MIXED MOCK MODULARITY OF SPECIAL DIVISORS

PHILIP ENGEL, FRANÇOIS GREER, AND SALIM TAYOU

ABSTRACT. We prove that the generating series of special divisors in toroidal compactifications of orthogonal Shimura varieties is a mixed mock modular form. More precisely, we find an explicit completion using theta series associated to rays in the cone decomposition. The proof relies on intersection theory at the boundary of the Shimura variety.

CONTENTS

1. INTRODUCTION

Let $(L, \cdot)$ be a non-degenerate even lattice of signature $(2, n)$ with quadratic form $Q(x) = \frac{1}{2} x \cdot x$. Associated to L is a type IV Hermitian symmetric domain $\mathbb{D}$, defined as a connected component of

$$\{\mathbb{C}x \in \mathbb{P}(L_{\mathbb{C}}) \,\big|\, x \cdot x = 0, \, x \cdot \bar{x} > 0\}.$$

It is a complex manifold of dimension n. Let $O^+(L)$ denote the group of orthogonal transformations of L which preserves the component $\mathbb{D}$, and let $\Gamma \subset O^+(L)$ be a finite index subgroup which acts trivially on the discriminant group $L^\vee/L$, where $L^\vee$ is the dual lattice.

The arithmetic quotient $X_\Gamma := \Gamma \backslash \mathbb{D}$ is a connected complex Shimura variety of orthogonal type. It parameterizes polarized Hodge structures on the lattice L with Hodge numbers $(1, n, 1)$. Such Hodge structures arise naturally in the study of polarized K3 surfaces when $n = 19$, and additionally of elliptic curves, abelian surfaces with real multiplication, and abelian surfaces, when $n = 1, 2, 3$, respectively.

There is a countable family of divisors, called *special divisors*[1], which are the images of orthogonal Shimura varieties associated to sublattices of L of signature $(2, n-1)$. Geometrically, they correspond to Noether–Lefschetz loci for the polarized variation of Hodge structure parameterized by X_Γ. For every $\beta \in L^\vee/L$ and $m \in -Q(\beta)+\mathbb{Z}$, there is an associated special divisor $Z^o(\beta, m) \subset X_\Gamma$ defined in Equation 3.1.1. We denote its class in the cohomology group $H^2(X_\Gamma, \mathbb{Q})$ by $[Z^o(\beta, m)]$. Define $[Z^o(0, 0)]$ to be the first Chern class of the dual of the Hodge bundle on X_Γ. The special cycles can be packaged in the following generating series with values in $H^2(X_\Gamma, \mathbb{Q}) \otimes \mathbb{C}[L^\vee/L]$:

$$\sum_{\beta \in L^\vee/L} \sum_{m \in -Q(\beta)+\mathbb{Z}} [Z^o(\beta, m)] \otimes v_\beta q^m. \tag{1.0.1}$$

In an influential paper [Bor99], Borcherds proved that Equation 1.0.1 is a vector-valued modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_L$ valued in $\mathrm{Pic}(X_\Gamma)$, where ρ_L denotes the Weil representation associated to the discriminant lattice $(L^\vee/L, Q)$. His proof relied on earlier work [Bor98], which constructed meromorphic modular forms on X_Γ whose zeroes and poles are special divisors. A cohomological version was known previously by the work of Kudla–Millson [KM90], see also [Gar18], and Hirzebruch-Zagier in the case of Hilbert modular varieties [HZ76].

This paper is concerned with the modularity behavior of Equation 1.0.1, in compactifications of the variety X_Γ. In general, X_Γ is non-compact and admits several compactifications. The first compactification was constructed by Satake [Sat60] and Baily-Borel [BB66] and is called the *Baily-Borel compacitification*. It is typically singular at the boundary, which has codimension at least $n-1$.

We will be interested in the *toroidal compactifications* $X_\Gamma \hookrightarrow X_\Gamma^\Sigma$, first constructed in [AMRT10]. They always have divisorial boundary, but are non-unique, depending on certain combinatorial data Σ. This data consists, for each isotropic line $I \subset L$, of a polyhedral cone decomposition Σ_I of the positive cone of the hyperbolic lattice $I^\perp/I$. These decompositions Σ_I must additionally be Γ-invariant. We refer to Section 3.3 for more details.

We will assume that the chosen polyhedral cone decomposition is simplicial, so that X_Γ^Σ has finite quotient singularities. It is also possible to choose Γ and Σ so that X_Γ^Σ is a smooth projective algebraic variety, although this is not necessary for our main results.

Theorem 1.1. *The generating series*

$$\sum_{\beta \in L^\vee/L} \sum_{m \in -Q(\beta)+\mathbb{Z}} [Z(\beta, m)] \otimes v_\beta q^m \tag{1.0.2}$$

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[1]or Heegner divisors

of cohomology classes of Zariski closures $Z(\beta, m) := \overline{Z^o(\beta, m)}$, valued in $H^2(X_\Gamma^\Sigma, \mathbb{Q}) \otimes \mathbb{C}[L^\vee/L]$, is a mixed mock modular form of weight $1 + n/2$ with respect to $\overline{\rho}_L$.

Mixed mock modular forms are holomorphic parts of mixed harmonic Maass forms which have been studied by [BF04, BFOR17] and which we define in Section 2.2.

To give a more precise form of the above theorem, we introduce some notation: for any isotropic plane $J \subset L$, the lattice $M := J^\perp/J$ is negative definite and has an associated theta series Θ_M of weight $\frac{n}{2} - 1$ and representation $\overline{\rho}_M$. The image of this theta series under an operator between Weil representations p_M^L is a modular form of the same weight, but for the representation $\overline{\rho}_L$.

