Integrality of Rigid Systems

X = smooth quasiproj $\mathbb{C}$-variety

$\overline{X}$ = good cptification

= projective smooth $\mathbb{C}$-variety $\quad j : X \hookrightarrow \overline{X}$, X dense in $\overline{X}$,
such that $D = \overline{X} - X$ is a strict normal crossings divisor.

$\mathcal{U} := \overline{X} - D_{sing}$ $\quad j : X \underset{a}{\hookrightarrow} \mathcal{U} \underset{b}{\hookrightarrow} \overline{X}$

Simpson's Motivicity Conjecture:

Conjecture: Rigid irred $\mathbb{C}$-loc sys w/ torsion det & quasiunipt monodromies "at ∞" are of geometric origin.
(around irred components of D)

In particular, $\Rightarrow$ Integrality of such rigid loc sys.

Thm (EG'18): Let $X, \overline{X}, D, j$ be as above.
Then irred, rigid $\mathbb{C}$-local sys w/ finite det & w/ quasi unipotent local monodromy at ∞ are integral.

Side note on rigidity:
$\mathcal{L}$ loc sys on X.

Def^n: $\mathcal{L}$ is cohomologically rigid if $H^1(\mathcal{U}, a_* \underline{End}^0(\mathcal{L})) = 0$

Def^n: $\mathcal{L}$ is rigid if it is an isolated point of the moduli stack.

These are equivalent (if smooth)

$\because$ Zariski Tangent Space of the moduli stack $H^1(\mathcal{U}, a_* \underline{End}^0 \mathcal{L})$.

For the rest of talk, when I say rigid I really mean coh rigid.

$$H^1(\overline{X}, j_{!*} \underline{End}^0 \mathcal{L}) = H^1(\mathcal{U}, a_* \underline{End}^0 \mathcal{L})$$

(look at Rmk 2.4 of [EG 18].

There is a generalization for G-loc sys, for G split reductive $/\mathbb{Z}$
Klevdal - Patrikis 2022

Thm (KP22): In prev. set up, G split connected reductive $/\mathbb{Z}$,
& $\rho : \pi_1^{top} \longrightarrow G(\mathbb{C})$ irred, coh rigid, quasi-unipotent local monodromy at ∞
such that image of $\pi_1^{top} \xrightarrow{\rho} G(\mathbb{C}) \longrightarrow A(\mathbb{C})$ has finite order
where A is the max ab. quotient of G.
Then, ($\overline{\operatorname{im}\rho}^{\circ}$ is ss &) ρ factors through $G(\mathcal{O}_L) \to G(\mathbb{C})$.

$G = GL_n$, recovers EG18.

We are going to use the following criterion:
Sps $\mathcal{V}$ is a (~~rigid~~) loc sys s.t. $\mathcal{V} = \mathcal{V}_{\mathcal{O}_{K,\Sigma}} \otimes \mathbb{C}$
where $\mathcal{V}_{\mathcal{O}_{K,\Sigma}}$ is a loc sys of proj $\mathcal{O}_{K,\Sigma}$-modules,
where K nbr field, & Σ is a finite set of finite places.
Criterion: $\mathcal{V}$ is integral iff $\forall \lambda' \in \Sigma$, $\mathcal{V}_{\overline{K}_{\lambda'}} = \mathcal{V}_K \otimes_K \overline{K}_{\lambda'}$ comes by ext^n of scalars from a local system $\mathcal{V}_{\mathcal{O}_{\overline{K}_{\lambda'}}}$ of free $\mathcal{O}_{\overline{K}_{\lambda'}}$-modules

Let's prove the main thm for X projective.

1) Fix $r, d \in \mathbb{N}$.

$N(r,d) := \{$rigid irred $\mathbb{C}$-loc sys, rank r, det of order $d\}$
is finite.

needs rigidity

Every such loc sys is defined over a nbr field K,
& so defined over $\mathcal{O}_{K,\Sigma}$ for some Σ.

$\because \pi_1^{top}$ is finitely generated.

2) Using some local arguments, for a fixed place $\boxed{\lambda \notin \Sigma, \lambda \nmid \ell}$ &
after spreading out X:

$$X_s \longrightarrow S$$

Take $\bar{s} \in S(\bar{\mathbb{F}}_p)$, for some specially chosen $p \neq \ell$.

