

ON THE NON-ABELIAN HODGE LOCUS I

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ABSTRACT. We partially resolve conjectures of Deligne and Simpson concerning \mathbb{Z} -local systems on quasi-projective varieties that underlie a polarized variation of Hodge structure. For local systems of “compact type”, we prove (1) a relative form of Deligne’s finiteness theorem, for any family of quasi-projective varieties, and (2) algebraicity of the corresponding non-abelian Hodge locus.

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1. INTRODUCTION

Let $\Pi = \pi_1(Y, *)$ be the fundamental group of a smooth quasi-projective variety. A fundamental result of Deligne [Del87] is that, up to conjugacy, only finitely many representations $\rho: \Pi \rightarrow \mathrm{GL}_n(\mathbb{Z})$ underlie a \mathbb{Z} -polarized variation of Hodge structure (\mathbb{Z} -PVHS) over Y .

We are primarily concerned with two questions here:

- (Q1) If instead, one has a family $\mathcal{Y} \rightarrow \mathcal{S}$ of smooth quasi-projective varieties, then do only finitely many representations of Π underlie a \mathbb{Z} -PVHS on some (unspecified) Y_s ?
- (Q2) In the relative moduli space of flat connections $M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$, is the locus underlying a \mathbb{Z} -PVHS algebraic?

The first question is due to Deligne [Del87, Question 3.13]. Simpson [Sim97, Conjecture 12.3] posed and made progress on the second question, proving that this locus is analytic.

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Note that the two questions are related: Q2 implies Q1 because an algebraic set will have only finitely many connected components, and the representation of Π is locally constant along a locus of flat connections underlying a \mathbb{Z} -PVHS.

We answer both questions, under the following assumption:

Definition 1.1. Let $\rho: \Pi \rightarrow \mathrm{GL}_n(\mathbb{Z})$ be a group representation and let $\mathbf{H}(\mathbb{R})$ denote the Zariski-closure of $\mathrm{im}(\rho)$ in $\mathrm{GL}_n(\mathbb{R})$. Let

$$\mathbf{H}(\mathbb{Z}) := \mathbf{H}(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{Z}).$$

We say that ρ is of *compact type* if $\mathbf{H}(\mathbb{Z}) \subset \mathbf{H}(\mathbb{R})$ is cocompact.

Theorem 1.2. *Let $\mathcal{Y} \rightarrow \mathcal{S}$ be a family of smooth quasi-projective varieties. Then the flat connections in $M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$ underlying a \mathbb{Z} -PVHS with compact type monodromy form an algebraic subvariety.*

*In particular, if $\Pi = \pi_1(Y_0, *)$ for some $0 \in \mathcal{S}$, then only finitely many compact type representations of Π underlie a \mathbb{Z} -PVHS on some fiber Y_s , up to an appropriate identification.*

The appropriate identification mentioned in the theorem above is explained in Definition 3.1.

A useful feature of the compact type case is that, due to Griffiths' generalization of the Borel extension theorem, a \mathbb{Z} -PVHS on Y_s extends over a projective, simple normal crossings compactification \overline{Y}_s . We may stratify \mathcal{S} into loci over which \mathcal{Y} admits a relative simple normal crossings compactification. This is achieved by induction on dimension, applying resolution of singularities over the generic point of each stratum. Note that both Q1 and Q2 are Zariski-local on \mathcal{S} . So both Q1 and Q2 (in the compact type case) reduce to families of smooth projective varieties. Hence, for the remainder of the paper, we assume that $\mathcal{Y} \rightarrow \mathcal{S}$ is smooth projective, and \mathcal{S} is quasiprojective.

Our result also answers a question asked by Landesman and Litt [LL22, Question 8.2.1], again in the cocompact case.

1.1. The non-abelian Hodge locus. In [Sim95], Simpson defined $M_{\mathrm{Dol}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$, resp. $M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$, the relative Dolbeault space, resp. the relative de Rham space: $M_{\mathrm{Dol}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$ is a relative moduli space of semistable Higgs bundles (\mathcal{E}, ϕ) with vanishing rational Chern classes and $M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$ is a relative moduli space of vector bundles with flat connection.

Let $N_{\mathrm{Dol}} \subset M_{\mathrm{Dol}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$ be the fixed point set of the \mathbb{G}_m -action $(\mathcal{E}, \phi) \mapsto (\mathcal{E}, t\phi)$ and let N_{dR} be its image in $M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$ under the non-abelian Hodge correspondence. Define

$$M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n(\mathbb{Z})) \subset M_{\mathrm{dR}}(\mathcal{Y}/\mathcal{S}, \mathrm{GL}_n)$$

to be the flat bundles having integral monodromy representations on a fiber of $\mathcal{Y} \rightarrow \mathcal{S}$. Following Simpson [Sim97, §12], we define the non-abelian Hodge locus, called the Noether-Lefschetz locus in *loc. cit.*,

$$\text{NHL}(\mathcal{Y}/\mathcal{S}, \text{GL}_n) := N_{\text{dR}} \cap M_{\text{dR}}(\mathcal{Y}/\mathcal{S}, \text{GL}_n(\mathbb{Z})).$$

These are the flat vector bundles underlying a \mathbb{Z} -PVHS.

The precise phrasing of Simpson's conjecture on the non-abelian Hodge locus [Sim97, Conjecture 12.3] is:

Conjecture 1.3. *NHL($\mathcal{Y}/\mathcal{S}, \text{GL}_n$) is an algebraic variety and the inclusions into $M_{\text{dR}}(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ and $M_{\text{Dol}}(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ are algebraic morphisms.*

When the base \mathcal{S} is projective, Conjecture 1.3 is a consequence of Serre's GAGA theorem, as explained in [Sim97, Corollary 12.2]. Furthermore, we have a decomposition

$$\text{NHL}(\mathcal{Y}/\mathcal{S}, \text{GL}_n) = \text{NHL}_c(\mathcal{Y}/\mathcal{S}, \text{GL}_n) \sqcup \text{NHL}_{nc}(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$$

according to whether the monodromy representation is compact type or non-compact type. Our main Theorem 1.2 proves $\text{NHL}_c(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ is algebraic. The case of non-compact type monodromy will be explored in future work of the authors.

1.2. Strategy of the proof. The proof splits into two parts, each of a rather different nature. First, Q1 is proven, using techniques from hyperbolic and metric geometry. Then, the resolution of Q1 is used to prove Q2, by applying more algebraic techniques.

1.2.1. Finiteness of monodromy representations. By slicing by hyperplanes, Q1 can be reduced to the case of curves, and in turn, to the universal family $\mathcal{C}_g \rightarrow \mathcal{M}_g$, for $g \geq 2$. Let

$$\Phi: C \rightarrow \Gamma \backslash \mathbb{D}$$

be the period map associated to a \mathbb{Z} -PVHS of compact type on some $C \in \mathcal{M}_g$. Every genus g Riemann surface C admits a hyperbolic metric, and Deligne's finiteness result relies critically on the length-contracting property of Φ [Gri70, 10.1]. But as the curve $C \in \mathcal{M}_g$ degenerates, the length-contracting property alone ceases to be useful: The monodromy representation will be determined by curves whose hyperbolic geodesic representatives have length growing to infinity.

These geodesics grow in length as they cross hyperbolic collars forming near the nodes of the limiting curve. Thus, our key lemma, see Proposition 3.16, is that the image of a length-decreasing harmonic map from a hyperbolic collar to a symmetric space is bounded, even as the transverse length to the collar grows to infinity.

1.2.2. *Algebraicity of $\text{NHL}_c(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$.* Our main tool for proving Q2 is an algebraization theorem for Douady spaces of Griffiths transverse, compact analytic subspaces of arithmetic manifolds $\Gamma \backslash \mathbb{D}$ which parameterize period images of \mathbb{Z} -PVHS's with big monodromy.

The local analytic branches of the non-abelian Hodge locus are the isomonodromic deformations of a fixed integral representation which underlie a \mathbb{Z} -PVHS. Hence the fibers of $\mathcal{Y} \rightarrow \mathcal{S}$ along such a branch admit a period map $\Phi_s: Y_s \rightarrow \Gamma \backslash \mathbb{D}$. The images $\Phi_s(Y_s)$ of such period maps are closed analytic spaces, tangent to the Griffiths distribution on $\Gamma \backslash \mathbb{D}$, of bounded volume with respect to the Griffiths line bundle.

When $\Gamma \backslash \mathbb{D}$ is compact, we prove that such period images are parameterized by a product of a compact Moishezon space and a sub-period domain of \mathbb{D} accounting for the factors where the monodromy representation is finite. We identify the non-abelian Hodge locus as a relative space of maps of bounded degree from \mathcal{Y}/\mathcal{S} to the universal family over the Moishezon space.

Then Q2 follows for period maps with a fixed target $\Gamma \backslash \mathbb{D}$. The set of such arithmetic quotients $\Gamma \backslash \mathbb{D}$ which can appear is bounded using the resolution of Q1. Theorem 1.2 follows.

1.3. **Organization of the paper.** In §2 we recall some background results on polarized variations of Hodge structures and period domains. In §3, we prove the relative version of Deligne's finiteness theorem, for representations of compact type. Then in §4, we introduce the Douady and Barlet spaces in the general context of polarized distribution manifolds and prove their key properties. In §5, we prove algebraicity of the compact type non-abelian Hodge locus.

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2. VARIATIONS OF HODGE STRUCTURES

We recall in this section some background results on polarized variations of Hodge structures and fix notations. Our main references are [GGK12, Kli17].

2.1. **Monodromy and Mumford-Tate group.** Let Y be a complex manifold and let $\mathbb{V} := (V_{\mathbb{Z}}, F^{\bullet}, \psi)$ be a polarized variation of Hodge structure of weight k on Y . Here $V_{\mathbb{Z}}$ is the \mathbb{Z} -local system, F^{\bullet} is the

Hodge filtration on $V_{\mathbb{Z}} \otimes \mathcal{O}_Y$, and ψ is the polarization. Let \mathbf{G} be the *generic Mumford-Tate* group of the variation and let \mathbf{H} be the algebraic monodromy group of \mathbb{V} .

We recall that \mathbf{G} is the Mumford–Tate group of the Hodge structure over a very general point of Y and \mathbf{H} is defined as follows: fix a base point $* \in Y$ and denote the monodromy representation associated to the local system $V_{\mathbb{Z}}$ by $\rho: \pi_1(Y, *) \rightarrow \mathrm{GL}(V_{\mathbb{Z},*})$, which lands in the subgroup $\mathrm{Sp}(V_{\mathbb{Z},*})$ or $\mathrm{O}(V_{\mathbb{Z},*})$ depending on the parity of the weight. Then \mathbf{H} is the identity component of the \mathbb{Q} -Zariski closure of the image of ρ . The groups \mathbf{G} and \mathbf{H} are reductive algebraic groups over \mathbb{Q} and by a classical theorem of Deligne and André, \mathbf{H} is a normal subgroup of $\mathbf{G}^{\mathrm{der}}$, the derived group of \mathbf{G} . It follows that we have a decomposition over \mathbb{Q} of the adjoint groups $\mathbf{G}^{\mathrm{ad}} = \mathbf{H}^{\mathrm{ad}} \times \mathbf{H}'$.