On the other hand, for any isotropic line $I \subset L$ and a ray $\mathbb{R}_{\geq 0}c$ in the cone decomposition Σ_I with $c \cdot c = 2N > 0$, the lattice $c^\perp \subset I^\perp/I$ is negative definite, and we can similarly consider the theta series $\Theta_{c^\perp}$ which is a vector-valued holomorphic modular form of weight $\frac{n-1}{2}$ and representation $\overline{\rho}_{c^\perp}$. The theta series Θ_N of the one dimensional lattice $\mathbb{Z}c$ also plays a role: there exists a weak harmonic Maass form F_N of weight $3/2$ and representation $\overline{\rho}_{\mathbb{Z}c}$ called the *Zagier Eisenstein series*, whose shadow is proportional to Θ_N, see Section 2.3. Our main theorem has the following more precise form.

Theorem 1.2. *The generating series*

$$\sum_{\beta \in L^\vee/L} \sum_{m \in Q(\beta)+\mathbb{Z}} [Z(\beta, m)] \otimes v_\beta q^m - \frac{1}{2} \sum_{(I,c)} p_{c^\perp \oplus \mathbb{Z}c}^L \left(\Theta_{c^\perp} \otimes F_N^+ \right) \otimes \Delta_{I,c}$$

$$+ \frac{1}{12} \sum_J E_2(q) \cdot p_M^L \left(\Theta_M \right) \otimes \Delta_J$$

valued in $H^2(X_\Gamma^\Sigma, \mathbb{Q}) \otimes \mathbb{C}[L^\vee/L]$ is a holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_L$.

Here Δ_J and $\Delta_{I,c} \subset X_\Gamma^\Sigma$ are the boundary divisors of the toroidal compactification X_Γ^Σ, associated to (Γ-orbits of) isotropic planes $J \subset L$ and rays $\mathbb{R}_{\geq 0}c \in \Sigma_I$. As $p_M^L(\Theta_M)$ and $p_{c^\perp \oplus \mathbb{Z}c}^L \left(\Theta_{c^\perp} \otimes F_N^+ \right)$ are mixed harmonic Maass forms, Theorem 1.2 implies Theorem 1.1.

Using Margulis' super-rigidity theorem, we prove that the first Betti number of X_Γ^Σ vanishes when $n \geq 3$ or if $n = 2$ and Γ is an irreducible lattice, see Section 4.1. We get the following corollary.

Corollary 1.3. *Assume either $n \geq 3$ or the Witt index of L is 1 and $n = 2$. Then the generating series in Equation 1.0.2 is also a mixed mock modular form in $\mathrm{Pic}(X_\Gamma^\Sigma)_\mathbb{Q}$ of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_L$.*

1.1. **Strategy of the proof.** We prove Theorem 1.2 by showing that the pairing of the generating series 1.0.2 with every homology class in $H_2(X_\Gamma^\Sigma, \mathbb{Q})$ is a holomorphic modular form of weight $1 + \frac{n}{2}$ and level $\overline{\rho}_L$.

For this, we prove a splitting theorem stating that the second homology of X_Γ^Σ is rationally generated by the homology of the open part X_Γ and the classes of algebraic curves $C \subset X_\Gamma^\Sigma$ contracted in the Baily-Borel compactification. For homology classes supported in the open part, we use Kudla–Millson's result [KM90] as an input to deduce that the generating series is modular.

Our second contribution is to compute the intersection of the special divisors with the curves $C \subset X_\Gamma^\Sigma$ in the boundary. The classes of such curves are generated by the one-dimensional toric boundary strata of X_Γ^Σ. We use tools from toric geometry to explicitly compute such intersection numbers.

The resulting generating series is a theta series of a real hyperbolic lattice, with a piecewise linear weighting determined by the cone decomposition Σ_I. We find suitable non-holomorphic completions à la Zwegers, and prove that they are non-holomorphic modular forms by a theorem of Vigneras [Vig77]. Finally, we determine the theta series explicitly, in terms of the rays in the cone decomposition. In some sense, our approach is closer to the original approach of Hirzebruch and Zagier [HZ76] for Hilbert modular surfaces.

1.2. **Comparison with earlier work.** A similar result has been proved by Bruinier and Zemel in [BZ22]. In their paper, they analyze the vanishing order of the automorphic forms constructed by Borcherds [Bor95], using an explicit expression of its Petersson metric from [Bru02, Bor98]. Bruinier and Zemel compute explicitly the vanishing orders of the Borcherds products along the boundary components to deduce the modularity statement.

Our desire to understand their paper from an intersection theoretic point of view, in the spirit of Hirzebruch–Zagier's seminal work [HZ76], was the starting point of our investigations. One question remaining from their paper is the rationality of the coefficients along the type III boundary divisors, to which our approach yields an positive answer. Our approach is more geometric, and yields unconditional results in cohomology when $n = 2$ and the Witt index of L is 2.

There has also been earlier work of [Pet15] in the context of moduli space of K3 surfaces of dimension 19 and with compacitifcations involving only type II boundary components. In the case of modular curves, our work generalizes results of Zagier [Zag75] and Funke [Fun02].

We are hopeful that this work might extend to modularity results for higher codimension cycles, as well as to other contexts where Borcherds products are not available but Kudla–Millson's results are still valid.

1.3. **Organization of the paper.** In Section 2, we review the Weil representation, introduce mixed mock modular forms, and prove the mixed modularity of theta series with piecewise linear weights on hyperbolic lattices. In Section 3, we review in detail the construction

of toroidal compactifications, and prove the main intersection formula of special divisors with toric boundary curves. In Section 4, we prove the splitting theorems for second homology that allow us to reduce the modularity computations to toric boundary curves of the toroidal compactification. We assemble the pieces of the proof of Theorem 1.2 in Section 5.

We work over $\mathbb{C}$ throughout.

2. Weil representation and vector-valued modular forms

In this section, we recall the Weil representation and define vector-valued mixed mock modular forms following [BF04], [BFOR17], and [Zag09]. Using a theorem of Vignéras [Vig77], we prove that certain theta series of hyperbolic lattices with piecewise linear weights are mixed mock modular forms. Then, we relate them to theta series of rays in the positive cone of the hyperbolic lattice.

