Dear Salim,

I thought a bit more about the connection of your approach to special divisors on toroidal compactifications of orthogonal Shimiura varieties and the one of Shaul and myself.

First of all, I think it is correct that if you replace in your computation of multiplicities of boundary divisors corresponding to inner rays the Zagier Eisenstein series by the weight 3/2 mock theta functions, you will get a generating series that is a *weakly holomorphic* modular form of weight $1 + n/2$. The result of Markus Schwagenscheidt and myself then shows that the multiplicities of the boundary divisors in your case are rational.

To understand the connection of the two approaches, I think Proposition 4.24 of the paper with Shaul [BZ] is the key. Here the $\Phi_m^K(\omega/|\omega|)$ are computed. These are the multiplicities that Shaul and I use. They are given by coefficients of $\theta_{K,\omega} \cdot G_N^+$, which are related to the multiplicities you use, plus a correction term, which is a regularized Petersson inner product.

To go from our modularity statement to yours and back, one can use a general lemma which I now describe. To simplify notation I only do this for scalar valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$, but it is clear that an analogous result also holds for vector valued forms for the Weil representation and for half integral weights.

Let k, l be non-negative integers with $0 \leq l \leq k$. For simplicity we also assume that $k > 2$. Let $\theta \in M_{k-l}$ be a holomorphic modular form of weight $k - l$. In addition, let $G = G^+ + G^- \in H_l$ be a harmonic Maass form of weight l (for $\mathrm{SL}_2(\mathbb{Z})$ with holomorphic part G^+ and non-holomorphic part G^-. If L denotes the Maass lowering operator, then $L(G)$ is a non-holomorphic modular form of weight $l-2$, which we assume to have moderate growth at the cusp.

For every positive integer m we write F_m for the unique harmonic Mass form in H_{2-k} (with cuspidal image under the ξ-operator) whose Fourier expansion starts with

$$F_m = q^{-m} + O(1).$$

Hence the principal part is simply q^{-m} plus a constant. Then the same argument as in Proposition 4.24 of [BZ] gives the following lemma.

Lemma 0.1. *The regularized integral*

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} \theta \cdot (LG) \cdot F_m \, d\mu(\tau)$$

is (up to constant factors I am suppressing) equal to

$$\mathrm{CT}(\theta G^+ \cdot F_m^+) + (\theta G, \xi(F_m))_{Pet}^{reg}.$$

Here $\mathrm{CT}(\cdot)$ *in the first summand denotes the constant term of a q-series, and the second summand is given by the regularized Petersson inner product.*

In Proposition 4.24 of [BZ] this lemma is applied to compute $\Phi_m^K(\omega/|\omega|)$. This quantity is given by the regularized integral of $\theta_{K,\omega} L(G_N) F_m$, hence the lemma is applied with $l = 3/2$, $k = 1 + n/2$.

Now the following lemma shows that the difference of the generating series of such regularized integrals and the mixed mock modular form θG^+ is weakly holomorphic.

Lemma 0.2. *For $m > 0$ let*

$$A(m) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} \theta \cdot (LG) \cdot F_m \, d\mu(\tau).$$

The q-series

$$B(q) = \sum_{m>0} A(m) q^m - \theta G^+$$

is a weakly holomorphic modular form in $M_k^!$.

Proof. We prove this using Borcherds' modularity criterion. That is for every weakly holomorphic modular form

$$f = \sum_n c(n) q^n \in M_{2-k}^!$$

we show that $\mathrm{CT}(f \cdot B(q)) = 0$.

To do so, we first notice that

$$f = \sum_{m>0} c(-m) F_m$$

(here we have used the simplifying assumption $k > 2$). Hence

$$\begin{aligned} \mathrm{CT}(f \cdot \sum_{m>0} A(m) q^m) &= \sum_{m>0} c(-m) A(m) \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} \theta \cdot (LG) \cdot \sum_{m>0} c(-m) F_m \, d\mu(\tau) \\ &= \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} \theta \cdot (LG) \cdot f \, d\mu(\tau). \end{aligned}$$

By the usual argument invoking Stokes' theorem we get

$$\begin{aligned} \mathrm{CT}(f \cdot \sum_{m>0} A(m) q^m) &\doteq \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} d(\theta \cdot G \cdot f \, d\tau) \\ &\doteq \mathrm{CT}(\theta \cdot G^+ \cdot f). \end{aligned}$$

Here the dot over the equality sign means that the equality is only true up to appropriate normalizing factors. This implies the assertion. □

In our application the $A(m)$ will be the multiplicities of the boundary divisors $B_{I,\omega}$ that Shaul and I use, while the coefficients of $\theta \cdot G^+$ will be the multiplicities that you use. Hence the lemma implies that the difference of the generating series in [BZ] and the generating series that you get is indeed a weakly holomorphic modular form. This difference can actually computed explicitly in terms of the regularized Petersson inner products appearing in the first lemma. This is where the non-rational multiplicities in [BZ] come from.

Please see if all this makes sense to you. If yes, it would be nice if you could include these explanations in the revision of you paper in some way.

Best regards,

Jan

References

[BZ] *J. H. Bruinierand S. Zemel*, Special cycles on toroidal compactifications of orthogonal Shimiura varieties, Math. Ann. **384** (2022), 1–63.

[Mi] *T. Miyake*, Modular Forms, Springer-Verlag (1989).