Let \mathbb{D} be the *Mumford-Tate domain* associated to the variation. It is a complex analytic space, homogeneous for $G := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ and it can be identified with a quotient G/K where $K \subset G$ is a compact subgroup. In terms of Hodge structures, K is the real subgroup preserving each $V^{p,q}$ and the Hodge pairing between $V^{p,q}$ and $V^{q,p}$. From the theory of symmetric spaces, \mathbb{D} is an analytic open subset of the *compact dual* \mathbb{D}^{\vee} , a projective subvariety of a symplectic or an orthogonal flag variety with specified Mumford-Tate group. There then exists a parabolic subgroup $P \subset G_{\mathbb{C}}$ such that $\mathbb{D}^{\vee} = G_{\mathbb{C}}/P$ and $P \cap G = K$.

The variation of Hodge structure \mathbb{V} on Y is completely described by its holomorphic *period map*:

$$\Phi : Y \rightarrow \Gamma \backslash \mathbb{D},$$

where $\Gamma \subset \mathbf{G}(\mathbb{Z})$ is a finite index subgroup preserving $V_{\mathbb{Z}}$ such that the monodromy representation factors through Γ . Up to taking a finite étale cover of Y , we can assume that Γ is neat, hence acting freely on \mathbb{D} . Then the quotient $X_{\Gamma} := \Gamma \backslash \mathbb{D}$ is a connected complex manifold, called a *connected Hodge manifold*, see [Kli17, Definiton 3.18]. It is the classifying space of polarized \mathbb{Z} -Hodge structures on $V_{\mathbb{Z}}$ whose generic Mumford-Tate group is contained in \mathbf{G} , with level structure corresponding to Γ .

In general, X_{Γ} is not algebraic unless \mathbb{D} is Hermitian symmetric. In that case, X_{Γ} is in fact quasiprojective by the Baily-Borel theorem [BB66], and Φ is algebraic by the Borel hyperbolicity theorem [Bor72], see also [BBT23] for another proof.

We can furthermore refine the period map by taking into account the algebraic monodromy group \mathbf{H} . The Mumford–Tate domain \mathbb{D} decomposes according to the decomposition $\mathbf{G}^{\mathrm{ad}} = \mathbf{H}^{\mathrm{ad}} \times \mathbf{H}'$ of adjoint

groups as $\mathbb{D} = \mathbb{D}_H \times \mathbb{D}_{H'}$ where \mathbb{D}_H is an $H := \mathbf{H}^{\text{ad}}(\mathbb{R})^+$ -homogeneous space. Up to a finite étale cover of Y , we can assume that the lattice Γ decomposes as $\Gamma = \Gamma_H \times \Gamma_{H'}$ where $\Gamma_H \subset \mathbf{H}(\mathbb{Z})$ and $\Gamma_{H'} \subset \mathbf{H}'(\mathbb{Z})$ are arithmetic subgroups. Then the projection of the period map Φ is constant on the second factor and hence the period map takes the following shape:

$$\Phi : S \rightarrow \Gamma_H \backslash \mathbb{D}_H \times \{t_Y\} \hookrightarrow \Gamma \backslash \mathbb{D},$$

where t_Y is a Hodge generic point in $\mathbb{D}_{H'}$. Then $X_{\Gamma_H} \times \mathbb{D}_{H'}$ serves as a classifying space of \mathbb{Z} -PVHS on a lattice isometric to $V_{\mathbb{Z},*}$ whose generic Mumford-Tate group is contained in \mathbf{G} , and whose monodromy factors through Γ_H . The classifying map for such a variation factors through the inclusion of $X_{\Gamma_H} \times \{t\}$ for some fixed t .

2.2. Automorphic vector bundles. Given any complex linear representation of $\chi: K \rightarrow \text{GL}(W)$, there is an associated holomorphic vector bundle $G \times_K W \rightarrow \mathbb{D}$ which is Γ -equivariant and hence descends to a holomorphic vector bundle over X_Γ . In particular, for any p , the natural representation of K on $V^{p,q}$ defines a holomorphic vector bundle on \mathbb{D} which is identified to the p th graded piece F^p/F^{p+1} of the Hodge filtration.

Any character $\chi: K \rightarrow \mathbb{S}^1$ defines an equivariant holomorphic line bundle $L_\chi \rightarrow \mathbb{D}$. For example, if the character χ is the determinant of the action of K on $V^{p,q}$, we get the line bundle $L_p = \det(F^p/F^{p+1})$. Any such equivariant line bundle admits a unique (up to scaling) left G -invariant hermitian metric

$$h: L_\chi \otimes \bar{L}_\chi \rightarrow \mathbb{C}.$$

Definition 2.1. The *Griffiths bundle* $L \rightarrow X_\Gamma$ is defined by

$$L := \bigotimes_{p \geq 0} (L_p)^{\otimes p}.$$

We denote the descent to X_Γ of the equivariant vector bundles F^p , line bundles L_p , and the hermitian metrics h by the same symbols.

Remark 2.2. While F^\bullet defines a filtration of holomorphic vector bundles over X_Γ , it does not, in general, define a \mathbb{Z} -PVHS over X_Γ for the tautological local system, because Griffiths' transversality fails.

Recall that the tangent space to the Grassmannian at a subspace $W \subset V$ is canonically isomorphic to $\text{Hom}(W, V/W)$. Since \mathbb{D} is an open subset of a flag variety \mathbb{D}^\vee , we have an inclusion

$$T\mathbb{D} \subset \bigoplus_p \text{Hom}(F^p, V/F^p).$$

The Griffiths transversality condition on a \mathbb{Z} -PVHS over Y implies that the differential $d\Phi$ of the period map lands in an appropriate subspace of the tangent space:

Definition 2.3. The *Griffiths distribution* $T^{\parallel} \subset T\mathbb{D}$ is the holomorphic subbundle of the tangent bundle defined by

$$T_{F^\bullet}^{\parallel} := T_{F^\bullet}\mathbb{D} \cap \bigoplus_p \text{Hom}(F^p, F^{p-1}/F^p).$$

It is G -invariant, and so descends to a distribution in TX_Γ which we also denote by T^{\parallel} .

The following proposition is [Gri70, Prop. 7.15].

Proposition 2.4. *Let $\omega_L := \frac{i}{2\pi} \partial\bar{\partial} \log h \in \Lambda^{1,1}(X_\Gamma, \mathbb{R})$ be the curvature form of the Hermitian metric h on L . Then $\omega_L|_{T^{\parallel}}$ is positive definite, in the sense that for any nonzero $v \in T_{\mathbb{R}}^{\parallel}$,*

$$\omega_L(v, Jv) > 0.$$

From this, Griffiths concluded that the image of Φ admits a holomorphic line bundle with positive curvature. In particular, using a generalization of the Kodaira embedding theorem due to Grauert, he proved, see [Gri70, Thm. 9.7]:

Theorem 2.5. *Let $\Phi: Y \rightarrow X_\Gamma$ be the period map of a \mathbb{Z} -PVHS on a compact, complex manifold Y . Then $\Phi(Y)$, with its reduced analytic space structure, is a projective algebraic variety.*

It seems though, that some conditions of Grauert's theorem do not always hold. In particular, it may not be the case that $T\Phi(Y) \subset T^{\parallel}$ due to singularities on $\Phi(Y)$. An independent proof and strengthening to the non-compact case was given in [BBT23, Thm. 1.1].

3. BOUNDEDNESS OF MONODROMY REPRESENTATIONS

Let \mathcal{S} be a smooth connected quasi-projective complex algebraic variety and let $\pi: \mathcal{Y} \rightarrow \mathcal{S}$ be a smooth projective morphism. Our goal in this section is to prove that there are only finitely representations $\pi_1(Y_0) \rightarrow \text{GL}_n(\mathbb{Z})$, up to conjugacy, which underlie a \mathbb{Z} -PVHS of compact type on some fiber Y_s of $\pi: \mathcal{Y} \rightarrow \mathcal{S}$, after an identification $\pi_1(Y_0, *) \simeq \pi_1(Y_s, *)$ moving the base point in the universal family.

Slicing \mathcal{Y} by hyperplanes, we can apply the Lefschetz theorem to reduce to the case of a relative smooth projective curve $\mathcal{C} \rightarrow \mathcal{S}$ (passing to a finite Zariski cover of \mathcal{S} if necessary). Then, we may as well assume that $\mathcal{S} = \mathcal{M}_g$ and that $\mathcal{C} = \mathcal{C}_g$ is the universal curve. This is a

particular instance of a question asked by Deligne, for representations of compact type, see [Del87, Question 3.13].

We can decompose \mathcal{M}_g into two subsets, the *thick* part and the *thin* part. Let $C \in \mathcal{M}_g$ be a Riemann surface of genus g and let $\gamma \in \pi_1(C)$ be a loop. Then C has a unique hyperbolic metric of constant curvature -1 , in the conformal equivalence class defined by the complex structure on C . There is a unique representative of the free homotopy class of γ which is a hyperbolic geodesic for this metric. Let $\ell_C(\gamma)$ denote its hyperbolic length. Then, the thick part of \mathcal{M}_g is a compact subset $\mathcal{M}_g^{\geq \epsilon} \subset \mathcal{M}_g$ consisting of all curves $C \in \mathcal{M}_g$, for which $\ell_C(\gamma) \geq \epsilon$ for all $\gamma \in \pi_1(C)$, see [Mum71, Cor. 3].

First, we deal with the thick part. The proof follows, nearly verbatim, Deligne's proof [Del87] of finiteness of monodromy representations underlying \mathbb{Z} -PVHS on a fixed curve C .

Definition 3.1. Let Π_g be the surface group:

$$\Pi_g = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g \mid \prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \rangle.$$

Fix a pointed Riemann surface $(C_0, *_{0}) \in \mathcal{M}_{g,1} = \mathcal{C}_g$ of genus g and an isomorphism $\pi_1(C_0, *_{0}) \simeq \Pi_g$. Then a path in \mathcal{C}_g connecting $(C_0, *_{0})$ to $(C, *)$ produces an identification

$$\pi_1(C, *) \simeq \pi_1(C_0, *_{0}) \simeq \Pi_g.$$

We call such an identification *admissible*.

Two such admissible identifications can be compared by an automorphism of Π_g induced by a path from $(C_0, *_{0})$ to itself, i.e., an element of $\pi_1(\mathcal{C}_g, (C_0, *_{0}))$. The paths connecting $(C_0, *_{0})$ to itself keeping $C_0 \in \mathcal{M}_g$ constant in moduli induce the inner automorphisms $\text{Inn}(\Pi_g)$. The paths connecting $(C_0, *_{0})$ to itself by moving $C_0 \in \mathcal{M}_g$ in moduli induce an inclusion of the mapping class group $\text{Mod}_g \subset \text{Out}(\Pi_g)$ as an index 2 subgroup of $\text{Out}(\Pi_g)$, corresponding to orientation. So any isomorphism $\pi_1(C, *) \simeq \Pi_g$ induced by an oriented homeomorphism $(C, *) \rightarrow (C_0, *_{0})$ is admissible.

Proposition 3.2. *Let $\rho: \pi_1(C, *) \rightarrow \text{GL}_n(\mathbb{Z})$ be the monodromy representation of a \mathbb{Z} -PVHS of rank n on some $C \in \mathcal{M}_g^{\geq \epsilon}$ in the thick part of the moduli space. There is an admissible identification $\pi_1(C, *) \simeq \Pi_g$ identifying ρ with one of a finite list of representations $\Pi_g \rightarrow \text{GL}_n(\mathbb{Z})$, up to conjugacy.*

Proof. A theorem of Procesi [Pro76] states that, up to conjugacy, a semisimple representation $\rho: \Pi \rightarrow \text{GL}_n(\mathbb{C})$ from any finitely generated

group Π is uniquely determined by the function

$$\begin{aligned} \{1, \dots, m\} &\rightarrow \mathbb{C} \\ j &\mapsto \text{tr}(\rho(\delta_j)) \end{aligned}$$

for some finite generating set $(\delta_j)_{1 \leq j \leq m}$ of the group, where m depends only on Π and n .