2.1. **Weil representation.** Let $(\Lambda, \cdot)$ be an even lattice of signature (b^+, b^-), with bilinear form

$$\Lambda \times \Lambda \to \mathbb{Z}$$
$$(x, y) \mapsto x \cdot y$$

and quadratic form $Q(x) = \frac{1}{2} x \cdot x \in \mathbb{Z}$. Let $\Lambda^\vee$ be the dual lattice of Λ, defined as

$$\Lambda^\vee := \{ x \in \Lambda_\mathbb{Q} \mid x \cdot y \in \mathbb{Z}, \, \forall y \in \Lambda \}.$$

The *discriminant lattice* $\Lambda^\vee / \Lambda$ is a finite abelian group of cardinality $D_\Lambda = |\det(e_i \cdot e_j)_{i,j}|$ where $\{e_i\}$ form a basis of Λ. It is endowed with a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form:

$$Q : \Lambda \times \Lambda \to \mathbb{Q}/\mathbb{Z}$$
$$x \mapsto Q(x).$$

Let $\mathrm{Mp}_2(\mathbb{R})$ be the double metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ whose elements consist of pairs (M, ϕ), where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and ϕ is a holomorphic function on the Poincaré upper half-plane $\mathbb{H}$ such that $\phi(\tau)^2 = c\tau + d$, for all $\tau \in \mathbb{H}$. The group $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the following elements (see [Ser77, P.78]):

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau \mapsto \sqrt{\tau} \right).$$

The *Weil representation* ρ_Λ associated to $(\Lambda, \cdot)$ is defined as the unique representation

$$\rho_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\Lambda^\vee / \Lambda])$$

such that the elements S and T act in the following way:

$$\rho_\Lambda(T)v_\gamma = e^{2\pi i Q(\gamma)}v_\gamma,$$

$$\rho_\Lambda(S)v_\gamma = \frac{i^{\frac{b^- - b^+}{2}}}{\sqrt{|\Lambda^\vee/\Lambda|}} \sum_{\delta \in \Lambda^\vee/\Lambda} e^{-2\pi i(\gamma\cdot\delta)}v_\delta, \tag{2.1.1}$$

where $\gamma \in \Lambda^\vee/\Lambda$ and $(v_\gamma)_{\gamma\in\Lambda^\vee/\Lambda}$ is the canonical basis of $\mathbb{C}[\Lambda^\vee/\Lambda]$.

Let $\Lambda_1 \subset \Lambda$ be a finite index sublattice. Then we have inclusions

$$\Lambda_1 \subseteq \Lambda \subseteq \Lambda^\vee \subseteq \Lambda_1^\vee.$$

Let $H = \Lambda/\Lambda_1$. It is a subgroup of $\Lambda_1^\vee/\Lambda_1$ and its orthogonal with respect to $\cdot$ is equal to $\Lambda^\vee/\Lambda_1$. Moreover, the subquotient $H^\perp/H$ is isomorphic to $\Lambda^\vee/\Lambda$, the discriminant lattice of Λ. Let $p : H^\perp \to \Lambda^\vee/\Lambda$ be the projection map. We can then define the following two maps:

$$p_{\Lambda_1}^\Lambda : \mathbb{C}[\Lambda_1^\vee/\Lambda_1] \to \mathbb{C}[\Lambda^\vee/\Lambda]$$

$$v_\gamma \mapsto \begin{cases} v_{p(\gamma)} & \text{if } \gamma \in H^\perp, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1.2}$$

$$p_\Lambda^{\Lambda_1} : \mathbb{C}[\Lambda^\vee/\Lambda] \to \mathbb{C}[\Lambda_1^\vee/\Lambda_1]$$

$$v_\delta \mapsto \sum_{\gamma\in H^\perp, p(\gamma)=\delta} v_\gamma. \tag{2.1.3}$$

These operators intertwine the Weil representations on both sides.

Let $I \subset \Lambda$ be a totally isotropic sublattice. The lattice $I^\perp/I$ is a non-degenerate lattice whose discriminant group can be realized as a subquotient of $\Lambda^\vee/\Lambda$: Let Λ_I be the lattice generated by Λ and $I_\mathbb{Q}\cap\Lambda^\vee$. Then the discriminant group of Λ_I is in fact isomorphic to the discriminant group of $I^\perp/I$. Hence we make the following definition.

Definition 2.1. If $I \subset \Lambda$ is a totally isotropic subspace, then we define

$$p_{I^\perp/I}^\Lambda := p_{\Lambda_I}^\Lambda \quad \text{and} \quad p_\Lambda^{I^\perp/I} := p_\Lambda^{\Lambda_I}.$$

These operators intertwine the Weil representations on both sides.

2.2. **Weak harmonic Maass forms and mixed mock modular forms.** Let $k \in \frac{1}{2}\mathbb{Z}$. Following [BF04, Section 3], recall the following definition.

Definition 2.2. A *weak harmonic Maass form* of weight k and representation ρ_Λ is a twice-differentiable function $f : \mathbb{H} \to \mathbb{C}[\Lambda^\vee/\Lambda]$ that satisfies:

(1) $f(\gamma.\tau) = \phi(\tau)^{2k}\rho_\Lambda(\gamma,\phi)f(\tau)$ for all $(\gamma,\phi) \in \mathrm{Mp}_2(\mathbb{Z})$;
(2) there exists $C > 0$ such that $f(\tau) = O(e^{Cy})$ as $y \to +\infty$, $\tau = x + iy$ (uniformly in x);

(3) $\Delta_k f = 0$ where

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right)$$

is the hyperbolic Laplace operator in weight k.

We denote by $H_k(\rho_\Lambda)$ the $\mathbb{C}$-vector space of weak harmonic Maass forms of weight k and representation ρ_Λ. By [BF04, Equation (3.2)], any such form f has a unique decomposition $f = f^+ + f^-$ where f^+ is holomorphic and f^- is called the *non-holomorphic part* of f.

