Non-abelian Hodge theory: from $\mathbb{C}$ to $\mathbb{F}_p$

The goal of this learning seminar is to understand the classical results on the correspondence between Higgs bundles, flat bundles and local systems on algebraic varieties. Our main references for the first part are the papers [Sim92], [Sim94], [Sim95], and [Sim97]. The second part will be devoted to the correspondence over $\mathbb{F}_p$ and the goal will to be to understand the integrality of rigid local systems following [EG18] and the results of [EG20].

1. Riemann-Hilbert correspondence. State the equivalence between representations of the fundamental group and vector bundles with integrable connections. Give examples with ordinary differential equations. If time permits, talk about the non-compact case, regular singularities. The reference for this talk are [Del70] and its account by Katz [Kat80].

2. Higgs bundles and local systems. Introduce the relation between local systems and Higgs bundles following [Sim92, §1]. Introduce Harmonic bundles and give an idea of the proof of Theorem 1. You can also have a look at [LP91, §§2, 3]. Can be split into two talks.

3. Moduli spaces I. Introduce the moduli space of Higgs bundles, local systems, and flat bundles following [Sim92, P.17]. Sketch their construction following [LP91, §5] and [Sim95, §5,§6], using the results of [Sim94] which will be explained in the next talk.

4. Moduli spaces II. Explain the general construction of moduli spaces of sheaves following [Sim94] and make the connection with the previous talk to construct $\mathcal{M}_{\mathrm{Betti}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{dR}}$.

5. Comparison of moduli spaces. The goal of this talk is to explain the different homeomorphisms between $\mathcal{M}_{\mathrm{Betti}}$, $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{dR}}$ and their regularity following [Sim92, §2], [Sim94, §5], and [Sim95, §7].

6. Extensions to semistable Higgs bundles. Explain the extension to a correspondence between semistable Higgs bundles (with vanishing Chern classes) and reducible representations following [Sim92, §3].

7. $\mathbb{C}$-PVHS and fundamental groups of Kähler manifolds. Follow [Sim92, §4] to introduce complex variations of Hodge structures as $\mathbb{C}^*$ fixed points, see also [LP91, §6]. Introduce rigid representations and the discussion about the fundamental groups of Kähler manifolds. Integrality of rigid representations will be discussed in another talk but you can state it here.

8. Hodge structures on the fundamental group. Explain the construction of Hodge structures on the fundamental group following §5 and §6 of [Sim92].

9. Flat connections in positive characteristic. Present the theory of of flat connections over perfect field following [EG20, §§2.2, 2.3] and introduce the notion of p-curvature. Present Ogus–Vologosky results results relating flat connections and with nilpotent p-curvature and nilpotent Higgs bundles, [EG20, §2.4, §2.5]. Present the connection to crystals [EG20, §2.6].

10. Rigid systems are p-adically nilpotent. Follow [EG20] to prove that rigid local systems are p-adically nilpotent.

11. Rigid systems are integral. Present [EG18] proving that cohomologically rigid local systems are integral.

References

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