THE NON-ABELIAN HODGE LOCUS: A SURVEY

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Abstract. The aim of this note is to survey recent work on the geometry of the non-abelian Hodge locus introduced by Simpson in [Sim97]. It is an extended version of my talk at the Workshop "Complex and p-Adic Simpson Correspondence" held in November 2025 at the Brin Mathematics Research Center.

1. Introduction

In his ICM address in 1962 [Š63], Shafarevitch famously asked the following: given a number field K and a finite set of non-archimedean places S of K, are there finitely many algebraic varieties of a given topological type over K with good reduction outside of S?

This question admits an analogue over the complex numbers that was first considered by Arakelov [Ara72] and Parshin [Par68] who proved the following seminal theorem.

Theorem 1.1. *Let $g \geq 1$ and let $\mathcal{S}$ be a smooth complex quasi-projective curve. There are finitely many isomorphism classes of smooth projective non-isotrivial families of curves $\mathcal{C} \to \mathcal{S}$ of genus g.*

Faltings proved a similar result for abelian varieties [Fal83b] and several finiteness results of this type have since been proved, see the introduction of [ELT25, KL10] for a brief history.

Inspired by the proofs of Faltings [Fal83a, Fal83b], Deligne proved in [Del87] a remarkable finiteness theorem for $\mathbb{Z}$-local systems on algebraic varieties that admit polarized variations of Hodge structures Theorem 2.1 and formulated an intriguing uniformity question, see 2.

Deligne's question is a consequence of a deep conjecture made by Simpson in [Sim97] on the algebraicity of the non-abelian Hodge locus, see Theorem 3.3.

The goal of this note is to explain the recent progress made in [ET23] on Deligne's question and on Simpson's conjecture on the algebraicity of the non-abelian Hodge locus.

Date: February 20, 2026.

2. Deligne's finiteness result

In [Del87], Deligne proved the following remarkable theorem.

Theorem 2.1. *Let X be a smooth quasi-projective algebraic variety over $\mathbb{C}$, $o \in X$ and let $N \geq 1$ be an integer. There are finitely many representations*

$$\rho : \pi_1(X, o) \to \mathrm{GL}_N(\mathbb{Z})$$

that underlie a polarized variation of Hodge structures.

Since the fundamental group of X only depends on the complex structure of X, Deligne asked [Del87, Question 3.13] whether the above result can be made uniform in the complex structure of X.

More precisely, assume that X is a curve of genus g with n punctures and let $\mathcal{X} \xrightarrow{u} \mathcal{S}$ be the universal family (with some level structure) of curves of genus g with n punctures such that $X \simeq \mathcal{X}_{u(o)}$ and $o \in X$. For $x \in \mathcal{X}$, the fundamental groups $\pi_1(\mathcal{X}_{u(x)}, x)$ define a local system on $\mathcal{X}$. Therefore, the choice of a path between two points x_1 and x_2 in $\mathcal{X}$ defines an isomorphism from $\pi_1(\mathcal{X}_{u(x_1)}, x_1)$ to $\pi_1(\mathcal{X}_{u(x_2)}, x_2)$. Such identification will be called *admissible*.

Let $\Sigma_{g,n} = \pi_1(\mathcal{X}_{u(o)}, o)$ be the fundamental group of a Riemann surface of genus g with n punctures. The group $\pi_1(\mathcal{S}, u(o))$ acts on $\Sigma_{g,n}$ as above and it is the *mapping class group action.*

Question. *Consider the set of representations $\rho : \Sigma_{g,n} \to \mathrm{GL}_N(\mathbb{Z})$ for which there exists an admissible identification $\gamma : \pi_1(\mathcal{X}_{u(a)}, a) \to \Sigma_{g,n}$ such that $\rho \circ \gamma$ admits a polarized variation of Hodge structures. Are there finitely many orbits of such representations ρ under mapping class group action?*

2.1. **A stronger finiteness conjecture.** We will formulate in this section a strengthening of Question 2. Let $\mathcal{S} \hookrightarrow \overline{\mathcal{S}}$ be a smooth compactification with normal crossings boundary. Let $s \in \overline{\mathcal{S}}$ and let U be an open neighborhood of s such that $U \cap \mathcal{S} \simeq (\Delta^*)^k \times \Delta^{n-k}$ for some $0 \leq k \leq d$ and d is the dimension of $\mathcal{S}$. The group $\pi_1(U \cap S, *) \simeq \mathbb{Z}^k$ acts on $\Sigma_{g,n}$ via local mapping class group action.