If $f^- = 0$ then f is a weakly holomorphic modular form. Denote by $M_k^!(\rho_\Lambda)$ the space of weakly holomorphic modular forms of weight k and representation ρ_Λ. The *shadow operator* on $H_k(\rho_\Lambda)$ is

$$\xi_k(f) = 2iy^k \overline{\partial_{\bar\tau} f} \in H_k(\overline{\rho}_\Lambda). \qquad (2.2.1)$$

By [BF04, Prop. 3.2], $\xi_k(f) \in M_{2-k}^!(\overline{\rho}_\Lambda)$ and the kernel of ξ_k is equal to $M_k^!(\rho_\Lambda)$. The following definition is inspired from [BFOR17, Definition 5.16] and is originally due to Zagier [Zag09]. We give here a generalization for vector-valued modular forms.

Definition 2.3. A *vector-valued mock modular form* of weight k is the holomorphic part f^+ of a weak harmonic Maass form f of weight k.

If $f : \mathbb{H} \to \mathbb{C}[\Lambda_1^\vee/\Lambda_1]$ and $g : \mathbb{H} \to \mathbb{C}[\Lambda_2^\vee/\Lambda_2]$ are two functions which satisfy relation (1) in Definition 2.2, the function

$$f \otimes g : \mathbb{H} \to \mathbb{C}[\Lambda_1^\vee/\Lambda_1] \otimes \mathbb{C}[\Lambda_2^\vee/\Lambda_2]$$

can be seen naturally as a function valued on $\mathbb{C}[\Lambda^\vee/\Lambda]$ where $\Lambda = \Lambda_1 \oplus \Lambda_2$. Then $f \otimes g$ satisfies the same invariance relation with respect to $\rho_{\Lambda^\vee/\Lambda} \simeq \rho_{\Lambda_1^\vee/\Lambda_1} \otimes \rho_{\Lambda_2^\vee/\Lambda_2}$. Furthermore, if $\Lambda \subseteq \Lambda'$ is a sublattice of finite index, then we have natural morphisms of vector spaces given by Equation 2.1.2 and Equation 2.1.3:

$$p_\Lambda^{\Lambda'} : H_k(\rho_\Lambda) \to H_k(\rho_{\Lambda'}) \text{ and}$$
$$p_{\Lambda'}^{\Lambda} : H_k(\rho_{\Lambda'}) \to H_k(\rho_\Lambda)$$

which commute with the shadow operator. In particular, they preserve the holomorphic and weakly holomorphic modular forms.

The following definition is a slight generalization of Definition 13.1 from [BFOR17].

Definition 2.4. A *mixed harmonic Maass form* of weight (k, ℓ) and representation ρ_Λ is a function h of the form

$$h = \sum_{j=1}^{N} p_{\Lambda_j \oplus \Lambda_j^\perp}^{\Lambda} (f_j \otimes g_j),$$

where $\Lambda_j \subseteq \Lambda$ is a non-degenerate sublattice, f_j is a weakly holomorphic vector-valued modular form of weight k with representation ρ_{Λ_j}

and g_j is vector-valued weak harmonic Maass form of weight l and representation $\rho_{\Lambda_j^\perp}$. If g_j^+ is the holomorphic part of g_j then by definition,

$$h^+ = \sum_{j=1}^N p_{\Lambda_j \oplus \Lambda_j^\perp}^\Lambda (f_j \otimes g_j^+),$$

is a *mixed mock modular form.*

Remark 2.5. One easily checks that h^+ is the holomorphic part of h and hence a different expression of h yields the same function h^+.

If we fix a weight $k \in \frac{1}{2}\mathbb{Z}$ and a representation ρ_Λ, then we can also define mixed mock modular forms of total weight k as follows.

Definition 2.6. A *mixed harmonic Maass form* of weight k and representation ρ_Λ is a function which can be expressed as a finite sum

$$\sum_{k_1+k_2=k} h_{k_1,k_2}$$

where each h_{k_1,k_2} is a mixed harmonic Maass form of weight (k_1, k_2) and representation ρ_Λ. By definition, its holomorphic part $\sum_{k_1+k_2=k} h_{k_1,k_2}^+$ is a mixed mock modular form of weight k and representation ρ_Λ.

2.3. **Zagier's Eisenstein series.** Let $N \geq 1$ and assume in this section that $\Lambda = \mathbb{Z}x$ is endowed with the bilinear form $x \cdot x = 2N$. The discriminant lattice is then isomorphic to $\mathbb{Z}/2N\mathbb{Z}$.

The vector-valued theta series associated to Λ is expressed as

$$\Theta_{\frac{1}{2},N} = \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{4N}} v_{[n]} \in M_{\frac{1}{2}}(\rho_N) , \qquad (2.3.1)$$

where $[n] \in \mathbb{Z}/2N\mathbb{Z}$ is the congruence class. It is a holomorphic modular form of weight $\frac{1}{2}$ and representation ρ_N.

Recall the shadow operator of Equation 2.2.1 is

$$\xi_{\frac{3}{2}} : H_{\frac{3}{2}}(\overline{\rho}_N) \to M_{\frac{1}{2}}^!(\rho_N) .$$

In their seminal paper [HZ76], see also [Zag75], [BFOR17, Theorem 6.3], Hirzebruch and Zagier construct for $N = 1$ a particular weak harmonic Maass form $F_1 \in H_{\frac{3}{2}}(\overline{\rho}_1)$, *Zagier's Eisenstein series*, satisfying

$$\xi_{\frac{3}{2}}(F_1) = -\frac{1}{8\pi}\Theta_{\frac{1}{2},1}.$$

The holomorphic part of this form has the important property that all its Fourier coefficients are rational numbers of arithmetic significance. In what follows, we recall the construction for general $N \geq 1$ following [BS17, Theorem 4.3] and discussion preceding it. We also refer the reader to [Bor98, Lemma 9.5].