Choose, for once and all, such a generating set $\delta_1, \dots, \delta_m$ for the surface group Π_g . We call this set the *Procesi generators*. Deligne's argument relies on the famous length-contracting property of period maps, due to Griffiths [Gri70, 10.1]:

Theorem 3.3. *There is a G -invariant metric on $\mathbb{D} = G/K$ for which any holomorphic, Griffiths transverse map $\Delta \rightarrow \mathbb{D}$ from a holomorphic disk is length-contracting for the hyperbolic metric on Δ .*

Choose a cover of $\mathcal{M}_g^{\geq c}$ by a finite number of contractible, compact subsets $\{V_i\}_{i \in I}$. Choosing a base-point consistently over V_i , the fundamental groups $\pi_1(C, *)$ for all $C \in V_i$ are uniquely identified, by the contractibility of V_i . Let $\pi_1(C, *) \simeq \Pi_g$ be an admissible identification, and consider the resulting family of Procesi generators $(\delta_j)_{1 \leq j \leq m}$ of $\pi_1(C, *)$ for $C \in V_i$. Then $\ell_C(\delta_j)$ is a continuous function on V_i which, by compactness, is bounded. Hence there exists some M for which $\ell_C(\delta_j) \leq M$ for all $1 \leq j \leq m$ and all $C \in V_i$.

Suppose that $\rho: \pi_1(C, *) \rightarrow \Gamma$ is the monodromy representation of a \mathbb{Z} -PVHS for some $C \in V_i$. Then, applying Theorem 3.3 to the hyperbolic uniformization $\Delta \rightarrow C$, we conclude that there exists a point $x \in \mathbb{D}$ for which $d_{\mathbb{D}}(x, \rho(\delta_j) \cdot x) \leq M$. In particular, x may be taken as the period image of some point on the lift to Δ of the hyperbolic geodesic representing δ_j . Thus, $\rho(\delta_j)$ has bounded translation length, and thus, bounded trace, by Lemma 3.4. See [Del87, Corollaire 1.9].

Lemma 3.4. *Let $g \in G$ and suppose that $d_{\mathbb{D}}(x, g \cdot x) \leq M$ for some $x \in \mathbb{D}$. There is a bound N , depending only on \mathbb{D} and M , for $\text{tr}(g)$.*

Proof. Fix a base point $x_0 \in \mathbb{D}$ and choose some $h \in G$ for which $h \cdot x_0 = x$. Then

$$d_{\mathbb{D}}(x, g \cdot x) = d_{\mathbb{D}}(h \cdot x_0, gh \cdot x_0) = d_{\mathbb{D}}(x_0, h^{-1}gh \cdot x_0) \leq M.$$

Since the closed ball of radius M around x_0 is compact, and the map $G \rightarrow G/K = \mathbb{D}$ has compact fibers, we conclude that the set

$$\{k \in G \mid d_{\mathbb{D}}(x_0, k \cdot x_0) \leq M\}$$

is compact. As the trace is a continuous function, we conclude that tr is bounded on the above set, in terms of M alone. We conclude that $\text{tr}(h^{-1}gh) = \text{tr}(g)$ is bounded. \square

Hence the trace $\text{tr}(\rho(\delta_j))$ is bounded in terms of $\ell_C(\delta_j) \leq M$, and hence it is bounded globally on V_i by some integer N . It is furthermore an integer, as ρ lands in $\text{GL}_n(\mathbb{Z})$. Since there are only finitely many possibilities for a map $\{1, \dots, m\} \rightarrow \{-N, \dots, N\}$, there are only finitely many monodromy representations achieved for a \mathbb{Z} -PVHS over any $C \in V_i$. Since the indexing set I is finite, we conclude the same over $\mathcal{M}_g^{\geq \epsilon}$, up to conjugacy. \square

Thus, it remains to consider the thin part of the moduli space $\mathcal{M}_g^{< \epsilon}$ consisting of smooth curves with systole less than ϵ .

Definition 3.5. A *collar* A is the Riemann surface with boundary

$$\left\{ r e^{i\theta} \in \mathbb{H} \mid \begin{array}{l} 1 \leq r \leq r_0 \\ \theta_0 \leq \theta \leq \pi - \theta_0 \end{array} \right\} / \sim$$

where $\tau \sim r_0 \tau$. A *half-collar* is the subregion where $\theta \leq \frac{\pi}{2}$.

We recall a famous result due to Keen [Kee74]. The sharpness is due to Buser [Bus78, Thm. C].

Lemma 3.6 (Collar Lemma). *Every simple closed geodesic γ of length ℓ on a complete hyperbolic surface C is contained a hyperbolic collar $A_\gamma \subset C$ of transverse length $\ln\left(\frac{e^{\ell/2} + 1}{e^{\ell/2} - 1}\right)$. Furthermore, any two such collars associated to disjoint geodesics are disjoint.*

The function

$$F(\ell) := \ln\left(\frac{e^{\ell/2} + 1}{e^{\ell/2} - 1}\right)$$

satisfies $\lim_{\ell \rightarrow 0^+} F(\ell) = +\infty$, and is monotonically decreasing towards zero as $\ell \rightarrow +\infty$. In terms of the constants r_0, θ_0 of Definition 3.5, we have $r_0 = e^\ell$ and $\theta_0 = \cos^{-1}(e^{-\ell/2})$. The perimeter of a boundary component of this collar is $\ell(1 - e^{-\ell})^{-1/2}$. More generally, the formula is $\text{Per}(A) = \frac{\ell}{\sin(\theta_0)}$.

For $C \in \mathcal{M}_g^{< \epsilon}$, let $\{\gamma_1, \dots, \gamma_k\}$ be the set of simple closed curves of hyperbolic length less than ϵ . Choosing ϵ smaller than the fixed point of the function $F(\ell)$, we conclude that all such curves are disjoint. So $k \leq 3g - 3$, with equality when $\{\gamma_1, \dots, \gamma_k\}$ form a pair-of-pants decomposition of C .

We now recall the result of Bers [Ber74, Ber85]:

Theorem 3.7. *There exists a constant B_g for which any hyperbolic surface of genus g admits a pair-of-pants decomposition, all of whose curves have length bounded above by B_g .*

By choosing ϵ so that $F(\epsilon) > B_g$, any such pair of pants decomposition *must* contain all simple closed curves of length less than ϵ , as any pair of pants decomposition not including γ_j would include a curve that crossed the collar of Lemma 3.6. Thus, we may extend the set $\{\gamma_1, \dots, \gamma_k\}$ to a full pair of pants decomposition $\{\gamma_1, \dots, \gamma_{3g-3}\}$ in such a way that $\ell_C(\gamma_j) \leq B_g$ for all j .

A pair of pants $P(\ell_1, \ell_2, \ell_3)$ is uniquely specified by the three cuff lengths $\ell_1, \ell_2, \ell_3 \in \mathbb{R}^+$. Two adjacent pairs of pants, glued along γ_i in a pants decomposition of C , contain a collar A_{γ_i} of transverse length at least $F(\ell(\gamma_i))$, but with the bounds B_g on the chosen pairs of pants, we can do better:

Proposition 3.8. *Suppose $P(\ell_1, \ell_2, \ell_3)$ is a pair of pants with $\ell_i \leq B_g$. There exists a constant $C_g > 0$ for which each cuff is contained in a half-collar of perimeter at least C_g .*

Proof. The key is to observe that even as $\ell_i \rightarrow 0$, the geometry of $P(\ell_1, \ell_2, \ell_3)$ converges, with the cuff γ_i limiting to a hyperbolic cusp, and the half-collars limiting to the horoball neighborhoods. Therefore $P(\ell_1, \ell_2, \ell_3)$ makes sense, for all $0 \leq \ell_i \leq B_g$. For each such surface, each cuff (resp. cusp) has a definite half-collar (resp. horoball) neighborhood of non-zero perimeter. So the maximal such perimeter is a continuous function on the compact set $[0, B_g]^3$, never equal to zero, and thus has a nonzero minimum. \square

Definition 3.9. The *truncated pair of pants* $P^o(\ell_1, \ell_2, \ell_3)$ (Fig. 1) is the complement of the half-collars in $P(\ell_1, \ell_2, \ell_3)$ with perimeter C_g .

If $\ell_i \geq C_g$ we need not truncate the corresponding cuff. Making C_g sufficiently small, we may assume that the (up to) three half-collars we cut are disjoint.

Remark 3.10. The issue with truncating pairs of pants by the universal (half-)collar of Lemma 3.6 is that the limit of its perimeter is

$$\lim_{\ell \rightarrow 0^+} \ell(1 - e^{-\ell})^{-1/2} = 0.$$

So the universal collar is not sufficient to bound the geometry (e.g. as measured by the hyperbolic diameter) of the truncated pair of pants, when $\ell \rightarrow 0$. Hence the need for Proposition 3.8.

Consider the three seam geodesics connecting cuffs of $P(\ell_1, \ell_2, \ell_3)$. These seams intersect each boundary component of $P^o(\ell_1, \ell_2, \ell_3)$ and $P(\ell_1, \ell_2, \ell_3)$ at two points. We call these (six total) points the *distinguished boundary points* of $P^o(\ell_1, \ell_2, \ell_3)$ and $P(\ell_1, \ell_2, \ell_3)$. Note that

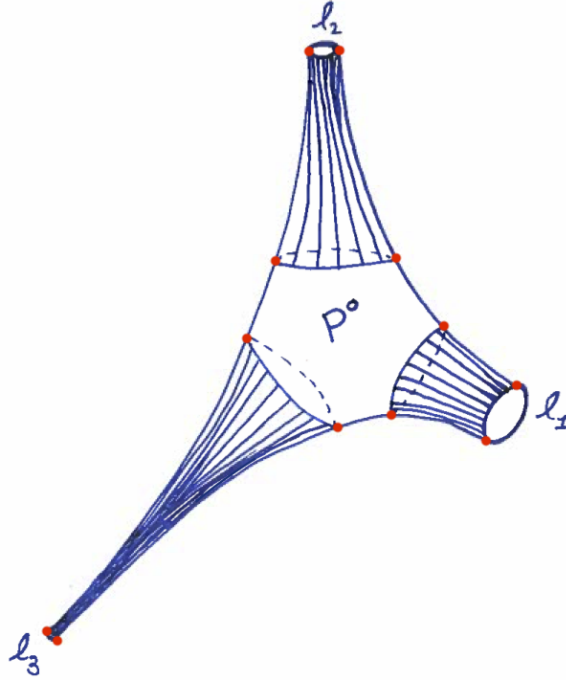


FIGURE 1. A hyperbolic pair of pants $P(l_1, l_2, l_3)$, and its truncation $P^o(l_1, l_2, l_3)$. Distinguished boundary points on P , P^o are shown in red.

the distinguished points on a given cuff are diametrically opposite. So when two pants are glued, the four total distinguished points on the cuff alternate which pants decomposition they come from.

Proposition 3.11. *Suppose $l_1, l_2, l_3 \leq B_g$ for some constant B_g . Let μ be a homotopy class of paths on the truncated pair of pants $P^o(l_1, l_2, l_3)$, terminating at two distinguished points of the boundary. Then, μ has a representative of bounded distance D_μ independent of l_i .*

Proof. The minimal length representative of μ on any truncated pair of pants is finite, and furthermore, this minimal length is continuous as one varies the l_i . This holds even when some $l_i = 0$, corresponding to cusped pairs of pants. The proposition follows because (l_1, l_2, l_3) is restricted to lie in the compact set $[0, B_g]^3$. \square

Thus, the length-contracting property will be useful inside the truncated pair of pants. But the following issue arises: As the core curve of a collar A shrinks, the length of any transverse geodesic grows. So the length-contracting property ceases to be useful on the collars, at least on its own. Thus, the next proposition is absolutely crucial.