Conjecture 2.2. *Let $N \geq 1$. For each $s \in \overline{\mathcal{S}}$, there exists an open neighborhood $s \in U \subset \overline{\mathcal{S}}$ such that the set of representations $\rho : \Sigma_{g,n} \to \mathrm{GL}_N(\mathbb{Z})$ that underlie a $\mathbb{Z}$-PVHS for some fiber $\mathcal{X}_a$, $a \in U \cap \mathcal{S}$ falls into finitely many orbits under the action of $\pi_1(U \cap S)$ and conjugacy. Furthermore, each orbit is finite.*

We explain how Conjecture 2.2 implies Question 2: by the compacteness of $\overline{\mathcal{S}}$, we can cover $\mathcal{S}$ by a finite union of open sets W_i which are products of disks and punctured disks and for which the conclusion of 2.2 holds. In each W_i, choose a base point a_i and a path γ_i from $u(o)$ to a_i. Using the identification given by γ_i, 2.2 asserts that the set of representations of $\Sigma_{g,n} \simeq \pi_1(\mathcal{X}_{a_i})$ that underlie a $\mathbb{Z}$-PVHS for some point in W_i falls into finitely many orbits under $\pi_1(W_i, a_i)$ and conjugacy. Since a different choice of γ_i yields an element of $\pi_1(\mathcal{S}, u(o))$, the set of all representations that underlie a $\mathbb{Z}$-PVHS for some point in W_i have finitely many orbits under $\pi_1(\mathcal{S}, u(o))$ and conjugacy. As the number of sets W_i is finite, question 2 admits a positive answer.

2.2. **Recent progress.**

Definition 2.3. *We say that a semi-simple representation $\rho : \Sigma_{g,n} \to \mathrm{GL}_N(\mathbb{Z})$ is $\mathbb{Q}$-anisotropic if the Zariski closure $\overline{\rho(\Sigma_{g,n})}^{\mathbb{Q}-\mathrm{Zar}}$ does not contain any non-trivial unipotent elements.*

An equivalent form of the $\mathbb{Q}$-anisotropy condition above is requiring that $\mathbf{H}(\mathbb{Z})\backslash\mathbf{H}(\mathbb{R})$ is compact where $\mathbf{H}$ is the semi-simple algebraic group over $\mathbb{Q}$ defined as $\overline{\rho(\Sigma_{g,n})}^{\mathbb{Q}\text{-Zar}}$. Notice also that being $\mathbb{Q}$-anisotropic is invariant under automorphisms of $\Sigma_{g,n}$.

In [ET23], we prove:

Theorem 2.4. *Question 2 admits a positive answer for $\mathbb{Q}$-anisotropic representations.*

We explain the proof of this theorem in Section 4.1.

3. Non-abelian Hodge locus

Non-abelian Hodge theory offers a powerful framework for studying representations of fundamental groups of algebraic varieties. Naturally, Theorem 2.1 and Question 2 admit a beautiful interpretation in this setting, which we will explain in this section. For background results on non-abelian Hodge theory, we refer to [Sim94, Sim95, Sim97]

Let X be a smooth projective algebraic variety over $\mathbb{C}$, $o \in X$, and let $N \geq 1$. Then Simpson introduces three moduli spaces on X:

- The Betti moduli space (or character variety): $\mathrm{M}_{\mathrm{Betti}}(X, \mathrm{GL}_N(\mathbb{C}))$ is the algebraic variety over $\mathbb{C}$ parameterizing conjugacy classes of semi-simple representations of $\pi_1(X, o)$ into $\mathrm{GL}_N(\mathbb{C})$.
- The de Rham moduli space: $\mathrm{M}_{\mathrm{dR}}(X, N)$ is the moduli space of holomorphic flat vector bundles of rank N on X.

- The Dolbeault moduli space: $\mathrm{M}_{\mathrm{Dol}}(X,N)$ is the moduli space of semistable Higgs bundles with vanishing rational Chern classes.