Let $r \in \mathbb{Z}/2N\mathbb{Z}$ and let $D \in \mathbb{Z}_{<0}$ such that $D^2 \equiv r^2\,(4N)$. Let $Q_{N,D,r}$ be the set of integral binary quadratic forms $Q : x \mapsto ax^2 + bxy + cy^2$ of

discriminant $D = b^2 - 4ac$ with $N|a$ and $b \equiv r \,(2N)$. The group $\Gamma_0(N)$ acts on this set with finitely many orbits and the order of a stabilizer $w_Q = \frac{1}{2}|\mathrm{Stab}_Q(\Gamma_0(N))|$ is finite. Define

$$H(D, r) := \sum_{Q_{N,D,r}/\Gamma_0(N)} \frac{1}{w_Q}.$$

For $t \geq 0$, let $\beta(t) = \frac{1}{2\pi} \int_1^\infty u^{-\frac{3}{2}} e^{-\pi t u} du$ and let $(v_r)_{r \in \mathbb{Z}/2N\mathbb{Z}}$ be the canonical basis of $\mathbb{C}[\Lambda^\vee/\Lambda]$. We define

$$F_N(\tau) = -\frac{1}{6} v_0 + \sum_{r \in \mathbb{Z}/2N\mathbb{Z}} \sum_{\substack{D \equiv r^2 \,(4N) \\ D < 0}}^{\infty} H(D, r) q^{\frac{-D}{4N}} v_r$$

$$+ \frac{\sqrt{N}}{\sqrt{y}} \sum_{n \in \mathbb{Z}} \beta\left(\frac{y n^2}{N}\right) q^{-\frac{n^2}{4N}} v_{[n]}. \quad (2.3.2)$$

A straightforward computation shows, see [BS17, Theorem 4.3 (1)],

Lemma 2.7. *The series F_N is an element of $H_{\frac{3}{2}}(\bar{\rho}_N)$. Moreover,*

$$\xi_{\frac{3}{2}}(F_N) = -\frac{\sqrt{N}}{8\pi} \Theta_{\frac{1}{2}, N}.$$

2.4. **Vignéras' modularity criterion.** We recall in this section Vignéras' modularity criterion [Vig77, Théorème 1], see also [FK17, Theorem 2.3], which will be our main tool for constructing mixed mock modular forms.

Let (Λ, Q) be a quadratic space of signature (p, q). If we choose a basis of $\Lambda_{\mathbb{R}}$ in which Q has the shape $Q(x) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$, then the Euler differential operator E and the Laplace differential operator Δ associated to Q are:

$$E = \sum_{i=1}^p x_i \frac{\partial}{\partial x_i}, \qquad \Delta = \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2}.$$

We have the following theorem, see [Vig77, Théorème 1] applied to the quadratic space $(\Lambda, -Q)$.

Theorem 2.8. *Let p be a twice differentiable function on $\Lambda_{\mathbb{R}}$ that satisfies the following assumptions:*

(1) *The differential equation*

$$(E - \tfrac{1}{4\pi}\Delta)p = kp$$

holds for an integer k.

(2) *Let $f : x \mapsto p(x) e^{2\pi Q(x)}$. Then $D(f)$ and $x \mapsto R(x)f(x)$ are in $L^2(\mathbb{R}^{p+q}) \cap L^1(\mathbb{R}^{p+q})$ for every derivation D of order at most 2 and every polynomial R of degree at most 2.*

Then the series

$$y^{\frac{k}{2}} \sum_{\lambda \in \Lambda^\vee} p(\lambda\sqrt{y}) q^{-Q(\lambda)} v_{[\lambda]}, \quad q = e^{2\pi i \tau},$$

converges absolutely and defines a non-holomorphic modular form of weight $k + \frac{p+q}{2}$ and representation $\overline{\rho}_\Lambda$.

Here $[\lambda] \in K^\vee / K$ denotes the class of λ in the discriminant group.

Remark 2.9. In [Vig77, Théorème 1], the above theorem is stated for scalar modular forms by introducing a level and a character, but the vector-valued version follows easily from the proof, especially formulas (1) and (3) in *loc. cit.*.

2.5. **Hyperbolic lattices.** We specialize our discussion from before to hyperbolic lattices. Let $(K, \cdot)$ be an even lattice of signature $(1, n-1)$. Let C_K be a connected component of $\{x \in K_\mathbb{R} \,|\, x \cdot x > 0\}$. Then for any v_1, v_2 in C_K, we have $v_1 \cdot v_2 > 0$.

Let $E(x) = 2 \int_0^x e^{-\pi t^2} dt$ be the Gauss error function and define $\psi(x) = xE(x) + \frac{1}{2\pi} E'(x)$. One can easily check that ψ satisfies the following differential equation:

$$x\psi'(x) + \frac{1}{2\pi}\psi''(x) = \psi. \tag{2.5.1}$$

Moreover, we have

$$E(x) = \operatorname{sgn}(x)(1 - 2\int_{|x|}^\infty e^{-\pi t^2} dt),$$

and we can thus write:

$$\psi(x) = xE(x) + \frac{1}{2\pi}E'(x) = |x| + \beta(x^2), \tag{2.5.2}$$

where for $t \geq 0$,

$$\beta(t) = \frac{1}{2\pi} \int_1^\infty u^{-\frac{3}{2}} e^{-\pi t u} du.$$

The importance of the functions E and ψ is that they are smooth approximations of the sign and the absolute value functions respectively, and they will help us build smooth approximations to the piecewise linear functions appearing in the intersection formulas of special cycles.

For a collection of vectors $\{c_i\}_{i\in S}$ in the closure of the positive cone C_K that satisfy $\sum_{i\in S} a_i c_i = 0$, define the function

$$p^+(\lambda) := \sum_{i \in S} a_i |\lambda \cdot c_i|$$

and consider the formal power series:

$$\Theta^+ = \sum_{\lambda \in K^\vee} p^+(\lambda) q^{-Q(\lambda)} v_{[\lambda]} \in \mathbb{C}[K^\vee / K][[q^{\frac{1}{D_K}}]].$$

Our goal in the next two sections is to analyze the convergence and the modularity properties of this series.

