $J(X)_U[\ell] \to U$ be the relative effective Cartier divisor of $J(X)_U \to U$ of ℓ-torsion points.

Let $J(X)_{U,prim}[\ell]$ be the relative effective Cartier divisor of non-zero ℓ-torsion points. Since $X_{\ell,U}$ is a $J(X)_U$-torsor over U, there is an induced map:

$$J(X)_{U,prim}[\ell] \times \mathcal{M}_{\ell,U} \to X_{\ell,U}. \tag{1}$$

The closure of the image in X_ℓ is a curve of X_ℓ which intersects $\mathcal{M}_\ell$ infinitely many times by the non-injectivity of η_ℓ. Hence $\mathcal{M}_\ell$ is isomorphic to an irreducible component of $J(X)_{U,prim}[\ell] \times_U \mathcal{M}_{\ell,U}$.

3.1. **Non-isotrivial case.** For ℓ large enough, $J(X)_{U,prim}[\ell]$ is irreducible by Proposition 2.5. Hence via its first projection, the above map is surjective over $J(X)_{U,prim}[\ell]$. Since there are $\ell^2 - 1$ torsion points in each fiber of $J(X)_{U,prim}[\ell]$ over U, this implies

$$d_{\mathcal{M}_\ell} = \ell d_X \geq \ell^2 - 1.$$

This is a contradiction by our assumption on ℓ.

3.2. **Isotrivial case.** We assume now that the elliptic fibration $X \to \mathbb{P}^1_k$ is isotrivial. Then the elliptic fibration $J(X) \to \mathbb{P}^1_k$ is also isotrivial. If the characteristic of k is different from 2 and 3, which will be assumed henceforth, then we can proceed following the lines of [Has03, Section 9]. The image of the étale fundamental group of U by ρ_ℓ factors through the automorphism group of the geometric generic fiber of $J(X) \to \mathbb{P}^1_k$ which is cyclic of order $2, 4$ or 6, see [Sil86, III.10]. Assume that the fibration $J(X) \to \mathbb{P}^1_k$ has n_0 degenerate fibers of type I_0^*, n_1' degenerate fibers of type I_a, $a > 0$, n_1'' degenerate fibers of type I_a^*, $a > 0$, n_2 fibers of type II or II^*, n_3 fibers of type III or III^*, and n_4 fibers of type IV or IV^*. For the definition of the type of singularities of fibers, see [Huy16, Chapter 11, Section 1.3].

By Equation (1), $\mathcal{M}_{\ell,U}$ is an irreducible component of a principal homogeneous space under $J(X)_{U,prim}[\ell]$. Using Riemann-Hurwitz as in the proof of [Has03, Theorem 9.9] and noticing that the computations of the ramification contributions of degenerate fibers from [Has03, Table 1, page 259] hold for ℓ large enough, see [N6́4, Chapitre III, 17], there exists $C > 0$ such that $g(\mathcal{M}_\ell) \geq C.c(J)$ where $g(\mathcal{M}_\ell)$ is the geometric genus of $\mathcal{M}_\ell$ and

$$c(J) = \frac{1}{2}n_0 + n_1' + n_1'' + \frac{5}{6}n_2 + \frac{3}{4}n_3 + \frac{2}{3}n_4 - 2.$$

Since $\mathcal{M}_\ell$ is a rational curve, we infer that $c(J) \leq 0$. We use now the method of [Has03, Proposition 9.6] to classify K3 surfaces that satisfy the last condition. By Shioda-Tate formula [SS10, Theorem 6.3]), we have :

$$\rho(X) = 2 + \sum_{s \in \mathbb{P}^1(k)} (r_s - 1) + r(X)$$

where r_s denotes the number of irreducible components of a fiber X_s for s a closed point in $\mathbb{P}^1_k$ and $r(X)$ is the rank of the Mordell-Weil group of $J(X)$. On the other hand, the ℓ-adic Euler formula ([Dol72, Theorem 1.1, Corollary 1.6][2]) implies that:

$$24 = \sum_{s \in \mathbb{P}^1(k)} [\chi(X_s)_\ell + \alpha_{s,\ell}] \tag{2}$$

where, for $s \in \mathbb{P}^1_k(k)$, $\chi(X_s)_\ell$ is the ℓ-adic Euler characteristic of the fiber X_s and $\alpha_{s,\ell}$ is its wild conductor defined in [Dol72, Section 1]. Recall that $r_s = \chi(X_s)_\ell$ if the fiber X_s has reduction type I_a and otherwise $r_s = \chi(X_s)_\ell - 1$. Since the characteristic of k is different from 2 and 3, all the wild conductors above vanish.

Combining the two previous formulas, we get:

$$\begin{aligned}\rho(X) &= 2 + \sum_{\substack{s \in \mathbb{P}^1_k(k) \\ \text{of type } I_a}} (\chi(X_s)_\ell - 1) + \sum_{\substack{s \in \mathbb{P}^1_k(k) \\ \text{not of type } I_a}} (\chi(X_s)_\ell - 2) + r(X) \\ &= 26 - n'_1 - 2N + r(X)\end{aligned}$$

[2]With the correct sign.

where $N = n_0 + n''_1 + n_2 + n_3 + n_4$. The assumption that $c(J) \leq 0$ implies that

$$18 + r(X) + 3n_1^{'} + 2n_1^{''} + \frac{4}{3}n_2 + n_3 + \frac{2}{3}n_4 \leq \rho(X).$$

Hence either X has Picard rank equal to 22, or $\rho(X) \leq 20$ and thus X is an element in the list given in [Has03, Proposition 9.6]. In all these cases, X is either a Kummer surface or its automorphism group is infinite. In both cases, X has infinitely many rational curves, see [BT05, Corollary 4.3] and [BT00, Lemma 4.9] for the second case.

3.3. **Situation in characteristic 2 and 3.** When the characteristic of k is equal to 2 or 3 and the elliptic fibration $X \to \mathbb{P}^1_k$ is isotrivial then the classification above must be modified to take into account the wild ramification factors in Equation (2) which do not vanish in general, apart from special cases, see [SS10, Section 4.6, Table 2]. For example, we could have a K3 surface with a single cusp of conductor 24 for which $c(J) = \frac{-7}{6}$ and $\rho(X) \geq 2$. It would be interesting to investigate these small rank situations.

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