RATIONAL CURVES ON ELLIPTIC K3 SURFACES

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ABSTRACT. We prove that any non-isotrivial elliptic K3 surface over an algebraically closed field k of arbitrary characteristic contains infinitely many rational curves. In the case when $\operatorname{char}(k) \neq 2, 3$, we prove this result for any elliptic K3 surface. When the characteristic of k is zero, this result is due to the work of Bogomolov-Tschinkel and Hassett.

1. INTRODUCTION

Let X be a K3 surface over an algebraically closed field k. In [BT00, Corollary 3.28], Bogomolov and Tschinkel prove that when the characteristic of k is zero and X admits a non-isotrivial elliptic fibration then X contains infinitely many rational curves. Later, Hassett in [Has03, Section 9] handled the general case of arbitrary elliptic complex K3 surfaces. In this note, we extend the above results to the case where k has positive characteristic.

Theorem 1.1. Let X be an elliptic K3 surface over an algebraically closed field k. Then X contains infinitely many rational curves in the following cases:

- (1) X admits a non-isotrivial elliptic fibration;
- (2) char(k) $\neq 2, 3$.

In characteristic zero, this is the content of [BT00, Corollary 3.28] and [Has03, Section 9]. When k has positive characteristic, the main ingredients in case (1) are a result on the image of ℓ -adic monodromy representations associated to non-isotrivial 1-dimensional families of elliptic curves, see Proposition 2.5. The proof is inspired from [BT00], though we simplify some arguments presented there. The proof in case (2) follows the arguments of Hassett in [Has03, Section 9]. This note is split into two parts. In the first section, some background on elliptic K3 surfaces is recalled. The main result is proved in the second section.

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2. Background on elliptic K3 surfaces

Let k be an algebraically closed field of positive characteristic and \mathbb{P}^1_k the projective line over k. We recall some facts about elliptic K3 surfaces. For a more comprehensive introduction, see [Huy16, Chapter 11].

An elliptic K3 surface is a K3 surface X which admits a surjective morphism $X \xrightarrow{\pi} \mathbb{P}^1_k$ whose generic fiber is a smooth integral curve of genus 1. If moreover the morphism π admits a section, then X is said to be a Jacobian elliptic K3 surface. The fibration is said to be non-isotrivial if not all the smooth fibers are isomorphic. For Jacobian elliptic K3 surfaces, the latter condition is equivalent to the fact that the *j*-invariant of the generic fiber is not in k.

2.1. Tate-Shafarevich group. Let $X \xrightarrow{\pi} \mathbb{P}_k^1$ be an elliptic K3 surface. For every integer $d \ge 0$, one can associate to X an elliptic K3 surface $J^d(X)$ as follows. If η denotes the generic point of \mathbb{P}_k^1 , then the generic fiber X_η over $k(\eta)$ is a smooth integral curve of genus 1. Then one can associate to it a smooth curve of genus 1, $Jac^d(X_\eta)$, which coarsly represents the étale sheafification of the functor

$$\operatorname{Pic}^d : (\operatorname{Sch}/k(\eta))^\circ \to (\operatorname{Sets}), S \mapsto \operatorname{Pic}^d(X_\eta \times S)/\sim .$$

Then $J^d(X) \to \mathbb{P}^1_k$ is defined as the unique relatively minimal smooth model of $Jac^d(X_\eta)$. For d = 0, we denote it simply J(X) and it is a Jacobian elliptic K3 surface, see[Huy16, Chap.11, Section 4.1] or [CD89, Thm. 5.3.1] for more details. For every smooth fiber $X_t, t \in \mathbb{P}^1_k$, the fiber $J(X)_t$ is isomorphic to the Jacobian elliptic curve associated to X_t . Let $J(X)^{sm} \subset J(X)$ be the open set of π -smooth points, viewed as a smooth group scheme over \mathbb{P}^1_k . Then the open π -smooth locus $X^{sm} \to \mathbb{P}^1_k$ is a $J(X)^{sm}$ -torsor over \mathbb{P}^1_k . Hence for an arbitrary Jacobian elliptic K3 surface $Y \to \mathbb{P}^1_k$, define the *Tate-Shafarevich group* III(Y) as the set of isomorphism classes of Y^{sm} -torsors over \mathbb{P}^1_k . The group structure on III(Y) depends on the choice of the section, however the isomorphism class does not.