Proposition 3.12. *Let (M, g) be a simply connected Riemannian manifold with non-positive sectional curvature and let $\Psi: A \rightarrow M$ be a length-contracting, harmonic map from a collar. Assume the perimeter of A is bounded above by C_g . Then, the image of A is contained in a ball of bounded radius $\frac{1}{2}(C_g + \pi)$.*

Proof. Recall that the collar A is parameterized by polar coordinates $(r, \theta) \in \mathbb{H}$ (Def. 3.5) where $r \in \mathbb{R}_{>0}/(r_0)\mathbb{Z}$ is the circle coordinate on the collar, and $\theta \in [\theta_0, \pi - \theta_0]$ is the transverse coordinate. Let p_0 be a point on the boundary component of A defined by $\theta = \theta_0$. Define

$$\begin{aligned} d: A &\rightarrow \mathbb{R}_{\geq 0} \\ q &\mapsto \text{dist}_g(\Psi(p_0), \Psi(q)). \end{aligned}$$

As M has non-positive sectional curvature and $\pi_1(M)$ is trivial, the distance function $\text{dist}_g(\Psi(p_0), \cdot): M \rightarrow \mathbb{R}_{\geq 0}$ is convex. The composition of a convex function with a harmonic function is subharmonic, so the function d is subharmonic. Let $S^1(q)$ denote the circle containing $q \in A$ (varying only the coordinate r) and define

$$d_{\max}(\theta) := \max_{q' \in S^1(q)} d(q'),$$

which is now circularly symmetric, and so is only a function of θ . It suffices to prove that d_{\max} is bounded.

Since the rotation action on A is conformal, the pullback along the rotation action of $d(q)$ is subharmonic. Thus $d_{\max}(\theta)$, as a maximum of subharmonic functions, is also subharmonic.

The hyperbolic metric is $y^{-2}(dx^2 + dy^2)$ on the upper half-plane, and so $g_{\text{hyp}}(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}) = 1$ when $\theta = \frac{\pi}{2}$. So the length-contracting property, along with the triangle inequality, implies

$$\begin{aligned} \left| \frac{\partial}{\partial \theta}(d(q)) \right| &\leq 1 \text{ when } \theta(q) = \frac{\pi}{2}, \text{ and so} \\ \left| \frac{d}{d\theta}(d_{\max}(\theta)) \right| &\leq 1 \text{ when } \theta = \frac{\pi}{2}. \end{aligned}$$

Thus $d_{\text{rel}}(\theta) := d_{\max}(\theta) - \theta$ has a non-positive derivative at $\theta = \frac{\pi}{2}$. On the other hand, θ is harmonic so $d_{\text{rel}}(\theta)$ is again subharmonic. As a subharmonic function with a non-positive derivative at $\frac{\pi}{2}$, we have that $d_{\text{rel}}(\theta)$ is bounded above by its value at the left endpoint p_0 for all $\theta \leq \frac{\pi}{2}$. Let $D \leq \frac{1}{2}\text{Per}(A) \leq \frac{1}{2}C_g$ denote the hyperbolic diameter of a boundary component of A . By the length-contracting property, we have $d_{\text{rel}}(\theta_0) \leq D - \theta_0$ so

$$d_{\max}(\theta) \leq D + (\theta - \theta_0) < \frac{1}{2}(C_g + \pi) \text{ for all } \theta \leq \frac{\pi}{2}.$$

Applying the same argument to a point p_0 on the other boundary component of the collar, we conclude that for a point p' on the core

curve, the ball of radius $\frac{1}{2}(C_g + \pi)$ about its image contains the image of the boundary of A entirely. We conclude the result by the maximum principle, as $q \mapsto \text{dist}_g(\Psi(p'), \Psi(q))$ is subharmonic. \square

Lemma 3.13. *There is a constant $\mu_n > 0$ depending only on n such that: For any arithmetic group Γ acting on a period domain \mathbb{D} classifying \mathbb{Z} -PVHS of rank at most n , and for any $p \in \mathbb{D}$, we have*

$$d_{\mathbb{D}}(p, \gamma(p)) > \mu_n \text{ for all } \gamma \in \Gamma \text{ non-quasi-unipotent.}$$

Proof. There are only finitely many possible spaces \mathbb{D} , corresponding to real Lie groups G of Hodge type and bounded rank, and compact subgroups $K \subset G$. Let $\chi_\gamma(t)$ denote the characteristic polynomial of γ . Since it is monic of degree n , we can apply the following effective form of Kronecker's theorem:

Theorem 3.14 ([BM71]). *Let α be an algebraic integer of degree $d \leq n$. Either α is a root of unity, or the largest Galois conjugate of α has absolute value at least*

$$c_n = 1 + \frac{1}{52n \log(6n)}.$$

Factoring $\chi_\gamma(t)$ into irreducible factors, this theorem bounds the norm of the largest eigenvalue of γ away from 1, whenever γ is non-quasi-unipotent. Let $\lambda_1, \dots, \lambda_n$ be these eigenvalues and let

$$L_\gamma := \inf_{p \in \mathbb{D}} d_{\mathbb{D}}(p, \gamma(p))$$

be the translation length. As L_γ is conjugation-invariant, it is solely a function $F_{\mathbb{D}}(\lambda_1, \dots, \lambda_n)$ of the eigenvalues.

Let $S = G/K_{\max}$ be the symmetric space associated to the real group G . Here $K_{\max} \subset G$ is a maximal compact subgroup containing K . Consider the map

$$\mathbb{D} = G/K \xrightarrow{\pi} G/K_{\max} = S.$$

For appropriate left G -invariant metrics, this map is length-contracting. Then, $L_\gamma \geq \inf_{p \in S} d_S(p, \gamma(p))$. The formula for the translation length on S is the same as for the symmetric space $\text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R})$:

$$\inf_{p \in S} d_S(p, \gamma \cdot p) = \sqrt{(\log |\lambda_1|)^2 + \dots + (\log |\lambda_n|)^2}.$$

Hence, taking $\mu_n < \log |c_n|$ and applying Theorem 3.14, we conclude that $L_\gamma > \mu_n$ for non-quasi-unipotent γ . \square

Corollary 3.15. *Consider a \mathbb{Z} -PVHS of rank n of compact type over a curve C . Up to passing to a finite étale cover of fixed degree, there is*

an $\epsilon > 0$ such that, for any $\gamma \in \pi_1(C)$ with $\ell_C(\gamma) < \epsilon$, the monodromy of γ is trivial: $\rho(\gamma) = I \in \Gamma$.

Proof. This follows from Lemma 3.13, the length-contracting property, and the fact that in the compact type case, the only quasi-unipotent elements of Γ are of finite order. Note that for all possible $\Gamma \subset \mathrm{GL}_n(\mathbb{Z})$, the torsion can be killed at a fixed finite level, since this holds for the entire group $\mathrm{GL}_n(\mathbb{Z})$. \square

Proposition 3.16. *Let (C, γ) and ϵ be as above, and let A be a hyperbolic collar on C containing γ , of perimeter C_g . Then the period map $A \rightarrow \Gamma \backslash \mathbb{D}$ lifts to a period map $\Phi: A \rightarrow \mathbb{D}$. Furthermore, the image of Φ is contained in a ball of bounded radius B .*

Proof. The restriction of the period map to A lifts to $\mathbb{D} = G/K$ by Corollary 3.15, because the monodromy of the core curve is trivial, and the core curve generates $\pi_1(A)$.

Define $\Psi = \pi \circ \Phi$ to be the composition of the period map $\Phi: H \rightarrow \mathbb{D}$ with the quotient map $\pi: \mathbb{D} \rightarrow S = G/K_{\max}$ to the symmetric space. Then π is harmonic. So Ψ , as the composition of a holomorphic and a harmonic map, is harmonic.

Applying Proposition 3.12, we conclude that for p, q two points on the two boundary components of A , the distance

$$d_S(\Psi(p), \Psi(q))$$

is bounded. Here we use that S is non-positively curved, simply connected, and that $\pi: \mathbb{D} \rightarrow S$ is distance-decreasing, so $\pi \circ \Phi$ is also distance-decreasing. The fibers of π are isometric, compact submanifolds $K_{\max}/K \subset \mathbb{D}$. We conclude that the distance between $\Phi(p)$ and $\Phi(q)$ is also bounded, for instance, by the above distance plus the covering radius of a fiber of π . \square

Theorem 3.17. *Up to admissible identification and conjugation, there are only finitely many representations $\rho: \Pi_g \rightarrow \mathrm{GL}_n(\mathbb{Z})$, of compact type, which underlie a \mathbb{Z} -PVHS on some curve in \mathcal{M}_g .*

Proof. Let $\Phi: C \rightarrow \Gamma \backslash \mathbb{D}$ be the period map of a \mathbb{Z} -PVHS of rank n on some curve $C \in \mathcal{M}_g^{<\epsilon}$. Take a Bers pair-of-pants decomposition of C as in Theorem 3.7.

There are only finitely many topological types of pants decomposition for surfaces of a given genus g . Fix a set of “reference” pants decompositions $\{R_k\}$, one for each possible topological type. We also fix a “reference” triple of seams on each pair of pants in R_k in such a way that the four points on each cuff alternate (or coincide in pairs; there are 4 topologically distinct ways to do this for each cuff).

On each reference decomposition, choose an admissible identification $\pi_1(R_k, *) \simeq \Pi_g$. This specifies a “reference” set of Procesi generators $(\delta_j)_{1 \leq j \leq m}$ for each topological type R_k of pants decomposition.

Then, we fix representatives of each Procesi generator δ_j which decompose into a collection of segments of the following two types:

- (1) segments contained in a pair of pants, which terminate at marked points on the cuffs, and
- (2) segments circling around the cuff which connect two marked points coming from adjacent pairs of pants.

Using the Bers pants decomposition of C , we may identify $C \rightarrow R_k$ with one of the references, by an oriented homeomorphism preserving the decomposition, and sending the hyperbolic seams to the reference seams. This induces an admissible identification with Π_g and the given representatives of the Procesi generators δ_j can be pulled back to C . Applying Proposition 3.8, we further decompose each generator δ_j into three types of segments:

- (1) segments geodesically crossing a half-collar of perimeter C_g ,
- (2) segments geodesically winding around a cuff, of a fixed homotopy class ν relative to two distinguished points coming from opposite pairs of pants, and
- (3) segments in a fixed homotopy class μ relative to two distinguished points on a truncated pair of pants $P^o(\ell_1, \ell_2, \ell_3)$ satisfying $\ell_i \leq B_g$.

Let $\tilde{\Phi}: \tilde{C} \rightarrow \mathbb{D}$ be the lift of the period map to the universal cover of C and let $[0, 1]$ be the lift of the loop δ_j to a segment in \tilde{C} . Then

$$d_{\mathbb{D}}(\tilde{\Phi}(0), \tilde{\Phi}(1)) \leq \sum_{\substack{\text{segments in} \\ \text{truncated pants}}} D_{\mu} + \sum_{\substack{\text{segments in} \\ \text{cuffs}}} L_{\nu} + 2e \max\{B, B'\} \text{ where}$$

- (1) D_{μ} bounds the length of a representative of a relative homotopy class μ in the truncated pairs of pants (Prop. 3.11),
- (2) $L_{\nu} = B_g \cdot \text{winding}(\nu)$ bounds the length of the geodesic representing ν purely in terms of the relative homotopy class,
- (3) B bounds the radius of a ball covering the image of a collar (Prop. 3.16) whose core curve has length less than ϵ ,
- (4) B' bounds the length of a transverse geodesic on a half-collar with core curve of length at least ϵ and perimeter C_g , and
- (5) e is the total number of collars crossed.