Show that $\mathcal{V}_1, \dots, \mathcal{V}_N$ descend to $\bar{K}_\lambda$-lisse sheaves

$$\mathcal{V}_{1,\lambda,s}, \dots, \mathcal{V}_{N,\lambda,s} \quad \text{on } X_s$$

for $s \in S(k)$, $k/\mathbb{F}_p$ finite, s lying under $\bar{s}$.

3) Use Drinfeld's Existence Thm for ℓ to ℓ' companions.

Thm (Drinfeld 2012):

Let Y be a smooth scheme $/\mathbb{F}_p$, & sps $E/\mathbb{Q}$ finite ext^n. Sps λ, λ' are finite places of E, not dividing p.

Let $\mathcal{E}_{\lambda'}$ be a lisse $\bar{E}_{\lambda'}$-sheaf on Y s.t. $\forall$ closed pts $y \in Y$, the polynomial
$$\det(1 - F_y t, \mathcal{E}_{\lambda'})$$
has coeffs in E, & its roots are λ-adic units (F_y is the Frobenius at y).

$\exists$ a lisse $\bar{E}_\lambda$-sheaf $\mathcal{E}_\lambda$ on Y compatible w/ $\mathcal{E}_{\lambda'}$.

In particular, can construct, for each $\lambda' \in \Sigma$

$$\mathcal{V}_{1,\lambda',s}, \dots, \mathcal{V}_{N,\lambda',s}$$

$\bar{K}_{\lambda'}$-lisse sheaves on X_s, & show that these satisfy all reqd conditions.

$\Downarrow$

Construct λ'-integral $\mathbb{C}$-loc sys

$$\mathcal{V}_1', \dots, \mathcal{V}_N'$$

on X.

Thus all such loc sys are integral at λ' $\forall \lambda' \in \Sigma$

Proof Sketch for q-proj X:

1) Fix $r, d, h \in \mathbb{N}$

$$\mathcal{S}(r,d,h) := \left\{ \begin{array}{c} \text{rank } r \text{ irred rigid } \mathbb{C}\text{-loc sys on } X \text{ w/ } \det^d = 1, \\ \text{local monodromy at } \infty \text{ is quasi unipotent w/} \\ \text{eigenvalues are } h^{th} \text{ roots of unity} \end{array} \right\}$$

is a finite set, & moreover $\exists$ nbr field K s.t. all elements of $\mathcal{S}(r,d,h)$ are defined over K.

As π_1^{top} is f.g., are all defined over $\mathcal{O}_{K,\Sigma}$ for some finite set of finite places Σ.

$\boxed{\text{WTS: } \mathbb{V}_{\overline{K}_{\lambda'}} \text{ defined } / \ \mathcal{O}_{\overline{K}_{\lambda'}} \quad \forall \mathbb{V} \in \mathcal{S}(r,d,h) \ \forall \lambda' \in \Sigma}$

2) Fix $\lambda \notin \Sigma$, $\lambda | \ell$

ρ_i^{top} $\quad 1 \le i \le \# \mathcal{S}(r,d,h)$ corresponding to elts of $\mathcal{S}(r,d,h)$.

$$\rho_i^{top} : \pi_1^{top} \xrightarrow[\rho_i^{\circ}]{} GL(r, \mathcal{O}_{K,\Sigma}) \hookrightarrow GL(r, \mathbb{C})$$

$$\rho_{i,\lambda}^{\circ} : \pi_1^{top} \xrightarrow{\wr} \underline{\underline{GL(r, \mathcal{O}_{K,\Sigma})}} \hookrightarrow GL(r, \mathcal{O}_{K_\lambda}).$$

$$\rho_{i,\lambda} : \pi_1^{ét} \longrightarrow GL(r, \mathcal{O}_{K_\lambda}) \hookrightarrow GL(r, K_\lambda).$$

$\exists$ connected regular scheme S of fin type $/\mathbb{Z}$ w/ a $\mathbb{C}$-generic pt s.t. the data $(X, j : X \hookrightarrow \overline{X}, D)$ come via base change by $\mathbb{C}$ from $(X_S, j_S : X_S \hookrightarrow \overline{X}_S, D_S)$.

Choose a geometric point $\bar{s} \in S(\overline{\mathbb{F}}_p)$ where p is prime to

$\left| \mathrm{Im}(\pi_1^{ét} \to GL(r, \mathcal{O}_{K_\lambda}) \twoheadrightarrow GL(r, k(\lambda)) \right|$, d, ℓ, h, all places in Σ

Here, we choose $\bar{s}$ to lie over a closed pt of S.