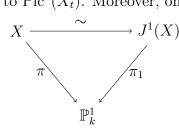
Proposition 2.1 (Chap.11, Section 5.2, 5.5(i), 5.6 [Huy16]). Let $X \to \mathbb{P}^1_k$ be a Jacobian elliptic K3 surface. The Tate-Shafarevich group $\operatorname{III}(X)$ is isomorphic to the Brauer group $\operatorname{Br}(X)$ of X and we have an injective map

$$\mathrm{III}(X) \hookrightarrow WC(X_{\eta}),$$

where $WC(X_{\eta})$ is the Weil-Châtelet group of the generic fiber of $X \to \mathbb{P}^{1}_{k}$.

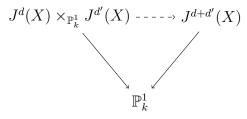
Recall that the Brauer group of X is defined as the étale cohomology group $H^2(X, \mathbb{G}_m)$ and recall also that for an elliptic curve E over a field K, the Weil-Châtelet group, denoted WC(E), is defined as the set of isomorphism classes of torsors under E over K, see [Huy16, Chapter 11, Section 5.1].

For every positive integer d and for every smooth fiber $X_t, t \in \mathbb{P}^1_k$, $J^d(X)_t$ is isomorphic to $\operatorname{Pic}^d(X_t)$. Moreover, one has an isomorphism



and $J(J^d(X)) \simeq J(X)$. In addition, the class $[J^d(X)]$ of $J^d(X)$ in Br(J(X)) is equal to d[X].

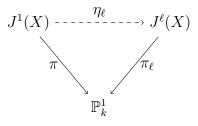
For every integers d, d', we have natural rational maps of algebraic varieties



For a positive integer ℓ , the diagonal embedding

$$J^{1}(X) \to \underbrace{J^{1}(X) \times_{\mathbb{P}^{1}_{k}} \cdots \times_{\mathbb{P}^{1}_{k}} J^{1}(X)}_{\ell \text{ times}}$$

composed with the rational map above defines a rational map η_{ℓ} which fits into the following commutative diagram



The map η_{ℓ} is defined over the smooth locus of π .

2.2. Rational curves. Let X be a K3 surface over k. A rational curve on X is an integral closed subscheme C of dimension 1 and of geometric genus 0. Recall the following existence result, attributed to Bogomolov and Mumford, with a refinement of Li and Liedtke ([LL12, Theorem 2.1]).

Proposition 2.2 (Bogomolov-Mumford). Let L be a non-trivial effective line bundle on a K3 surface X over k. Then L is linearly equivalent to a sum of effective rational curves.

2.3. Relative effective Cartier divisors.

Definition 2.3. Let $X \to \mathbb{P}^1_k$ be an elliptic K3 surface. A relative effective Cartier divisor on X/\mathbb{P}^1_k is a closed subscheme \mathcal{M} on X such that $\mathcal{M} \to \mathbb{P}^1_k$ is finite flat. If moreover \mathcal{M} is irreducible, it is called a multisection.

Given an elliptic K3 surface X and a multisection \mathcal{M} on X, the map $\mathcal{M} \to \mathbb{P}^1_k$ is finite flat and its degree is by definition the degree of \mathcal{M} .

Let X_0 be a smooth fiber of $X \to \mathbb{P}^1_k$ over a point $0 \in \mathbb{P}^1_k$. Then we have a map given by the intersection product

$$\operatorname{Pic}(X) \xrightarrow{(X_0,)} \mathbb{Z}.$$

It sends any multisection to its degree. The image of the above map is a non-zero subgroup of \mathbb{Z} , of finite index. Denote by d_X its index. It is called the degree of the elliptic fibration $X \to \mathbb{P}^1_k$. Remark that an elliptic fibration is Jacobian if and only if its degree is equal to one.

Lemma 2.4. Let $X \to \mathbb{P}^1_k$ be an elliptic K3 surface.

- (1) The order of [X] in Br(J(X)) is equal to d_X .
- (2) There exists a multisection of degree $d_{\mathcal{M}} = d_X$ which is a rational curve.
- (3) There exists at least one multisection \mathcal{M} such that $d_{\mathcal{M}} = d_X$ and which is moreover generically étale over \mathbb{P}^1_k .