Thus $d_{\mathbb{D}}(\tilde{\Phi}(0), \tilde{\Phi}(1))$ is bounded. We conclude by Lemma 3.4 that in turn, the trace $\text{tr}(\rho(\delta_j))$ is bounded. Then, the theorem follows as in Proposition 3.2. \square

Corollary 3.18. *Let \mathcal{S} be a smooth connected quasi-projective complex algebraic variety and let $\pi : \mathcal{Y} \rightarrow \mathcal{S}$ be a smooth projective morphism. There are only finitely representations $\pi_1(Y_0, *) \rightarrow \mathrm{GL}_n(\mathbb{Z})$, up to conjugacy, which underlie a \mathbb{Z} -PVHS of compact type on some fiber Y_s of $\pi : \mathcal{Y} \rightarrow \mathcal{S}$, after an identification $\pi_1(Y_0, *) \simeq \pi_1(Y_s, *)$ induced by moving $*$ in \mathcal{Y} .*

Proof. This follows from the discussion at the beginning of the section, using the Lefschetz hyperplane theorem. \square

4. DOUADY SPACES OF POLARIZED DISTRIBUTION MANIFOLDS

In this section we abstract some key elements of the Hodge manifolds, in the case where Γ is cocompact.

Definition 4.1. A *distribution manifold* (X, T^{\parallel}) is a compact, complex manifold X , together with a holomorphic subbundle $T^{\parallel} \subset TX$ of its tangent bundle (i.e. a holomorphic distribution).¹

Let $L \rightarrow X$ be a holomorphic line bundle and let h be a Hermitian metric on L . We say that (L, h) is *positive* on (X, T^{\parallel}) if the $(1, 1)$ -form $\omega_L := \frac{i}{2\pi} \partial \bar{\partial} \log h$ satisfies $\omega_L|_{T^{\parallel}} > 0$. We call (L, h) a *polarization* of the distribution manifold (X, T^{\parallel}) .

We now recall fundamental results on the analogues of the Hilbert and Chow varieties for complex manifolds and analytic spaces.

Definition 4.2. An *analytic cycle* on X is a finite formal \mathbb{Z} -linear combination $\sum_i n_i [Z_i]$ of irreducible, closed, reduced analytic subspaces $Z_i \subset X$ of a fixed dimension. An analytic cycle is *effective* if $n_i \geq 0$.

We have then the following fundamental result of Barlet, see [Bar75].

Theorem 4.3. *Effective analytic cycles on X are parameterized by a countable union of analytic spaces, locally of finite type.*

Call a connected component \mathfrak{B} of this analytic space a *Barlet space*.

Remark 4.4. In general, a Barlet space of an analytic space X of finite type need not be of finite type, even if X is a smooth, proper \mathbb{C} -variety. A famous counterexample is due to Hironaka: let $C, D \subset M$ be two smooth curves in a smooth projective 3-fold T , with $C \cap D = \{p, q\}$. We can consider the variety

$$\widehat{M} := Bl_{\widehat{C}} Bl_D(M \setminus q) \cup Bl_{\widehat{D}} Bl_C(M \setminus p),$$

that is, we blow up M along C and D , but in opposite orders at p and q . If f is a fiber of one of the exceptional divisors, then the Barlet space

¹We do not require the distribution to be integrable.

containing f is not of finite type, as f admits a deformation to a cycle of the form $f + (Z_1 + Z_2)$ where Z_1 and Z_2 are the strict transforms of the fibers at p and q of the first blow-up in the second blow-up.

Definition 4.5. Let (X, T^{\parallel}) be a distribution manifold. A *parallel Barlet space* \mathfrak{B}^{\parallel} of (X, T^{\parallel}) is a connected component of the sublocus of \mathfrak{B} defined by the following property:

$$\sum_i n_i [Z_i] \in \mathfrak{B}^{\parallel} \text{ iff there is a dense open set } \\ Z^{\circ} \subset \cup_i Z_i \text{ for which } TZ^{\circ} \subset T^{\parallel}.$$

This is visibly a locally closed analytic condition on the Barlet space. In fact, much more is true:

Theorem 4.6. *Let (X, T^{\parallel}, L, h) be a polarized distribution manifold. Any parallel Barlet space \mathfrak{B}^{\parallel} is a proper analytic space.*

Furthermore, there are only finitely many Barlet spaces parameterizing cycles of pure codimension d on which $c_1(L)^{n-d}$ is bounded.

Proof. Let g be an arbitrary hermitian metric on X , for instance, we can construct g via a partition of unity. Define a smooth distribution $T^{\perp} \subset TX$ by $T_x^{\perp} := (T_x^{\parallel})^{\perp g}$. Then, we have a g -orthogonal splitting $TX = T^{\parallel} \oplus T^{\perp}$ as smooth \mathbb{C} -vector bundles. Let g^{\perp} denote the degenerate, semi-positive hermitian form on TX which is defined by $(0, g|_{T^{\perp}})$ with respect the decomposition $TX = T^{\parallel} \oplus T^{\perp}$.

Let $N > 0$ and define a symmetric tensor by

$$\tilde{g}(v, w) := \omega_L(v, Jw) + Ng^{\perp}(v, w) \in S^2 T^* X.$$

We claim that \tilde{g} is a Hermitian metric on X for sufficiently large N . This follows from $\omega_L(v, Jw)$ being positive-definite on T^{\parallel} , g^{\perp} vanishing on T^{\parallel} and being positive definite on T^{\perp} , and compactness of X .

For any codimension d analytic cycle $Z := \sum_i n_i [Z_i] \in \mathfrak{B}^{\parallel}$, define

$$\text{vol}_L(Z) = \sum_i n_i \int_{Z_i} c_1(L)^{n-d} = [Z] \cdot c_1(L)^{n-d}.$$

Observe that $c_1(L)^{n-d}$ is pointwise positive on $Z_i^{\circ} \subset Z_i$. Furthermore $\text{vol}_L(Z)$ is constant on a connected component of \mathfrak{B}^{\parallel} because it is given as the intersection number on the right. Next, we define

$$\text{vol}_{\tilde{g}}(Z) := \sum_i n_i \int_{Z_i} \text{vol}_{\tilde{g}|_{Z_i}}$$

and observe $\text{vol}_{\tilde{g}}(Z) = \text{vol}_L(Z)$ because $\tilde{g}(\cdot, \cdot)|_{T^{\parallel}} = \omega_L(\cdot, J\cdot)|_{T^{\parallel}}$ and $TZ_i^{\circ} \subset T^{\parallel}$. Thus, X admits a hermitian metric \tilde{g} in which $\text{vol}_{\tilde{g}}(Z)$ is constant on a connected component of \mathfrak{B}^{\parallel} , equal to $[Z] \cdot c_1(L)^{n-d}$.

Let $Z^{(1)}, Z^{(2)}, \dots$ be a countable sequence of effective analytic cycles in (possibly different) connected components $\mathfrak{B}^{\parallel, (i)}$, for which $\text{vol}_L = \text{vol}_{\tilde{g}}$ remains bounded. By a theorem of Harvey-Schiffman, [HS74, Thm. 3.9] we can extract a convergent subsequence that converges to an effective analytic cycle $Z^{(\infty)}$ for which $\text{vol}_{\tilde{g}}(Z^{(i)})$ converges to $\text{vol}_{\tilde{g}}(Z^{(\infty)})$. Such convergence defines the topology on \mathfrak{B} .

By [Fuj78, Prop. 2.3], the $Z^{(i)}$ converge in the sense of currents of integration to $Z^{(\infty)}$, and in particular, the integrals $\int_{Z^{(i)}} \omega_L^{n-d}$ must converge to $\int_{Z^{(\infty)}} \omega_L^{n-d}$ and so remain bounded. Additionally, we have $\text{vol}_{\tilde{g}}(Z^{(\infty)}) = \text{vol}_L(Z^{(\infty)})$ and this equality holds for any choice N in the definition $\tilde{g} = \omega_L + Ng^\perp$. We conclude that there is a Zariski-dense open subset $Z^o \subset Z^\infty$ for which $TZ^o \subset (T^\perp)^\perp = T^\parallel$, as otherwise $\text{vol}_{\tilde{g}}(Z^{(\infty)})$ would increase as N increases.

Thus, the union of all components \mathfrak{B}^\parallel for which $c_1(L)^{n-d}$ is bounded is sequentially compact. Hence each component of \mathfrak{B}^\parallel is a compact analytic space, and there are only finitely many components with bounded vol_L . The theorem follows. \square

We now consider the analogue of Hilbert spaces. A *Douady space* of X is an analytic space \mathfrak{D} parametrizing flat families of closed analytic subspaces of X , see [Dou66, §9.1] for a precise definition. By the main theorem of Douady [Dou66, pp. 83-84], there is a universal analytic subspace $\mathcal{Z} \subset \mathfrak{D} \times X$ which is flat over X , and any flat family parameterized by a base M is the pullback along an analytic classifying morphism $M \rightarrow \mathfrak{D}$.

In general, a Douady space may only be locally of finite type, for similar reasons as the Barlet spaces.

Given a sub-analytic space $Z \subset X$, we can define an effective analytic cycle $[Z] \in \mathfrak{B}$ called the *support*. It is the positive linear combination $\sum_i n_i [Z_i]$ where Z_i are the irreducible components of the reduction of Z that have top-dimensional set-theoretic support, and n_i is the generic order of non-reducedness of Z along Z_i , see [Fuj78, Sec. 3.1]. There is an analogue, the *Douady-Barlet morphism* $[\cdot]: \mathfrak{D} \rightarrow \mathfrak{B}$, of the Hilbert-Chow morphism, sending an analytic space to its support.

Theorem 4.7 ([Fuj78, Prop. 3.4]). *The Douady-Barlet morphism is proper on each component \mathfrak{D} of the Douady space.*

Definition 4.8. *A parallel Douady space \mathfrak{D}^\parallel is a connected component of the sublocus of $Z \in \mathfrak{D}$ for which $[Z] \in \mathfrak{B}^\parallel$.*

Remark 4.9. It is important to note that the Zariski tangent space of $Z \in \mathfrak{D}^\parallel$ is not required to lie in T^\parallel . For instance, consider a flat family

$\mathcal{Z}^* \rightarrow C^* = C \setminus 0$ of complex submanifolds of X , with the tangent bundle $T\mathcal{Z}_t$ lying in T^\parallel for all $t \in C^*$. The flat limit Z_0 over the puncture might be nilpotently thickened in directions outside of T^\parallel , if the total space of the family itself does not have a tangent bundle $T\mathcal{Z}^*$ lying in T^\parallel , and this could even occur generically along Z_0 .

Corollary 4.10. *Let (X, T^\parallel, L, h) be a polarized distribution manifold. Then, each connected component of \mathfrak{D}^\parallel is a proper analytic space.*

Proof. This follows directly from Theorem 4.7 and Theorem 4.3. \square

Theorem 4.11. *Let $Z \in \mathfrak{D}^\parallel$ lie in a parallel Douady space. Then Z is projective, and $L|_Z$ is an ample line bundle.*

Proof. A simplification of the proof in [BBT23, Thm. 1.1] applies. It follows from Siu and Demailly's resolution [Siu84, Siu85, Dem87] of the Grauert-Riemenschneider conjecture, applied to a resolution of Z , that Z is Moishezon. We have the following lemma:

Lemma 4.12. *Let S be an open stratum of the singular stratification of $\cup_i Z_i$. Then $TS \subset T^\parallel$.*

Proof. By assumption, there is a dense open $Z^o \subset \cup_i Z_i$ for which $TZ^o \subset T^\parallel$. We claim that $TS \subset \overline{TZ^o}$ lies in the Zariski closure of TZ^o in TX . Then the result will follow as T^\parallel is Zariski-closed in TX .