These three spaces are related by the following isomorphisms:

(1) $\mathrm{M}_{\mathrm{Betti}}(X,\mathrm{GL}_N(\mathbb{C}))$ is bi-holomorphically isomorphic to $\mathrm{M}_{\mathrm{dR}}(X,N)$ by the Riemann-Hilbert correspondence [Del70].
(2) $\mathrm{M}_{\mathrm{dR}}(X,N)$ is real-analytically isomorphic to $\mathrm{M}_{\mathrm{Dol}}(X,N)$ by the non-abelian Hodge correspondence.

Remark 3.1. The three spaces $\mathrm{M}_{\mathrm{Betti}}(X,\mathrm{GL}_N(\mathbb{C}))$, $\mathrm{M}_{\mathrm{dR}}(X,N)$, and $\mathrm{M}_{\mathrm{Dol}}(X,N)$ have structures as quasi-projective algebraic varieties but these structures are not compatible in general with the isomorphisms above.

The moduli space $\mathrm{M}_{\mathrm{Dol}}(X,N)$ admits an algebraic action of $\mathbb{C}^\times$ determined by

$$t\cdot(E,\theta)=(E,t\cdot\theta)$$

for $t\in\mathbb{C}^\times$ and (E,θ) Higgs bundle. The fixed points of the $\mathbb{C}^\times$ action are called $\mathbb{C}$-polarizable variations of Hodge structures.

Let $\mathrm{M}_{\mathrm{Betti}}(X,\mathrm{GL}_N(\mathbb{Z}))$ denote the locus of local systems with integral monodromy, up to conjugacy. Using the various correspondences above, we can consider the intersection inside the moduli of flat bundles:

$$\mathrm{M}_{\mathrm{Betti}}(X,\mathrm{GL}_N(\mathbb{Z}))\cap\mathrm{M}_{\mathrm{Dol}}(X,N)^{\mathbb{C}^\times}\ .$$

This intersection corresponds to local systems which underlie a variation of Hodge structures and a consequence of Theorem 2.1 is that this intersection is finite.

3.1. **The non-abelian Hodge locus.** The moduli spaces that appeared in the previous section can be constructed in families following Simpson [Sim95]. Let $\mathcal{X}\to\mathcal{S}$ be a smooth projective morphism. Then we have the following three relative moduli spaces:

(1) The relative moduli space of semi-simple local systems

$$\mathrm{M}_{\mathrm{Betti}}(\mathcal{X}/\mathcal{S},\mathrm{GL}_N(\mathbb{C}))\to\mathcal{S}\ .$$

Its fiber over a point $s\in\mathcal{S}$ is $\mathrm{M}_{\mathrm{Betti}}(\mathcal{X}_s,\mathrm{GL}_N(\mathbb{C}))$.
(2) The relative moduli space of flat bundles

$$\mathrm{M}_{\mathrm{dR}}(\mathcal{X}/\mathcal{S},N)\to\mathcal{S}\ .$$

Its fiber over a point $s\in\mathcal{S}$ is $\mathrm{M}_{\mathrm{dR}}(\mathcal{X}_s,N)$.
(3) The relative moduli space of Higgs bundles

$$\mathrm{M}_{\mathrm{Dol}}(\mathcal{X}/\mathcal{S},N)\to\mathcal{S}\ .$$

Its fiber over a point $s\in\mathcal{S}$ is $\mathrm{M}_{\mathrm{Dol}}(\mathcal{X}_s)$.

The three moduli spaces above are similarly related by the Riemann-Hilbert correspondence and the non-abelian Hodge correspondence. The relative moduli space $\mathrm{M}_{\mathrm{Dol}}(\mathcal{X}/\mathcal{S}, N)$ admits an algebraic action of $\mathbb{C}^\times$ which is compatible with its action on the fibers over $\mathcal{S}$.

Let $\mathrm{M}_{\mathrm{Betti}}(\mathcal{X}/\mathcal{S}, \mathrm{GL}_N(\mathbb{Z}))$ denote the relative moduli space of local systems with integral monodromy.