Proof. For (2), let \mathcal{M} be a multisection of degree d_X . By Proposition 2.2, \mathcal{M} is linearly equivalent to a sum of rational curves $\sum_i C_i$. Then there exists a unique curve C_i which is horizontal and all the others are vertical. Then C_i satisfies the desired properties.

For (1), notice that X_{η} is a torsor under the elliptic curve $J(X)_{\eta}$ and that d_X is the index of X_{η} , i.e is the greatest common divisor of the degrees of residue fields of closed points of X_{η} (see [Lic68, 1]). Since the order of X_{η} in $WC(J(X)_{\eta})$ is equal to its index by [Lic68, Theorem 1], it implies that the order of [X] is exactly d_X . By [Lic68, Section 5, Theorem 4]¹, it is also equal to the minimal degree of residue fields of separable closed points. Hence there exists a closed separable point

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¹More precisely, see the proof given there.

in X_{η} of degree d_X . Taking its closure yields a separable multisection. This proves (3).

2.4. **Monodromy.** Let $X \xrightarrow{\pi} \mathbb{P}^1_k$ be an elliptic K3 surface. Let U be the largest Zariski open subset of \mathbb{P}^1_k over which the map π is smooth. Thus $X_U \to U$ is a torsor under the smooth group scheme $J(X)_U \to U$. For $b \in U$ a closed point and m prime to $p := \operatorname{char}(k)$, the étale fundamental group $\pi_1^{\text{ét}}(U, b)$ of U acts on the group of m-torsion points in $J(X)_b$ and defines a group morphism

$$\rho: \pi_1^{\text{\'et}}(U, b) \to \operatorname{Aut}\left(\lim_{\overset{\leftarrow}{\operatorname{gcd}}(m, p)=1} J(X)_b[m]\right) = \prod_{\operatorname{gcd}} \operatorname{Aut}(\operatorname{T}_\ell J(X)_b).$$

This action preserves the Weil paring and factors as follows:

$$\rho: \pi_1^{\text{\acute{e}t}}(U, b) \to \prod_{\ell \land p=1} \operatorname{SL}(\operatorname{T}_\ell J(X)_b).$$

For every prime ℓ , we denote by $\rho_{\ell^{\infty}}$ the representation of $\pi_1^{\text{\acute{e}t}}(U, b)$ on the Tate module $T_{\ell}J(X_b)$ and denote by ρ_{ℓ} its reduction modulo ℓ . Then $\rho_{\ell^{\infty}}$ is simply the projection on the ℓ -factor in the previous map. The monodromy group Γ is the image of $\pi_1^{\text{\acute{e}t}}(U, b)$ under ρ . The next result on the image of the monodromy group will be crucial in the proof of Theorem 1.1.

Proposition 2.5 ([CH05]). If the elliptic fibration is not isotrivial, then there exists a constant c(k) depending only on k, such that for every $\ell > c(k)$ the morphism ρ_{ℓ} is surjective.

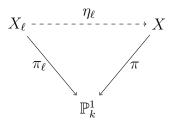
This is the content of [CH05, Theorem 1.1] where the surjectivity is proven for the reduction modulo ℓ , then one uses Lemma 2 in [Ser98, IV-23]. Notice that in [CH05, Theorem 1.1], the base field is supposed to be finite but one can check that the proof given there works for perfect fields, as mentioned in the discussion after Theorem 1.1 in *loc.cit*.

3. Proof of Theorem 1.1

If X has Picard rank $\rho(X)$ at least 20, then the automorphism group of X is infinite and hence X contains infinitely many rational curves, see [Huy16, Chap.13, Remark 1.6] and [BT00, Theorem 4.1]. Hence we assume that $\rho(X) \leq 19$.