2.6. **Case of non-isotropic vectors.** Let $c \in C_K$ and define

$$\phi_c(\lambda) := \sqrt{\frac{c \cdot c}{2}} \psi \left(\frac{\sqrt{2}\, \lambda \cdot c}{\sqrt{c \cdot c}} \right) \quad \text{for } \lambda \in K_{\mathbb{R}}.$$

Then using Equation (2.5.1), one can check that:

$$\left(E - \tfrac{1}{4\pi}\Delta \right) \phi_c = \phi_c. \tag{2.6.1}$$

Now consider the special case where $(c_i)_{i \in S}$ is a finite set of primitive integral elements of the positive cone $C_K \cap K$, such that $\sum_{i \in S} a_i c_i = 0$, with $a_i \in \mathbb{Z}$ non-zero integers. Let $c_i \cdot c_i = 2N_i > 0$. Define

$$p(\lambda) := \sum_{i \in S} a_i \phi_{c_i}(\lambda)$$

$$= \sum_{i \in S} a_i \sqrt{N_i}\, \psi \left(\frac{\lambda \cdot c_i}{\sqrt{N_i}} \right). \tag{2.6.2}$$

Proposition 2.10. *Let c_i be a collection of (non-isotropic) primitive integral vectors of C_K which satisfy $\sum_i a_i c_i = 0$ for some integers c_i. Then the function p satisfies the conditions of Theorem 2.8.*

The proof of this proposition is inspired from the proof of Proposition 2.4 in [Zwe02]. We first prove the following lemma.

Lemma 2.11. *Let $(c_i)_{i \in S}$ be a finite set of distinct integral elements of $\overline{C}_K$ which satisfies a relation of the form $\sum_i a_i c_i = 0$, where a_i are non-zero numbers and let $p^+(\lambda) = \sum_i a_i |\lambda \cdot c_i|$. Let U denote the open set of elements $\lambda \in K_{\mathbb{R}}$ where the function $\lambda \mapsto p^+(\lambda)$ does not vanish. Then there exists an euclidean norm $|| \cdot ||$ on $K_{\mathbb{R}}$ such that:*

(1) If all $c_i \in C_K$, then $-Q(\lambda) \geq ||\lambda||^2$ for all $\lambda \in U$.
(2) In general, $-Q(\lambda) \geq ||\lambda||$ for all $\lambda \in U \cap K$.

Proof. Notice that S has cardinality at least 3 and that whenever $\lambda \cdot c_i$ have the same sign for all $i \in S$, then the function $\lambda \mapsto p^+(\lambda)$ vanishes. Hence U is contained in the set where the linear forms $\lambda \mapsto \lambda \cdot c_i$ don't have all the same sign. In particular, U is a finite union over (i, j) of the sets $U_{(i,j)}$ where $\lambda \cdot c_i \geq 0$ and $\lambda \cdot c_j \leq 0$. So it suffices to prove the lemma for each $U_{(i,j)}$. Fix such pair and call it $(1, 2)$.
First case: Assume that c_1 and c_2 are not isotropic. Write

$$\lambda = \lambda_{c_1^{\perp}} + \frac{\lambda \cdot c_1}{c_1 \cdot c_1} c_1. \text{ Then}$$

$$0 \geq \lambda \cdot c_2 = \lambda_{c_1^{\perp}} \cdot c_2 + \frac{\lambda \cdot c_1}{c_1 \cdot c_1} c_1 \cdot c_2 \implies$$

$$\frac{(\lambda \cdot c_1)(c_1 \cdot c_2)}{c_1 \cdot c_1} \leq -\lambda_{c_1^{\perp}} \cdot c_2.$$

Let $\widetilde{c}_2$ be the orthogonal projection of c_2 to $c_1^\perp$. By Cauchy–Schwarz inequality, we have

$$(\lambda_{c_1^\perp} \cdot c_2)^2 = (\lambda_{c_1^\perp} \cdot \widetilde{c}_2)^2 \le (\lambda_{c_1^\perp} \cdot \lambda_{c_1^\perp})(\widetilde{c}_2 \cdot \widetilde{c}_2).$$

Hence we get

$$\frac{(\lambda \cdot c_1)^2}{|c_1 \cdot c_1|} \le \frac{|c_1 \cdot c_1|}{(c_1 \cdot c_2)^2}(\lambda_{c_1^\perp} \cdot \lambda_{c_1^\perp})\left(c_2 \cdot c_2 - \frac{(c_1 \cdot c_2)^2}{c_1 \cdot c_1}\right)$$

$$\le \left(1 - \frac{(c_2 \cdot c_2)(c_1 \cdot c_1)}{(c_1 \cdot c_2)^2}\right)|\lambda_{c_1^\perp} \cdot \lambda_{c_1^\perp}|.$$

Let $\mu > 0$ such that $\frac{1-\mu}{1+\mu} = 1 - \frac{(c_2 \cdot c_2)(c_1 \cdot c_1)}{(c_1 \cdot c_2)^2}$. Then one can check that

$$-Q(\lambda) \ge \mu Q_{c_1}(\lambda), \tag{2.6.3}$$

where Q_{c_1} is a positive definite quadratic form defined by:

$$Q_{c_1}(\lambda) := -Q(\lambda_{c_1^\perp}) + \frac{(\lambda \cdot c_1)^2}{2(c_1 \cdot c_1)}.$$

Second case: Assume now that c_1 is isotropic and c_2 and is still anisotropic. The lattice (c_1, c_2) has signature $(1, 1)$ so its orthogonal $M \subset K$ is negative definite. We can write

$$\lambda = n_1 c_1 + n_2 c_2 + v,$$

where $v \in M^\vee$. Then by assumption $x \cdot c_1 = n_2(c_1 \cdot c_2) \ge 0$ which implies that $n_2 \ge 0$ and $\lambda \cdot c_2 = n_1(c_1 \cdot c_2) + n_2(c_2 \cdot c_2) \le 0$. So

$$n_2(c_2 \cdot c_2) \le -n_1(c_1 \cdot c_2),$$

and in particular, $n_1 \le 0$.