Definition 3.2. *The non-abelian Hodge locus is the intersection inside* $\mathrm{M}_{\mathrm{dR}}(\mathcal{X}/\mathcal{S}, N)$*:*

$$\mathrm{NAHL}(\mathcal{X}/\mathcal{S}) = \mathrm{M}_{\mathrm{Betti}}(\mathcal{X}/\mathcal{S}, \mathrm{GL}_N(\mathbb{Z})) \cap \mathrm{M}_{\mathrm{Dol}}(\mathcal{X}/\mathcal{S}, N)^{\mathbb{C}^\times} .$$

Simpson proved that the irreducible components of this locus are complex analytic and made the following remarkable conjecture [Sim97, Conjecture 12.3].

Conjecture 3.3. *The locus* $\mathrm{NAHL}(\mathcal{X}/\mathcal{S})$ *admits an algebraic structure compatible with the morphisms to* $\mathrm{M}_{\mathrm{dR}}(\mathcal{X}/\mathcal{S}, N)$ *and* $\mathrm{M}_{\mathrm{Dol}}(\mathcal{X}/\mathcal{S}, N)$*.*

Notice that Theorem 3.3 yields a positive answer to Deligne's question 2: indeed algebraicity implies that $\mathrm{NAHL}(\mathcal{X}/\mathcal{S})$ has only finitely many connected components. Each component can be seen as an isomonodromic deformation of a representation underlying a polarized variation of Hodge structures. Therefore, the representations of $\pi_1(X, o)$ are constant on these components, up to admissible identification, which answers question 2.

3.2. **Analogies with the Hodge locus.** Theorem 3.3 can be thought of as the analogue of the classical algebraicity of the Hodge locus which is a deep result due to Cattani-Deligne-Kaplan [CDK95], see also recent work [BKT20]. As remarked in the introduction of [CDK95], the fact that the Hodge locus is a countable union of algebraic varieties is a consequence of the Hodge conjecture.

In a similar vein, Simpson [Sim97, Conjecture 12.3] formulates the non-abelian Hodge conjecture:

Conjecture 3.4. *Let* $\mathbb{V}_{\mathbb{Z}}$ *be a local system on a smooth quasi-projective* X *that underlies a* $\mathbb{Z}$*-PVHS. Then* $\mathbb{V}_{\mathbb{Z}}$ *is motivic.*

We offer a heuristic explaining that Theorem 3.4 implies a weaker version of Theorem 3.3: the non-abelian Hodge locus is a *countable union of algebraic varieties.*

For integers N, d and M, consider the scheme $\mathcal{F}_{N,d,M}(X)$ parameterizing closed subvarieties $Y \subset \mathbb{P}^N(\mathbb{C}) \times X$ such that $Y \to X$ is flat of relative dimension d, smooth over some Zariski open $U \subset X$ and for each $x \in X$, Y_x has volume bounded by M in $\mathbb{P}^N(\mathbb{C})$. This corresponds to a morphism $\phi : X \to \mathrm{Hilb}_{d,M}(\mathbb{P}^N(\mathbb{C}))$, the Hilbert scheme of closed subschemes of $\mathbb{P}^N(\mathbb{C})$ of dimension d and volume bounded by M such that the image of ϕ intersects the locus of smooth subschemes of $\mathbb{P}^N(\mathbb{C})$. To get a finite type scheme, we impose furthermore that ϕ has image of volume bounded by M with respect to a fixed polarization on the quasi-projective variety $\mathrm{Hilb}_{\mathrm{d,M}}(\mathbb{P}^N)$. This ensures that $\mathcal{F}_{N,d,M}(X)$ is a finite type algebraic variety over $\mathbb{C}$. A local system as in Theorem 3.4 yields a $\mathbb{C}$ point in $\mathcal{F}_{N,d,M}(X)$ for some N, d, M.

Given now a family smooth (projective for simplicity)$\mathcal{X} \to \mathcal{S}$ which is smooth projective for simplicity over a quasi-projective base $\mathcal{S}$, we can form the relative space $g_{N,d,M} : \mathcal{F}_{N,d,M}(\mathcal{X}/\mathcal{S}) \to \mathcal{S}$ as finite type morphism of algebraic varieties. One can see by a Baire category argument that any analytic component of the non-abelian Hodge locus in $\mathcal{S}$ corresponds to the image of the morphism $g_{N,d,M}$ for some N, d, M, hence it is algebraic. Therefore, the non-abelian Hodge locus is a countable union of algebraic varieties.