The elliptic surface X defines a class in the Tate-Shafarevich group $\operatorname{III}(J(X))$ of J(X), which is isomorphic to the Brauer group $\operatorname{Br}(J(X))$ by Proposition 2.1. This class is a sum of two elements $\alpha_p + \alpha$, where α has torsion prime to p and α_p is torsion of order p^a , for some integer a. Here p is the characteristic of k. We will construct infinitely many multisections on X which are rational curves and whose degrees tend to infinity. This will be enough to prove Theorem 1.1. Denote by d_X the degree of X and let ℓ be a prime number with residue 1 (mod p^a) and

such that $\ell > \max(d_X, c(k))$, where c(k) is given by Proposition 2.5. The prime to p torsion part of $\operatorname{Br}(J(X))$ is a divisible group by [Huy16, Chap. 18, Example 1.5]. The Kummer exact sequence and the assumption on the Picard rank ensures furthermore that it is not trivial (see formula (1.8) *loc. cit*). We can thus find an elliptic K3 surface $\pi_{\ell} : X_{\ell} \to \mathbb{P}^1$ such that $J(X_{\ell}) \simeq J(X)$, $\ell[X_{\ell}, \pi_{\ell}] = [X, \pi]$ in $\operatorname{Br}(J(X))$ and $d_{X_{\ell}} = \ell d_X$. Take for instance $\alpha_p + \alpha_{\ell}$, where α_{ℓ} is a non-trivial element in $\operatorname{Br}(J(X))$ which satisfies $\ell.\alpha_{\ell} = \alpha$. Hence $J^{\ell}(X_{\ell}) \simeq X$ and we have a rational map defined at the end of section 2.1:



By Lemma 2.4, X_{ℓ} contains a rational multisection \mathcal{M}_{ℓ} of degree $d_{\mathcal{M}_{\ell}} = d_{X_{\ell}} = \ell d_X$. If the restriction of η_{ℓ} to \mathcal{M}_{ℓ} is isomorphic to its images above \mathbb{P}^1_k then $\eta_{\ell}(\mathcal{M}_{\ell})$ is a rational curve on X of degree divisible by ℓ which is the desired result. Otherwise, since the multiplication by ℓ map is étale (by [Gro62, Théorème 2.5]), there exists infinitely many closed points b in the maximal open subset $U \subset \mathbb{P}^1_k$ where π is smooth, $\mathcal{M}_{\ell,U} \to U$ is smooth and two distinct points P_1, P_2 in $X_{\ell,b} \cap \mathcal{M}_{\ell}$ such that $\ell.(P_1 - P_2) = 0$ in $J(X)_b$. Thus, the point $P_1 - P_2$ is a ℓ -primitive torsion point in $J(X)_b$. Let $J(X)_U[\ell] \to U$ be the relative effective Cartier divisor of $J(X)_U \to U$ of ℓ -torsion points.

Let $J(X)_{U,prim}[\ell]$ be the relative effective Cartier divisor of non-zero ℓ -torsion points. Since $X_{\ell,U}$ is a $J(X)_U$ -torsor over U, there is an induced map:

(1)
$$J(X)_{U,prim}[\ell] \times \mathcal{M}_{\ell,U} \to X_{\ell,U}.$$

The closure of the image in X_{ℓ} is a curve of X_{ℓ} which intersects \mathcal{M}_{ℓ} infinitely many times by the non-injectivity of η_{ℓ} . Hence \mathcal{M}_{ℓ} is isomorphic to an irreducible component of $J(X)_{U,prim}[\ell] \times_U \mathcal{M}_{\ell,U}$.

3.1. Non-isotrivial case. For ℓ large enough, $J(X)_{U,prim}[\ell]$ is irreducible by Proposition 2.5. Hence via its first projection, the above map is surjective over $J(X)_{U,prim}[\ell]$. Since there are $\ell^2 - 1$ torsion points in each fiber of $J(X)_{U,prim}[\ell]$ over U, this implies

$$d_{\mathcal{M}_{\ell}} = \ell d_X \ge \ell^2 - 1.$$

This is a contradiction by our assumption on ℓ .

3.2. Isotrivial case. We assume now that the elliptic fibration $X \to \mathbb{P}^1_k$ is isotrivial. Then the elliptic fibration $J(X) \to \mathbb{P}^1_k$ is also isotrivial. If the characteristic of k is different from 2 and 3, which will be assumed

henceforth, then we can proceed following the lines of [Has03, Section 9]. The image of the étale fundamental group of U by ρ_{ℓ} factors through the automorphism group of the geometric generic fiber of $J(X) \to \mathbb{P}_k^1$ which is cyclic of order 2, 4 or 6, see [Sil86, III.10]. Assume that the fibration $J(X) \to \mathbb{P}_k^1$ has n_0 degenerate fibers of type I_0^* , n'_1 degenerate fibers of type I_a^* , a > 0, n''_1 degenerate fibers of type I_a^* , a > 0, n_2 fibers of type II or II*, n_3 fibers of type III or III*, and n_4 fibers of type IV or IV*. For the definition of the type of singularities of fibers, see [Huy16, Chapter 11, Section 1.3].