Let Z_i be an irreducible component containing S and consider the map $d\pi_i: T\tilde{Z}_i \rightarrow TX$ from a resolution. Let $\tilde{Z}_i^o := \pi_i^{-1}(Z_i \cap Z^o)$. As $d\pi_i$ is continuous and $d\pi_i(T\tilde{Z}_i^o) \subset TZ^o$, we have $\text{im}(d\pi_i) \subset \overline{TZ^o}$. The claim follows if we can show $\text{im}(d\pi_i) \supset TS'$, for a dense open $S' \subset S$, i.e. can we lift a generic tangent vector of S' to \tilde{Z}_i ? This is immediate from the generic smoothness of $\pi_i|_{\pi_i^{-1}(S)^{\text{red}}}$. \square

Lemma 4.12 implies we have $L^d \cdot V > 0$ for any subvariety V of dimension d , because TV is generically contained in the tangent bundle of some singular stratum S and $\frac{i}{2\pi}\partial\bar{\partial}\log(h)$ is positive definite on T^\parallel . So Z satisfies the Nakai-Moishezon criterion. Then, a theorem of Kollár [Kol90, Thm. 3.11] implies that Z is projective. \square

Definition 4.13. Let $\mathfrak{C} \subset (\mathfrak{D}^\parallel)^{\text{red}}$ be an irreducible component of a parallel Douady space. For $Z_t \in \mathfrak{C}$ let $L_t := L|_{Z_t}$.

We say that \mathfrak{C} has *maximal variation* if there exists an analytic open set $U \subset \mathfrak{C}$ for which $(Z_s, L_s) \not\cong (Z_t, L_t)$ for all $s, t \in U$, $s \neq t$.

Theorem 4.14. *Let \mathfrak{C} be an irreducible component of a parallel Douady space of (X, T^\parallel, L, h) with maximal variation. Then \mathfrak{C} is Moishezon.*

Proof. Let $u: \mathfrak{Z}^{\parallel} \rightarrow \mathfrak{C}$ be the universal flat family and let $\mathfrak{L} \rightarrow \mathfrak{Z}^{\parallel}$ be the universal polarizing line bundle. For any fixed $n \in \mathbb{N}$, the locus $\mathfrak{C}_n \subset \mathfrak{C}$ of projective (Thm. 4.11) schemes $Z \in \mathfrak{C}$ on which $nL = n\mathfrak{L}|_Z$ is not very ample is closed. Taking the sequence

$$\cdots \subset \mathfrak{C}_{3!} \subset \mathfrak{C}_{2!} \subset \mathfrak{C}_{1!} \subset \mathfrak{C}$$

gives a nested sequence of closed analytic subspaces. The intersection is empty since for all $Z \in \mathfrak{C}$, there is some $n_Z \in \mathbb{N}$ for which $n_Z L$ is very ample, and $n_Z \mid i!$ for all $i \geq n_Z$. We conclude some \mathfrak{C}_n is empty for large enough n , so $|nL|$ is a projective embedding for all $Z \in \mathfrak{C}$.

Furthermore, the locus on which $H^i(Z, nL)$ jumps in dimension is also closed, and so by the same argument, we may assume $h^i(Z, nL) = 0$ for all $i > 0$ and all $Z \in \mathfrak{C}$. Then $u_*(n\mathfrak{L})$ is a vector bundle of rank

$$N + 1 := \chi(Z, nL) = h^0(Z, nL).$$

It is a vector bundle because χ is constant in (analytic) flat families.

Let $\mathbb{P} \rightarrow \mathfrak{C}$ be the projective frame bundle of $u_*(n\mathfrak{L})$, a principal holomorphic $J = \mathrm{PGL}(N + 1)$ -bundle. Points of \mathbb{P} correspond to some $Z \subset X$, and a basis of sections of $H^0(Z, nL)$, modulo scaling. We have an analytic map

$$\phi: \mathbb{P} \rightarrow \mathcal{H}$$

where $\mathcal{H} \subset \mathrm{Hilb}(\mathbb{P}^N)$ is the component of the Hilbert scheme with Hilbert polynomial χ , sending $(Z, [s_0 : \cdots : s_N]) \in \mathbb{P}$ to the closed subscheme of \mathbb{P}^N with the given embedding. Note \mathcal{H} is projective and ϕ is equivariant with respect to the natural J -action on both sides.

Consider the set of algebraic cycles $\mathcal{O} := \{\overline{J \cdot x} \mid x \in \mathcal{H}\} \subset \mathrm{Chow}(\mathcal{H})$. A point of \mathcal{O} uniquely determines a J -orbit, since a J -orbit is recoverable from its closure. Since the action of J is algebraic on \mathcal{H} , the space \mathcal{O} is stratified by algebraic varieties

$$\mathcal{O} = \mathcal{O}_1 \sqcup \cdots \sqcup \mathcal{O}_m$$

with each \mathcal{O}_j an irreducible, locally closed set of some component $\mathrm{Chow}_j(\mathcal{H})$ of the Chow variety. Let $\mathcal{H}_j \subset \mathcal{H}$ be the locally closed set of points x for which $\overline{J \cdot x} \in \mathcal{O}_j$ and choose the j such that \mathcal{H}_j is the largest-dimensional space intersecting $\phi(\mathbb{P})$. These are the “generic” J -orbits which arise from choosing a basis of $H^0(Z, nL)$.

Since \mathbb{P} is irreducible, we have $\phi(\mathbb{P}) \subset \overline{\mathcal{H}_j}$. Observe that there is a rational map (a morphism on \mathcal{H}_j)

$$\begin{aligned} \psi: \overline{\mathcal{H}_j} &\dashrightarrow \overline{\mathcal{O}_j} \\ x &\mapsto \overline{J \cdot x} \end{aligned}$$

with the closure of the latter taken in $\mathrm{Chow}_j(\mathcal{H})$, which is projective.

Let $U \subset \mathfrak{C}$ be a small analytic open around a given point. There is a local analytic section of $\mathbb{P}|_U \rightarrow U$, call it s_U . Then, $\phi \circ s_U: U \rightarrow \overline{\mathcal{H}}_j$ is analytic and ψ is rational, so the composition

$$\psi \circ \phi \circ s_U: U \dashrightarrow \overline{\mathcal{O}}_j$$

is a meromorphic map. Furthermore, since ψ collapses J -orbits, and ϕ is J -equivariant, we conclude that this local meromorphic map is independent of choice of local section s_U . So these maps patch together to give a meromorphic map $\alpha: \mathfrak{C} \dashrightarrow \overline{\mathcal{O}}_j$. Since α is meromorphic, by Hironaka, there is a resolution of indeterminacy

$$\mathfrak{C} \xleftarrow{\beta} \tilde{\mathfrak{C}} \xrightarrow{\gamma} \overline{\mathcal{O}}_j$$

of $\alpha = \gamma \circ \beta^{-1}$ with β bimeromorphic.

Finally, we apply the assumption of maximal variation: There exists some analytic open $U \subset \mathfrak{C}$ for which $(Z_s, L_s) \not\cong (Z_t, L_t)$ for all $s, t \in U$, $s \neq t$. This implies that $(Z_t, nL_t) \not\cong (Z_s, nL_s)$ for all $s \neq t$ in a possibly smaller neighborhood. Thus, the J -orbits in \mathcal{H} corresponding to (Z_t, nL_t) are distinct in an analytic open set. Hence γ is generically finite. We conclude that $\tilde{\mathfrak{C}}$ and thus \mathfrak{C} is Moishezon. \square

Remark 4.15. The assumption of maximal variation is necessary. For instance, let X be an arbitrary compact, complex manifold, and consider the distribution manifold for which $T^{\parallel} = 0$. It admits a polarization by setting $L = \mathcal{O}_X$ with h the trivial metric. Then, the Douady space of points in X is a parallel Douady space, isomorphic to X itself. But of course, X need not be Moishezon, so not all parallel Douady spaces are Moishezon in this generality.

Meta-Definition 4.16. We define *data of GAGA type* on X to be a collection of holomorphic data Data_X to which the GAGA theorem applies, upon restriction to a projective scheme $Z \in \mathfrak{D}^{\parallel}$.

Example 4.17. An example of data of GAGA type would be $\text{Data}_X = (F^{\bullet}, \nabla)$ where F^{\bullet} is a descending filtration of holomorphic vector bundles on X and ∇ is a holomorphic connection on F^0 . For any parallel analytic space $Z \in \mathfrak{D}^{\parallel}$, the restriction of F^{\bullet} to Z is a filtration F_Z^{\bullet} of algebraic vector bundles, by Serre's GAGA theorem [Ser56].

Similarly, a well-known extension of GAGA implies that the restriction of ∇ to a connection ∇_Z on F_Z^0 is an algebraic connection.

Meta-Theorem 4.18. *Let Data_X be data of GAGA type on X .*

We say that an irreducible, closed analytic subspace $\mathfrak{D}_0 \subset \mathfrak{D}^{\parallel}$ has maximal variation with respect to Data_X if the isomorphism type of

the restriction of this data to $Z \in \mathfrak{D}_0$ is determinative in an analytic open set $U \subset \mathfrak{D}_0$: $(Z_s, \text{Data}_s) \not\cong (Z_t, \text{Data}_t)$ for all $s \neq t \in U$.

Then Theorem 4.14 still holds: \mathfrak{D}_0 is Moishezon.

Proof. By GAGA, the restriction of Data_X to any $Z \in \mathfrak{D}_0$ is algebraic data, denoted Data_Z . The general form of such algebraic data, together with Z , is parameterized by an algebraic variety (adding rigidifying data corresponding to an algebraic group action as necessary), admitting an algebraic compactification $\mathcal{H}_{\text{Data}}$. Then, we apply the same argument as in Theorem 4.14 to the classifying map

$$\begin{aligned} \mathfrak{D}_0 &\dashrightarrow \mathcal{H}_{\text{Data}} \\ Z &\mapsto (Z, \text{Data}_Z) \end{aligned}$$

to conclude that \mathfrak{D}_0 is Moishezon. \square

Example 4.19. For the data of GAGA type (F^\bullet, ∇) discussed in Example 4.16, $\mathcal{H}_{\text{Data}}$ can be concretely constructed as follows.

Let \mathfrak{D}_0 be an irreducible component of \mathfrak{D}^{\parallel} containing Z and with maximal variation with respect to (F^\bullet, ∇) . Denote by $\pi : \mathfrak{Z} \rightarrow \mathfrak{D}_0$ the universal flat family and $f : \mathfrak{L} \rightarrow \mathfrak{Z}$ the universal polarizing bundle.