In [ET23], we make the following progress towards Simpson's conjecture.

Theorem 3.5. *The analytic components of the non-abelian Hodge locus corresponding to $\mathbb{Q}$-anisotropic local systems are algebraic.*

Let us also mention the recent work on codimension bounds that have been obtained in the work of Morris [Mor25].

4. Ideas of the proofs

We explain in this section the ideas that go into the proofs of Theorem 2.4 and Theorem 3.5.

4.1. **Deligne's finiteness.**

The proof of Theorem 2.4 is inspired from Deligne's original treatment in [Del87] which we recall briefly. The theorem can be reduced to curves using Lefschetz hyperplane section theorem and we will therefore only consider curves.

Let X be a smooth curve of genus g with n punctures and $o \in X$. Given an integer $N \geq 1$, a theorem of Procesi [Pro76] asserts that the set of semisimple representations $\rho : \pi_1(X, o) \to \mathrm{GL}_N(\mathbb{C})$ is determined by the traces of a finite set of elements $\gamma_1, \ldots, \gamma_r \in \pi_1(X, o)$. Each free homotopy class of an element γ_i contains a loop with minimal length with respect to the hyperbolic metric on the Riemann surface X. Let $\ell(\gamma_i)$ be this length. Then an important step in Deligne's argument is to prove the following bound

$$|\rho(\gamma_i)| \leq C_N \ell(\gamma_i) \tag{4.1}$$

where the constant C_N only depends on N. This step crucially uses the length-contracting property of period maps. As the trace $\rho(\gamma_i)$ is an integer, it can achieve only finitely many values and therefore only finitely many representations appear.

We now explain the proof of Theorem 2.4. The $\mathbb{Q}$-anisotropic condition allows us to reduce to compact curves. Consider then the universal family (with some level structures) of curves of genus g, $\mathcal{C}_g \to \mathcal{M}_g$.

Let $\epsilon > 0$ and decompose $\mathcal{M}_g = \mathcal{M}_g^{<\epsilon} \cup \mathcal{M}_g^{\geq\epsilon}$ where $\mathcal{M}_g^{\geq\epsilon}$ is the locus of Riemann surfaces where the systole has length at least ϵ. By a theorem of Mumford [Mum71], $\mathcal{M}_g^{\geq\epsilon}$ is compact.

The first observation is that since $\mathcal{M}_g^{\geq\epsilon}$ is compact, we can cover it by a finite union of contractible sets and on each set, there is a well defined length functional $\ell(\gamma_i)$. The fact that the systole is lower bounded means that $\ell(\gamma_i)$ is upper bounded and length-contracting property of period maps allows to use the same bound as in Equation (4.1).

However, in the set $\mathcal{M}_g^{<\epsilon}$, the Riemann surface will acquire singularities and loops transverse to this singularity will have geodesic length going to infinity, as the Riemann surface approaches the boundary of $\mathcal{M}_g$. Here, length-contracting property of period maps is no longer enough to get the Equation (4.1).

Let γ denote a vanishing loop. Then by assumption of $\mathbb{Q}$-anisotropy, the monodromy $\rho(\gamma)$ is quasi-unipotent. Up to base change, it is in fact unipotent and therefore must be trivial. Let C_γ be a collar centered around γ. The period map restricts to C_γ lifts to the period domain $\mathcal{D}$ defined by the variation of Hodge structure, since the monodromy on C_γ is trivial.

The following proposition is crucial [ET23, Proposition 3.12].

Theorem 4.1. *There exists a set $K \subset D$ of bounded diameter which depends only on g, such that $\rho(C_\gamma) \subset K$*

An important consequence of the previous theorem is that, even though the transverse length of the collar goes to infinity as the core loop shrinks, its image is still contained in a compact subset. Therefore, by subdividing the Riemann surface into regions where the geometry is bounded and regions where the image under the period map lands in a bounded set, we obtain a similar bound to Equation (4.1) which allows us to conclude.

4.2. **Algebraicity of the non-abelian Hodge locus.** Theorem 3.3 is proved based on our answer to question 2.