Grothendieck constructed

$$sp : \pi_1^{ét}(X) \longrightarrow\!\!\!\!\!\rightarrow \underbrace{\pi_1^{ét,t}(X_{\bar{s}})}_{\text{tame quotient of } \pi_1^{ét}}$$

& induces an isomorphism on prime to p quotients

$$\pi_1^{ét,p'}(X) \cong \pi_1^{ét,p'}(X_{\bar{s}}).$$

sp induces an equivalence

$$\left\{\begin{matrix}\text{(tame) monodromy prime to } p \\ \text{lisse sheaves on } X\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{- - - -} \\ \text{- - on } X_{\bar{s}}\end{matrix}\right\}.$$

$$\rho_{i,\lambda} : \pi_1^{ét}(X) \longrightarrow GL(r, K_\lambda) \quad \text{factors as}$$

$$\rho_{i,\lambda} : \pi_1^{ét}(X) \xrightarrow{sp} \pi_1^{ét,p'}(X) \cong \pi_1^{ét,p'}(X_{\bar{s}}) \xrightarrow[\rho_{i,\lambda,\bar{s}}]{} GL(r, K_\lambda)$$

Prop^n 3.1 (EG'18):

There exists a finite field $k/\mathbb{F}_p$, a point $s \in S(k)$, & a factorization

$$\begin{tikzcd}
\pi_1^{ét,p'}(X_{\bar{s}}) \arrow[r] \arrow[d, "\rho_{i,\lambda,\bar{s}}"'] & \pi_1^{ét,p'}(X_s) \arrow[dl, dashed, "\rho_{i,\lambda,s}"] \\
GL(r, K_\lambda) &
\end{tikzcd}$$

s.t. $\det(\rho_{i,\lambda,s})$ is finite.

Lemma 3.3: $\rho_{i,\lambda,s}$ tame, quasi unipt monodromy at ∞ w/ eigenvalues killed by h.

Step 3: Point of Prop^n 3.1 is that Drinfeld's thm is now applicable to the $\rho_{i,\lambda,s}$ $(1 \leq i \leq \# \mathcal{S}(r,d))$. ⇝ $\boxed{\mathcal{V}_{i,\lambda,s}}$ $\bar{K}_\lambda$-lisse sheaves on X_s

Pick $\lambda' \in \Sigma$, we can construct companions of $\bar{K}_\lambda$-lisse sheaves on X_s

$$\mathcal{V}_{i,\lambda',s} \qquad 1 \leq i \leq \# \mathcal{S}(r,d)$$

$$\left(\sigma : \hat{\bar{\mathbb{Q}}}_\lambda \simeq \hat{\bar{\mathbb{Q}}}_{\lambda'}\right)$$

Integral basically by defn.

One can check $\mathbb{V}_{i,\lambda',s}$ satisfies all of our requirements, i.e. irred, tame, pairwise non-isomorphic, condition for local monodromy, etc.

To show cohomological rigidity, one needs to use purity argument, local acyclicity, L-functions of trace 0 endomorphisms of companions, etc. show that the $\mathbb{V}_{i,\lambda',s}$ are coh rigid $\forall\ 1 \le i \le \# \mathcal{M}(r,d,h)$.

$$\left(\text{Using} \quad \pi_1^{top}(X) \hookrightarrow \pi_1^{ét}(X) \xrightarrow{sp} \pi_1^{ét}(X_{\bar{s}}) \right)$$

we can construct $\underbrace{\mathbb{V}_{i,\lambda'}}$, λ' adic local systems on X. pairwise non-isom, integral, irred, coh rigid, local monodromy ✓, etc.

$$\rho_{i,\lambda'} : \underset{\substack{\uparrow \\ \text{fin gen}}}{\pi_1^{top}} \longrightarrow \underbrace{GL(\mathcal{O}_{L,\Sigma'})}_{\substack{\downarrow \\ GL(\mathbb{C})}} \hookrightarrow GL(\mathcal{O}_{K_\lambda}).$$

We have constructed reqd number of λ-integral rigid $\mathbb{C}$-loc sys on X satisfying our conditions.

$\hookrightarrow$ rigidity follows from Betti-Étale comparison thm

$\therefore$ all such rigid loc sys are integral. ☒

Appendix: Drinfeld's existence thm:

Thm (Drinfeld 2012):

Let Y be a smooth scheme $/\mathbb{F}_p$, & sps $E/\mathbb{Q}$ finite ext^n. Sps λ, λ' are finite places of E, not dividing p.