Let \mathcal{H} be the component of the Hilbert scheme that $|nL|$ maps Z into. The Hilbert polynomials P^\bullet of the vector bundles F_Z^\bullet which arise from restricting F^\bullet are constant along $Z \in \mathfrak{D}_0$. We may choose integers $m_p, n_p \gg 0$ for which any vector bundle (even coherent sheaf) with Hilbert polynomial P^p over any $Z \in \mathcal{H}$ is a quotient of the form

$$(-m_p L)^{\oplus n_p} \twoheadrightarrow F_Z^p.$$

For instance, choose m_p uniformly over all of \mathcal{H} so that $F_Z^p(m_p L)$ is globally generated with vanishing higher cohomology. Then for a fixed n_p , there is a surjection $\mathcal{O}_Z^{\oplus n_p} \twoheadrightarrow F_Z^p(m_p L)$ corresponding to a basis of global sections. Furthermore, this quotient is uniquely determined by the induced surjection

$$H^0(Z, (k_p L)^{\oplus n_p}) \twoheadrightarrow H^0(Z, F_Z^p((m_p + k_p)L))$$

for all k_p large enough. We can ensure that $h^0(Z, k_p L)$ is constant over all of \mathcal{H} . So this defines an embedding of the relative moduli space of coherent sheaves with Hilbert polynomial P^p over \mathcal{H} into a Grassmannian bundle $\text{Gr}(V_p)$ of the vector bundle $V_p := \pi_*(k_p L)^{\oplus n_p}$. This is the standard construction, due to Grothendieck [Gro60], of an embedding of the quot-scheme into a Grassmannian, performed relatively over \mathcal{H} .

The inclusion $F_Z^p \hookrightarrow F_Z^{p-1}$ is an element $H^0(Z, (F_Z^p)^* \otimes F_Z^{p-1})$. This vector space includes into $H^0(Z, (m_p L)^{\oplus n_p} \otimes F_Z^{p-1})$ and by choosing $m_p \gg m_{p-1}$, we can ensure that the latter receives a surjection from

$H^0(Z, (m_p L)^{\oplus n_p} \otimes (-m_{p-1} L)^{\oplus n_{p-1}})$. Thus, the inclusion $F_Z^p \hookrightarrow F_Z^{p-1}$ is determined by an $n_p \times n_{p-1}$ -matrix of global sections of $(m_p - m_{p-1})L$, uniquely up to a subspace of this vector space of matrices. Choosing k_p so that $k_{p-1} + m_{p-1} = k_p + m_p$ we can insure that $F_Z^p \hookrightarrow F_Z^{p-1}$ is induced by an inclusion $V_{p,Z} \hookrightarrow V_{p-1,Z}$ of the fibers over $Z \in \mathcal{H}$.

Thus, the isomorphism type of F_Z^\bullet as a filtered vector bundle can be rigidified in terms of flag-like data $\mathrm{Fl}(F_Z^\bullet)$ involving subspaces of, and morphisms between, the V_p vector bundles. Furthermore, the isomorphism type of F_Z^\bullet is uniquely determined by a J' -orbit on $\mathrm{Fl}(F_Z^\bullet)$, for J' an algebraic group. Concretely, J' is the group of changes-of-basis of $H^0(Z, F_Z^p(m_p L))$ and changes-of-lift of the inclusions $F_Z^p \hookrightarrow F_Z^{p-1}$.

Let $\mathcal{H}_{\mathrm{filt}}$ be the principal J' -bundle consisting of a filtered vector bundle F_Z^\bullet on some $Z \in \mathcal{H}$ with Hilbert polynomial P^\bullet , together with its rigidifying data in $\mathrm{Fl}(F_Z^\bullet)$. We have a forgetful map $\mathcal{H}_{\mathrm{filt}} \rightarrow \mathcal{H}$.

Over $\mathcal{H}_{\mathrm{filt}}$, we construct the relative moduli space $\mathcal{H}_{\mathrm{Data}}^o \rightarrow \mathcal{H}_{\mathrm{filt}}$ of algebraic connections ∇ on F^0 . We could also work with the algebraic subloci of $\mathcal{H}_{\mathrm{Data}}^o$ for which ∇ is flat, or $\nabla(F^p) \subset F^{p-1} \otimes \Omega^1$ on $(Z^{\mathrm{red}})_{\mathrm{sm}}$. Take an algebraic compactification $\mathcal{H}_{\mathrm{Data}}^o \hookrightarrow \mathcal{H}_{\mathrm{Data}}$.

As in Theorem 4.14, we have a principal J -bundle $\mathbb{P} \rightarrow \mathfrak{D}_0$ with $J = \mathrm{PGL}(N+1)$ corresponding to changes of basis of $H^0(Z, nL)$. Over \mathbb{P} , we have a principal J' -bundle $\mathbb{P}' \rightarrow \mathbb{P}$ consisting of the space of all rigidifying data for F_Z^\bullet as above. We also have a flat, algebraic connection ∇_Z on F_Z^0 . So there is a holomorphic classifying map $\mathbb{P}' \rightarrow \mathcal{H}_{\mathrm{Data}}$, which is J' - and J -equivariant for the actions on the source and target. The remainder of the argument of Theorem 4.14 applies.

5. ALGEBRAICITY OF THE NON-ABELIAN HODGE LOCUS

We now apply the general results of the previous section to the polarized distribution manifold $(X_\Gamma, T^{\parallel}, L, h)$ where $X_\Gamma = \Gamma \backslash \mathbb{D}$ for Γ cocompact, T^{\parallel} is the Griffiths distribution, L is the Griffiths line bundle, and h is the equivariant hermitian metric. Let $G = G_1 \times \cdots \times G_k$ be the decomposition of the semisimple group $G = \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ into \mathbb{R} -simple factors. These give the \mathbb{C} -simple factors of $G_{\mathbb{C}}$ by [Sim92, 4.4.10].

We have a decomposition $\mathbb{D} = \mathbb{D}_1 \times \cdots \times \mathbb{D}_k$ and on each factor \mathbb{D}_i we have a filtered vector bundle with flat connection. Let (F_i^\bullet, ∇_i) be the pullbacks of these to \mathbb{D} . Then, they descend to X_Γ even when Γ does not split as a product of lattices $\Gamma_i \subset G_i$. Let V_i denote the \mathbb{C} -local system on X_Γ of flat sections of (F_i^0, ∇_i) .

Definition 5.1. We define the *Hodge data of GAGA type*

$$\mathrm{Hodge}_{X_\Gamma} = \{(F_i^\bullet, \nabla_i)\}_{i=1, \dots, k}$$

to be this k -tuple of filtered flat vector bundles.

Remark 5.2. It is important to remark that the universal filtered flat vector bundle $(F^\bullet, \nabla) = \bigoplus_i (F_i^\bullet, \nabla_i)$ is not the same data of GAGA type as above! For instance, it may be impossible to tell how (F^\bullet, ∇) splits, upon restriction to some $Z \subset X_\Gamma$.

Remark 5.3. Let $Z \in \mathfrak{D}^{\parallel}$ be reduced and irreducible. Suppose $\tilde{Z} \rightarrow Z$ is a resolution of singularities. Then \tilde{Z} admits a \mathbb{Z} -PVHS by pulling back $(V_{\mathbb{Z}}, F^\bullet, \nabla)$. The pullback of $\text{Hodge}_{X_\Gamma} = \{(F_i^\bullet, \nabla_i)\}_{i=1, \dots, k}$ constitutes the data of a splitting of the corresponding \mathbb{R} -VHS into factors. Let V be the local system of flat sections of $\nabla_{\tilde{Z}}$.

The \mathbb{Z} -PVHS on \tilde{Z} , and thus, the period map $\Phi: \tilde{Z} \rightarrow X_\Gamma$, is recoverable from (Z, Hodge_Z) and one critical missing piece of information: the location of the integral lattice $V_{\mathbb{Z},*} \hookrightarrow V_*$ in a fiber over some base point $* \in \tilde{Z}$ —this is the only data which cannot be captured coherently on X_Γ itself, and to which GAGA cannot be applied.

Now, we leverage the fact that the lattice $V_{\mathbb{Z},*}$ must be invariant under parallel transport.

Proposition 5.4. *Let $Z \in \mathfrak{D}^{\parallel}$ be irreducible and reduced, and suppose $\tilde{Z} \rightarrow Z$ is a resolution of singularities. Let $(V_{\mathbb{Z}}, F^\bullet)$ be the corresponding pullback \mathbb{Z} -PVHS and let $* \in \tilde{Z}$ be a base point. Let*

$$\rho: \pi_1(\tilde{Z}, *) \rightarrow \text{GL}(V_{\mathbb{Z},*})$$

be the monodromy representation and let $H = \prod_{i \in I} G_i \subset G$ be the collection of simple factors in which $\text{im } \rho$ is Zariski-dense. Fixing a frame of $V_{\mathbb{Z},}$, the infinitesimal changes-of-frame which give rise to a lattice preserved by ρ are contained in*

$$\prod_{i \in I} \mathbb{C} \times \prod_{i \notin I} \mathfrak{gl}(V_i).$$

Proof. An infinitesimal change-of-frame $a \in \mathfrak{gl}(V_*)$ resulting in a new monodromy-invariant lattice is exactly a matrix commuting with $\text{im}(\rho)$, and thus commuting with $\mathbf{H}(\mathbb{R})$. Since V_i is an irreducible representation of $(G_i)_{\mathbb{C}}$, Schur's lemma implies that a acts by a scalar λ_i on each summand $V_i \subset V$ for which $G_i \subset H$. \square

Definition 5.5. Given any analytic subspace $Z \subset X_\Gamma$ we define Γ_Z as the image of $\pi_1(\tilde{Z}) \rightarrow \Gamma$ for some resolution of singularities $\tilde{Z} \rightarrow Z^{\text{red}}$.

Lemma 5.6. *Let $Z^\nu \rightarrow Z^{\text{red}}$ be the normalization. Then, $\Gamma_Z \subset \Gamma$ is the image of $\pi_1(Z^\nu)$. It is also the image of $\pi_1(U)$ for any dense open subset $U \subset (Z^{\text{red}})_{\text{sm}}$.*

Proof. Let Z_{sm}^ν denote the nonsingular locus. Then $\pi_1(Z_{\text{sm}}^\nu) \rightarrow \pi_1(Z^\nu)$ is surjective. The same property holds for the inverse image of Z_{sm}^ν or U in any desingularization. Thus, $\pi_1(\tilde{Z})$, $\pi_1(Z_{\text{sm}}^\nu)$, $\pi_1(Z^\nu)$, $\pi_1(U)$ all have the same image in $\Gamma = \pi_1(X_\Gamma)$. \square

Proposition 5.7. *Let $Z \in \mathfrak{D}^{\parallel}$ be irreducible and reduced. The group Γ_Z only jumps in size, in an open neighborhood of $Z \in \mathfrak{D}^{\parallel}$.*

Proof. Let $(C, 0) \rightarrow \mathfrak{D}^{\parallel}$ be an analytic arc, and consider the pullback family $\mathfrak{Z} \rightarrow (C, 0)$, with $\mathfrak{Z}_0 = Z$. Let $\mathcal{W} = \mathfrak{Z}^\nu$ be the normalization of the total space. The general fiber \mathcal{W}_t is normal, so $\Gamma_{Z_t} = \text{im}(\pi_1(\mathcal{W}_t))$ by Lemma 5.6. This is the same group for all $t \in C \setminus 0$ if we assume (as we may) that \mathcal{W} is a fiber bundle over $C \setminus 0$. There is a deformation-retraction $\mathcal{W} \rightarrow \mathcal{W}_0$ to the central fiber. Tracing an element of $\pi_1(\mathcal{W}_t)$ through the retraction, we get a free homotopy from any $\gamma_t \in \pi_1(\mathcal{W}_t)$ to an element $\gamma_0 \in \pi_1(\mathcal{W}_0)$.