Given a smooth projective morphism $\mathcal{X} \to \mathcal{S}$, and a representation $\rho : \pi_1(\mathcal{X}_s, o) \to \mathrm{GL}_N(\mathbb{C})$ which is $\mathbb{Q}$-anisotropic and which admits a polarized variation of Hodge structure. The analytic component B in NAHL that contains (s, ρ) corresponds to isomonodromic deformations of ρ that still admits a polarized variation of Hodge structure and we will prove that it is an algebraic variety.

Let $\varphi : \mathcal{X}_s \to \Gamma\backslash\mathcal{D}$ be the monodromy period map. Then $\rho(\pi_1(\mathcal{X}, o))$ and Γ have the same $\mathbb{Q}$-Zariski closure and the quotient $\Gamma\backslash\mathcal{D}$ is compact. Our approach is to show that the analytic component B is the locus where the period map $\Phi_s : \mathcal{X}_s \to \Gamma\backslash\mathcal{D}$ can be deformed in $\mathcal{X}$.

The quotient of the period domain $Y_\Gamma := \Gamma\backslash\mathcal{D}$ is endowed with a holomorphic distribution $T^h \subset T$, the Griffiths distribution, such that φ is tangent to T^h by Griffiths transversality axiom on polarized variations of Hodge structures. Therefore, we only need to consider deformations of φ that are still tangent to T^h.

Let $\mathcal{L}$ be the *Griffiths line bundle* Y_Γ with its hermitian metric. Its curvature is positive definite in the directions of T^h and negative definite in the orthogonal directions. It follows that any closed analytic subvariety of Y_Γ that is tangent to T^h is algebraic and polarized by $\mathcal{L}$ [Gri70, BBT23].

The length-contracting property of period maps allows to bound the volume of the image of $\Phi(\mathcal{X}_s)$ with respect to the line bundle $\mathcal{L}$ solely in terms of the topology of $\mathcal{X}_s$, which is constant in the family. Deforming of the period map Φ amounts to deforming the image in $\Gamma\backslash\mathcal{D}$ in such way that the Zariski tangent space is generically contained in T^h. We need therefore to parametrize the space of deformations of the image.

Theorem 4.2. *There exists a compact Moishezon space $\mathcal{Z}$ that parametrizes closed analytic subspaces W of Y_Γ whose Zariski tangent space is generically contained in T^h, whose monodromy is Zariski dense, and whose volume with respect to $\mathcal{L}$ is bounded by a fixed constant C.*

It follows from the previous theorem that we have a universal family $\mathcal{W} \to \mathcal{Z}$ over a Moishezon base parameterizing cycles in Y_Γ which arise as period images from a member of the family $\mathcal{X} \to \mathcal{S}$ with monodromy contained in Γ and Zariski dense. Up to a modification, we can assume that $\mathcal{Z}$ is projective and the morphism $\mathcal{W} \to \mathcal{Z}$ is also projective polarized by the line bundle $\mathcal{L}$.

Consider the space

$$\mathrm{Hom}_{\mathrm{fiber}}(\mathcal{X}/\mathcal{S}, \mathcal{W}/\mathcal{Z}) = \{\{(s, z, f)|s \in \mathcal{S}, z \in \mathcal{Z}, f : \mathcal{X}_s \to \mathcal{W}_z\}$$

of maps from a fiber of the family $\mathcal{X} \to \mathcal{S}$ to a fiber of the family $\mathcal{W}/\mathcal{Z}$.

It is in fact an algebraic locally of finite type over $\mathcal{S} \times \mathcal{Z}$ and B is identified to an irreducible component of this space. Therefore, B is a Moishezon space. The morphism to $\mathrm{M}_{\mathrm{dR}}(\mathcal{X}/\mathcal{S}, N)$ is also algebraic, as we can algebraize the flat connection by GAGA.

We therefore conclude that components of the non-abelian Hodge locus corresponding to $\mathbb{Q}$-anisotropic representations are algebraic, which yields Theorem 3.5.

Acknowledgment. We would like to thank Philip Engel for helpful suggestions and the Institute for Advanced Study for its Hospitality. This work is supported by an NSF award 2503815 and by Burke Research Initiation Award from Dartmouth College.

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