On the other hand,

$$-Q(\lambda) = -n_1 n_2(c_1 \cdot c_2) - n_2^2\frac{c_2 \cdot c_2}{2} - Q(v)$$

$$= -\frac{n_1 n_2(c_1 \cdot c_2)}{2} - n_2\frac{n_1(c_1 \cdot c_2) + n_2(c_2 \cdot c_2)}{2} - Q(v)$$

$$\ge n_2^2|c_2 \cdot c_2| - Q(v).$$

If $n_2 \ne 0$, then since λ and c_1 are in K, we have that $|n_2(c_1 \cdot c_2)| \ge 1$, hence we get also $-Q(\lambda) \ge \frac{|n_1|}{2}$ and therefore

$$-3Q(\lambda) \ge |n_1| + n_2^2|c_2 \cdot c_2| - Q(v) \gg ||\lambda||$$

for the euclidean norm $||\lambda||^2 = n_1^2 + n_2^2 - Q(v)$.

If $n_2 = 0$, then there exists c_3 such that $\lambda \cdot c_3 > 0$, otherwise all $\lambda \cdot c_i$ would have the same sign for $i \ne 1$ and hence

$$p(\lambda) = 2a_1|\lambda \cdot c_1| = 2a_1|n_2(c_1 \cdot c_2)|,$$

contradicting the fact that p does not vanish on λ. From the inequality $\lambda \cdot c_3 > 0$ we get then:

$$n_1(c_1 \cdot c_3) + n_2(c_2 \cdot c_3) + \widetilde{c_3} \cdot v > 0,$$

where $\widetilde{c_3}$ is the projection of c_3 to M. From Cauchy-Schwartz inequality, we have:

$$|n_1(c_1 \cdot c_3)| \leq |n_2(c_2 \cdot c_3)| + \sqrt{(v \cdot v)(\widetilde{c_3} \cdot \widetilde{c_3})},$$

from which it follows that there exists a constant $\mu_1 > 0$ such that

$$-Q(\lambda) \geq \mu_1 \left(n_1^2 + n_2^2 - Q(v) \right),$$

and we can choose again the euclidean norm $||\lambda||^2 = n_1^2 + n_2^2 - Q(v)$.

Last case: Assume now that both c_1 and c_2 are isotropic. Then we can introduce $c_2' = c_1 + c_2$ which is not isotropic and notice that $U_{1,2}$ is contained in $U_{1,2'} \cup U_{2',2}$ and hence we use the previous analysis on each of them. This finishes the proof of the lemma. $\square$

We are now ready to prove Proposition 2.10.

Proof Proposition 2.10. Condition (1) follows from Equation 2.6.1. We only need to prove condition (2). By Equation 2.5.2, we have the decomposition for $x \in \mathbb{R}$

$$\psi(x) = |x| + \beta(x^2).$$

This corresponds to a decomposition

$$p(\lambda) = p^+(\lambda) + p^-(\lambda),$$

where $p^+(\lambda) = \sum_i a_i |\lambda \cdot c_i|$. From the estimate $\beta(t) \ll e^{-\pi t}$, for $t \geq 0$, we get that

$$|p^-(\lambda)e^{\pi(\lambda \cdot \lambda)}| \ll \sum_{i \in S} e^{-\pi \left[-\lambda \cdot \lambda + 2\frac{(\lambda \cdot c_i)^2}{c_i \cdot c_i} \right]}$$

$$\ll \sum_{i \in S} e^{-2\pi Q_{c_i}(\lambda)},$$

where as before, $Q_c(\lambda) = -Q(\lambda_{c^\perp}) + \frac{(\lambda \cdot c)^2}{2(c \cdot c)}$ is the positive quadratic form associated to a positive integral vector c.

The function p^+ vanishes if all the linear forms $\lambda \mapsto \lambda \cdot c_i$ have the same sign when $i \in S$. Hence, in the region where it is non-vanishing, by (1) in Lemma 2.11, there exists an Euclidean norm on $K_\mathbb{R}$ such that:

$$|p^+(\lambda)e^{\pi(\lambda \cdot \lambda)}| \leq |p^+(\lambda)e^{-\pi||\lambda||^2}|.$$

Combining the two previous estimates, this proves that for any polynomial R, the function $\lambda \mapsto R(\lambda)p(\lambda)e^{\pi(\lambda \cdot \lambda)}$ is in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

For the first order derivatives, notice that

$$\psi'(x) = E(x) = \text{sgn}(x) - \text{sgn}(x) \cdot 2 \int_{|x|}^{\infty} e^{-\pi t^2} dt.$$

For any non-zero vector $v \in K_{\mathbb{R}}$, the partial derivative $\partial_v p$ can then be decomposed accordingly as

$$(\partial_v p)^+(\lambda) + (\partial_v p)^-(\lambda).$$

One can easily check that the function $(\partial_v p)^+$ vanishes on the region where all the $\lambda \mapsto \lambda \cdot c_i$ have the same sign. Furthermore, we have the estimate

$$\int_{|x|}^{\infty} e^{-\pi t^2} dt \ll e^{-\pi x^2},$$

which implies in a similar way that

$$|(\partial_v p)^-(\lambda) e^{\pi(\lambda \cdot \lambda)}| \ll \sum_{i \in S} e^{-2\pi Q_{c_i}(\lambda)}.$$

For the second order derivative, since $\psi''(x) = e^{-\pi x^2}$, we get immediately for any two non-zero vectors $v, v' \in K_{\mathbb{R}}$ the estimate

$$|\partial_v \partial_{v'} p(\lambda) e^{-\pi(\lambda \cdot \lambda)}| \ll \sum_{i \in S} e^{-\pi\left[\lambda \cdot \lambda - 2\frac{(\lambda \cdot c_i)^2}{c_i \cdot c_i}\right]} \le \sum_{i \in S} e^{-2\pi Q_{c_i}(\lambda)}.$$

Hence the result. $\square$

As it appeared in the previous proof, the function p is a smooth approximation of the function $p^+(\lambda) = \sum_{i \in S} a_i |\lambda \cdot c_i|$. By Theorem 2.8,

$$\Theta(\tau) = \frac{1}{\sqrt{y}} \sum_{\lambda \in K^\vee} p(\sqrt{y}\lambda) q^{-Q(\lambda)} v_{[\lambda]}$$

$$= \frac{1}{\sqrt{y}} \sum_{\gamma \in K^\vee/K} \sum_{\lambda \in \gamma+K} p(\sqrt{y}\lambda) q^{-Q(\lambda)} v_\gamma$$

transforms like a non-holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_K$, where $(v_\gamma)_{\gamma \in K^\vee/K}$ is the usual basis of the $\mathbb{C}$-vector space $\mathbb{C}[K^\vee/K]$.