Let $\mathcal{E}_{\lambda'}$ be a lisse $\bar{E}_{\lambda'}$-sheaf on Y s.t. $\forall$ closed pts $y \in Y$, the polynomial
$$\det(1 - F_y t, \mathcal{E}_{\lambda'})$$
has coeffs in E, & its $\underline{\text{roots are } \lambda\text{-adic units}}$ (F_y is the Frobenius at y).

$\exists$ a lisse $\bar{E}_\lambda$-sheaf $\mathcal{E}_\lambda$ on Y compatible w/ $\mathcal{E}_{\lambda'}$.

WLOG. Y a scheme of finite type $/\mathbb{Z}[\ell^{-1}]$, $\ell \neq p$

Let F be some finite extn of $\mathbb{Q}_\ell$.

$LS_r^F(Y) = \{F\text{-lisse sheaves on } Y\} /$ isomorphic semisimplifications

contravariant functor in Y.

$\|Y\| = \{$ isom classes of pairs (α, k) for k a finite field, $\alpha \in Y(k)\}$

$P_r(R) = \{1 + c_1 t + \dots + c_r t^r : c_i \in R, c_r \in R^\times\}$

$P_r \cong \mathbb{G}_a^{r-1} \times \mathbb{G}_m$

$\mathbb{N}$ - action on $\|Y\|$ & $P_r(R)$.

$$\mathbb{N} \times \|Y\| \longrightarrow \|Y\|$$
$$(n, (\alpha, k)) \longmapsto (\alpha', k')$$

k' is unique deg n ext^n of k,
& $\alpha' \in Y(k')$ corresponding to α.

$$\mathbb{N} \times P_r(R) \longrightarrow P_r(R)$$
$$n, \quad \prod(1 - \beta_i t) \longmapsto \prod(1 - \beta_i^n t)$$

$\widetilde{LS}_r^F(Y) = \{\mathbb{N}\text{ - equivariant maps } \|Y\| \longrightarrow P_r(\mathcal{O}_F)\}$.

$\cong \quad \{|Y| \longrightarrow P_r(\mathcal{O}_F)\}$.

non functorially!

$$LS_r^F \longrightarrow \widetilde{LS}_r^F$$
$$\mathcal{E} \longmapsto \left((\alpha, k) \mapsto \text{char poly of Frob} \in Gal_k \text{ acting on } \alpha^* \mathcal{E}\right).$$

Čebotarev density $\Rightarrow$ if Y reduced & normal, then this is injective.
(if Y_{red} is normal)

$$LS_r^F(Y) \subseteq \widetilde{LS}_r^F(Y)$$

Drinfeld proves:

(Thm 2.5) Sps Y regular finite type $/ \mathbb{Z}[\ell^{-1}]$, fix $f \in \widetilde{LS}_r^F(Y)$.
Then $f \in LS_r^F(Y)$ iff

(i) $\forall$ regular scheme C of fin type $/\mathbb{Z}$ of pure dim 1, &
$\varphi : C \to Y$, we want $\varphi^*(f) \in LS_r^F(C)$.

(ii) $\exists$ dominant étale map $Y' \to Y$ s.t. $\forall$ smooth sep curves $C /$ finite field,
& every morphism $C \to Y'$, the image of f in $\widetilde{LS}_r^F(C)$ comes

from a tame lisse sheaf.

Proof Drinfeld Existence Thm :-

Conditions on $\mathcal{E}_{\lambda'}$ $\Rightarrow$ it defines $f \in \widetilde{LS}_r^{E_\lambda}(Y)$.

In fact, by Thm 2.5, $f \in LS_r^{E_\lambda}(Y)$, & so corresponds to $\mathcal{E}_\lambda$, a K_λ-lisse sheaf on Y.

This is the λ-companion of $\mathcal{E}_{\lambda'}$. //

Rmk :- Drinfeld's Existence Thm for case of curves follows immediately from L. Lafforgue's work on function field Langlands correspondence

$\sigma : \overline{\mathbb{Q}}_\ell \simeq \overline{\mathbb{Q}}_{\ell'}$

$\mathcal{E}_{\lambda'}$ defines $f_{\lambda'} \in LS_r^{E_{\lambda'}}(Y)$, then

$\sigma \circ f_{\lambda'} \in \widetilde{LS}_r^{E_\lambda}(Y)$, & then show that $\sigma \circ f_{\lambda'} \in LS_r^{E_\lambda}$.