Conversely, we can lift any element of $\pi_1(\mathcal{W}_0)$ to an element of $\pi_1(\mathcal{W}_t)$: We have $\pi_1(\mathcal{W}_0) = \pi_1(\mathcal{W}) = \pi_1(\mathcal{W} \setminus ((\mathcal{W}_0)_{\text{sing}} \cup \mathcal{W}_{\text{sing}}))$ because \mathcal{W} is normal and $(\mathcal{W}_0)_{\text{sing}} \cup \mathcal{W}_{\text{sing}}$ has codimension 2. Thus, any element of $\pi_1(\mathcal{W}_0) = \pi_1(\mathcal{W})$ can be represented by a loop in \mathcal{W} avoiding both $(\mathcal{W}_0)_{\text{sing}}$ and $\mathcal{W}_{\text{sing}}$. Then, this loop can be deformed off its intersection with $(\mathcal{W}_0)_{\text{sm}}$ as $(\mathcal{W}_0)_{\text{sm}}$ is a locally smooth divisor in \mathcal{W}_{sm} . So we can represent the loop in $\mathcal{W} \setminus \mathcal{W}_0$. Finally, $\pi_1(\mathcal{W} \setminus \mathcal{W}_0)$ is a \mathbb{Z} -extension of $\pi_1(\mathcal{W}_t)$ because it is a fiber bundle over the punctured disk $C \setminus 0$.

Thus, $\Gamma_{Z_t} = \text{im}(\pi_1(\mathcal{W}_0))$. Then the natural morphism $\mathcal{W}_0 \rightarrow \mathfrak{Z}_0 = Z$ is a finite birational morphism because Z is reduced. Thus, it factors the normalization $Z^\nu \rightarrow \mathcal{W}_0 \rightarrow Z$ and so $\text{im}(\pi_1(Z^\nu)) = \Gamma_Z \subset \Gamma_{Z_t} = \text{im}(\pi_1(\mathcal{W}_0))$. Thus Γ_Z only jumps in size. \square

Remark 5.8. The same statement holds, up to passing to a finite index subgroup of Γ_Z , when Z is generically non-reduced.

Theorem 5.9. *If $Z \subset X_\Gamma$ is irreducible, reduced, and Γ_Z is Zariski-dense in G , then any irreducible component $\mathfrak{C} \subset \mathfrak{D}^{\parallel}$ containing Z has maximal variation with respect to Hodge_{X_Γ} . In particular, \mathfrak{C} is Moishezon by Meta-Theorem 4.18.*

Proof. We must find an analytic open set $U \subset \mathfrak{C}$ for which

$$(Z_s, \{(F_i^\bullet, \nabla_i)\}) \not\cong (Z_t, \{(F_i^\bullet, \nabla_i)\})$$

for all $s \neq t \in U$. Choose U to be a small neighborhood of $Z \in \mathfrak{C}$. Since Z is irreducible and reduced, we can assume that Z_t is irreducible and reduced for all $t \in U$. Applying Proposition 5.7, we may ensure that all $Z_t \in U$ satisfy the property that Γ_{Z_t} is Zariski-dense in G . It

suffices to show there are no holomorphic arcs $C \rightarrow U$ for which the isomorphism type of $(Z_t, \{(F_i^\bullet, \nabla_i)\})$ is constant over all $t \in C$.

Choose a smooth base point $* \in Z_t$. Then by Proposition 5.4, the only deformations of the lattice $V_{Z,*} \subset V_*$ which remain invariant under $\nabla_{\tilde{Z}_t}$ are those which differ by scaling each summand of $V = \bigoplus V_i$ by some $\lambda_i \in \mathbb{C}^*$. But such scaling does not change the period map, as the Hodge flag $F^\bullet = \bigoplus F_i^\bullet$ is also preserved by this scaling action.

Thus, Hodge_{X_Γ} is determinative on U —if it were non-determinative, the fixed data $(Z_t, \{(F_i^\bullet, \nabla_i)\})$ would admit a flat deformation of the local system $V_{\mathbb{Z}}$ which induced a non-isomorphic Hodge filtration. \square

Remark 5.10. One could as easily have worked with Barlet spaces, since the support morphism $[\cdot]: \mathfrak{C} \rightarrow \mathfrak{B}^{\text{II}}$ will be bimeromorphic onto its image, under the assumptions of Theorem 5.9. The disadvantage is that the embedding into a compact, algebraic parameter space, as in Example 4.19, is unclear for Barlet spaces.

Theorem 5.11. *Let $\mathcal{Y} \rightarrow \mathcal{S}$ be a smooth projective family over a quasiprojective variety \mathcal{S} . Then the non-abelian Hodge locus of compact type $\text{NHL}_c(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ is algebraic.*

Proof. Let Y_s be a fiber. As we saw in Section 2, the data of a \mathbb{Z} -PVHS on Y_s with generic Mumford-Tate group $\mathbf{G} \subset \text{GL}_n$ and monodromy \mathbf{H} is completely determined by

- (1) a holomorphic, Griffiths transverse period map $\Phi_s : Y_s \rightarrow X_{\Gamma_H}$ whose monodromy image is Zariski-dense, and
- (2) a point in $\mathbb{D}_{H'}$ corresponding to a summand on which the \mathbb{Z} -PVHS is locally constant.

Thus, up to passing to a finite index subgroup of fixed level, the monodromy representation of such a \mathbb{Z} -PVHS has a reduction of structure to the product $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ where the corresponding local system has trivial monodromy on the summand associated to \mathbf{H}' .

Hence, possibly passing to a smaller value of n , we can restrict our attention to the $(Y_s, \nabla_s) \in \text{NHL}_c(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ which underlie a \mathbb{Z} -PVHS \mathbb{V} with Zariski-dense monodromy in the generic Mumford-Tate group.

By Corollary 3.18, only finitely many representations of $\pi_1(Y_s)$ of compact type can appear in this manner. Thus, there is a finite list of compact Hodge manifolds X_Γ which receive all the period maps for such (Y_s, ∇_s) . So to prove the theorem, we may restrict our attention to a single compact period target $\Gamma \backslash \mathbb{D} = X_\Gamma$.

It remains to show: The space of pairs (Y_s, Φ_s) of a fiber of $\mathcal{Y} \rightarrow \mathcal{S}$, together with a Griffiths' transverse map $\Phi_s : Y_s \rightarrow X_\Gamma$ with Zariski-dense monodromy is an algebraic variety (and the maps into the relative

de Rham and Dolbeault spaces are algebraic). We first prove that each irreducible analytic component of the space of pairs (Y_s, Φ_s) is algebraic, then we prove that the number of components is finite.

Fix an irreducible analytic component $B \subset \text{NHL}_c(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$. There is an analytic Zariski open subset $B^\circ \subset B$ on which $\text{im}(\Phi_s)$, taken with its reduced scheme structure, form a flat family of closed analytic subspaces of X_Γ over B° . So there is an irreducible component $\mathfrak{C} \subset \mathfrak{D}^{\parallel}$ for which $\text{im}(\Phi_s) \in \mathfrak{C}$ for $(Y_s, \Phi_s) \in B^\circ$.

Since Y_s is smooth, the morphism $Y_s \rightarrow \Phi_s(Y_s)$ factors through the normalization $Y_s \rightarrow \Phi_s(Y_s)^\nu$. Thus, $\Gamma_{\text{im}(\Phi_s)}$ contains the image of $\pi_1(Y_s)$ in Γ . Since we have restricted to the case where the monodromy is Zariski-dense, \mathfrak{C} is Moishezon by Theorem 5.9.

Let $\mathfrak{Z} \rightarrow \mathfrak{C}$ be the universal family. For all $(Y_s, \Phi_s) \in B^\circ$, the period mapping Φ_s factors through the inclusion $\text{im}(\Phi_s) \hookrightarrow \mathfrak{Z}$ as a fiber of the universal family. That is, we have a map $\Xi: \mathcal{Y} \times_{\mathcal{S}} B^\circ \rightarrow \mathfrak{Z}$ for which $\Phi = \pi_{X_\Gamma} \circ \Xi$.

The analytic deformations of (Y_s, Φ_s) in B are exactly the isomonodromic deformations of the local system $V_{\mathbb{Z}}$ on Y_s to nearby fibers, which underlie a \mathbb{Z} -PVHS. But for $(Y_s, \Phi_s) \in B^\circ$, these are exactly the ways to deform the inclusion $\Xi_s: Y_s \hookrightarrow \mathfrak{Z}$ of fibers. Since $\mathcal{Y} \rightarrow \mathcal{S}$ is algebraic and $\mathfrak{Z} \rightarrow \mathfrak{C}$ is Moishezon, the irreducible component of $\text{Hom}_{\text{fiber}}(\mathcal{S}, \mathfrak{Z})$, the space of morphisms from a fiber of \mathcal{Y} to a fiber of \mathfrak{Z} , which contains $(Y_s, \Xi_s) \in B^\circ$, is Moishezon.

The inclusion into $M_{\text{dR}}(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ is Moishezon because ∇_s is the pull back along Ξ_s of the relative connection on F^0 on the universal family over $\mathfrak{Z} \rightarrow \mathfrak{C}$. The relative connection on F^0 is Moishezon, by GAGA. Thus, B° and its closure B are algebraic, as they are Moishezon subsets of an algebraic variety. The inclusion into $M_{\text{Dol}}(\mathcal{Y}/\mathcal{S}, \text{GL}_n)$ is Moishezon by the same reasoning, applied to the associated graded of the universal Hodge flag over $\mathfrak{Z} \rightarrow \mathfrak{C}$, equipped with its Higgs field.

Finally, it remains to prove that (1) only finitely many irreducible components \mathfrak{C} of the parallel Douady space appear, and (2) for each one that appears, the number of irreducible components of the space $\text{Hom}_{\text{fiber}}(\mathcal{Y}, \mathfrak{Z})$ is finite.

Let F^\bullet be the Hodge filtration on Y_s coming from a period map $\Phi_s: Y_s \rightarrow X_\Gamma$ and let $A \rightarrow \mathcal{Y}$ be an ample line bundle on the universal family. Then by Simpson [Sim94, Lemma 3.3], the vector bundles F^p enjoy the following property: If m_s is an integer for which $T_{Y_s}(m_s A)$ is globally generated, then $\mu_A(F^{p+1}) \leq \mu_A(F^p) + m_s n$. Here μ_A is the slope with respect to A . Note that $\mu_A(F^0) = 0$ because F^0 has a flat structure. We may choose an $m_s = m$ uniformly over all of \mathcal{S} . We

conclude that the slopes $\mu_A(F^p)$ are bounded, in a way depending only on $\mathcal{Y} \rightarrow \mathcal{S}$. In turn, $A^{d-1} \cdot \det(F^p)$ is bounded for all p , and so there is an a priori bound on $A^{d-1} \cdot L$, where L is the Griffiths bundle. It follows that $A^{d-r} \cdot L^r$ is bounded for any r .

This bounds the Griffiths volume of the image $\Phi_s(Y_s)$ of any period map, and so by Theorem 4.6, only finitely many components of the parallel Barlet space \mathfrak{B}^{\parallel} of X_Γ occur as the support of period images from Y_s . The same finiteness holds for relevant components \mathfrak{C} of the parallel Douady space, as we are taking period images with their reduced scheme structure, see Remark 5.10.

Finally, the bounds on $A^{d-r} \cdot L^r$ also bound the volume of the graph $\Gamma(\Xi_s)$ of a morphism $(Y_s, \Xi_s) \in \text{Hom}_{\text{fiber}}(\mathcal{Y}, \mathfrak{Z})$, viewed as a subvariety of $\mathcal{Y} \times \mathfrak{Z}$. We conclude that there must be only finitely many components of $\text{Hom}_{\text{fiber}}(\mathcal{Y}, \mathfrak{Z})$. \square

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