By Equation (1), $\mathcal{M}_{\ell,U}$ is an irreducible component of a principal homogeneous space under $J(X)_{U,prim}[\ell]$. Using Riemann-Hurwitz as in the proof of [Has03, Theorem 9.9] and noticing that the computations of the ramification contributions of degenerate fibers from [Has03, Table 1, page 259] hold for ℓ large enough, see [N64, Chapitre III, 17], there exists C > 0 such that $g(\mathcal{M}_{\ell}) \geq C.c(J)$ where $g(\mathcal{M}_{\ell})$ is the geometric genus of \mathcal{M}_{ℓ} and

$$c(J) = \frac{1}{2}n_0 + n'_1 + n''_1 + \frac{5}{6}n_2 + \frac{3}{4}n_3 + \frac{2}{3}n_4 - 2$$

Since \mathcal{M}_{ℓ} is a rational curve, we infer that $c(J) \leq 0$. We use now the method of [Has03, Proposition 9.6] to classify K3 surfaces that satisfy the last condition. By Shioda-Tate formula [SS10, Theorem 6.3]), we have :

$$\rho(X) = 2 + \sum_{s \in \mathbb{P}^1(k)} (r_s - 1) + r(X)$$

where r_s denotes the number of irreducible components of a fiber X_s for s a closed point in \mathbb{P}^1_k and r(X) is the rank of the Mordell-Weil group of J(X). On the other hand, the ℓ -adic Euler formula ([Dol72, Theorem 1.1, Corollary 1.6]²) implies that:

(2)
$$24 = \sum_{s \in \mathbb{P}^1(k)} \left[\chi(X_s)_{\ell} + \alpha_{s,\ell} \right]$$

where, for $s \in \mathbb{P}_k^1(k)$, $\chi(X_s)_\ell$ is the ℓ -adic Euler characteristic of the fiber X_s and $\alpha_{s,\ell}$ is its wild conductor defined in [Dol72, Section 1]. Recall that $r_s = \chi(X_s)_\ell$ if the fiber X_s has reduction type I_a and otherwise $r_s = \chi(X_s)_\ell - 1$. Since the characteristic of k is different from 2 and 3, all the wild conductors above vanish.

Combining the two previous formulas, we get:

$$\rho(X) = 2 + \sum_{\substack{s \in \mathbb{P}_k^1(k) \\ \text{of type } I_a}} (\chi(X_s)_{\ell} - 1) + \sum_{\substack{s \in \mathbb{P}_k^1(k) \\ \text{not of type } I_a}} (\chi(X_s)_{\ell} - 2) + r(X)$$
$$= 26 - n'_1 - 2N + r(X)$$

 2 With the correct sign.

where $N = n_0 + n''_1 + n_2 + n_3 + n_4$. The assumption that $c(J) \leq 0$ implies that

$$18 + r(X) + 3n'_{1} + 2n''_{1} + \frac{4}{3}n_{2} + n_{3} + \frac{2}{3}n_{4} \le \rho(X).$$

Hence either X has Picard rank equal to 22, or $\rho(X) \leq 20$ and thus X is an element in the list given in [Has03, Proposition 9.6]. In all these cases, X is either a Kummer surface or its automorphism group is infinite. In both cases, X has infinitely many rational curves, see [BT05, Corollary 4.3] and [BT00, Lemma 4.9] for the second case.

3.3. Situation in characteristic 2 and 3. When the characteristic of k is equal to 2 or 3 and the elliptic fibration $X \to \mathbb{P}^1_k$ is isotrivial then the classification above must be modified to take into account the wild ramification factors in Equation (2) which do not vanish in general, apart from special cases, see [SS10, Section 4.6, Table 2]. For example, we could have a K3 surface with a single cusp of conductor 24 for which $c(J) = \frac{-7}{6}$ and $\rho(X) \geq 2$. It would be interesting to investigate these small rank situations.

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