The holomorphic part of this series is equal to the series introduced at the end of Section 2.4:

$$\Theta^+(\tau) = \sum_{\lambda \in K^\vee} p^+(\lambda) q^{-Q(\lambda)} v_{[\lambda]}.$$

The non-holomorphic part is equal to

$$\sum_{i \in S} a_i \Theta_K^{c_i}(\tau),$$

where $c_i \cdot c_i = 2N_i$ and the series $\Theta_K^{c_i}$ is an absolutely convergent series given as follows:

$$\Theta_K^{c_i}(\tau) = \frac{\sqrt{N_i}}{\sqrt{y}} \sum_{\lambda \in K^\vee} \beta\left(\frac{y(\lambda \cdot c_i)^2}{N_i}\right) q^{-Q(\lambda)} v_{[\lambda]}$$

$$= \frac{\sqrt{N_i}}{\sqrt{y}} p_{c_i^\perp \oplus \mathbb{Z}c_i}^K \left(\sum_{\substack{\lambda = \delta + \frac{n}{2N_i} c_i \\ \delta \in (c_i^\perp)^\vee, n \in \mathbb{Z}}} \beta\left(\frac{yn^2}{N_i}\right) q^{-\frac{n^2}{4N_i^2}Q(c_i) - Q(\delta)} v_{[\delta]} \otimes v_{[n]} \right)$$

$$= \frac{\sqrt{N_i}}{\sqrt{y}} p_{c_i^\perp \oplus \mathbb{Z}c_i}^K \left[\left(\sum_{\delta \in (c_i^\perp)^\vee} q^{-Q(\delta)} v_{[\delta]} \right) \otimes \left(\sum_{n \in \mathbb{Z}} \beta\left(\frac{yn^2}{N_i}\right) q^{-\frac{n^2}{4N_i}} v_{[n]} \right) \right]$$

$$= p_{c_i^\perp \oplus \mathbb{Z}c_i}^K \left(\Theta_{c_i^\perp} \otimes F_{N_i}^- \right)$$

Here $\Theta_{c_i^\perp}$ is the theta series of the negative definite lattice $c_i^\perp$ which is a holomorphic modular form of weight $\frac{n-1}{2}$ and representation $\overline{\rho}_{c_i^\perp}$ and $F_{N_i}^-$ is the non-holomorphic part of the generalized Zagier Eisenstein series introduced in Section 2.3. In particular, the non-holomorphic part of

$$\Theta - \sum_{i \in S} a_i p_{c_i^\perp \oplus \mathbb{Z}c_i}^K (\Theta_{c_i^\perp} \otimes F_{N_i}).$$

vanishes. Therefore, it is a holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_K$. We have thus proven the following theorem.

Theorem 2.12. *Let $(c_i)_{i \in S}$ be a finite set of integral vectors of C_K with a relation $\sum_{i \in S} a_i c_i = 0$, $a_i \in \mathbb{Z}$. Then the function Θ^+ is a mixed harmonic Maass form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_K$. More precisely, the function*

$$\Theta^+ - \sum_{i \in S} a_i p_{c_i^\perp \oplus \mathbb{Z}c_i}^K (\Theta_{c_i^\perp} \otimes F_{N_i}^+).$$

is a holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_K$.

2.7. **Case of isotropic vectors.** We now analyze the general case where the vectors c_i are allowed to be isotropic. For each isotropic vector c_i, let $M_i := c_i^\perp / c_i$ be the associated negative definite lattice.

We can write

$$\sum_{i \in S_1} a_i c_i = -\sum_{i \in S_2} a_i c_i,$$

where S_1 is the set of indices where $a_i > 0$ and S_2 is the set of indices where $a_i < 0$. Define $c := \sum_{i \in S_1} a_i c_i$, so that c is a rational vector in the positive cone C_K.

For $\epsilon > 0$, define

$$c_i(\epsilon) := \begin{cases} c_i + \epsilon \frac{c}{\sum_{\ell \in S_1} a_\ell} & \text{if } i \in S_1 \\ c_i - \epsilon \frac{c}{\sum_{\ell \in S_2} a_\ell} & \text{if } i \in S_2. \end{cases}$$

Notice that the relation $\sum_i a_i c_i(\epsilon) = 0$ still holds and for ϵ small enough, the vectors $c_i(\epsilon)$ all lie in the positive cone C_K.

For $\lambda \in K_{\mathbb{R}}$, let $p^+(\lambda) = \sum_i a_i |\lambda \cdot c_i|$ and consider again the series:

$$\Theta^+ = \sum_{\lambda \in K^\vee} p^+(\lambda) q^{-Q(\lambda)} v_{[\lambda]} \in \mathbb{C}[K^\vee/K][[q^{\frac{1}{D_K}}]].$$

Our goal in this section is to prove the following theorem.

Theorem 2.13. *Let* $(c_i)_{i \in S}$ *be a finite set of integral vectors of* $\overline{C}_K$ *with a relation* $\sum_i a_i c_i = 0$, $a_i \in \mathbb{Z}$. *Then the series* Θ^+ *converges absolutely for* $\tau \in \mathbb{H}$ *where* $q = e^{2\pi i \tau}$ *to a holomorphic function which is a mixed mock modular form. More precisely, the function*

$$\Theta^+ - \sum_{Q(c_i)>0} a_i p^K_{c_i^\perp \oplus \mathbb{Z}c_i}(\Theta_{c_i^\perp} \otimes F^+_{N_i}) + \frac{1}{6} \sum_{Q(c_i)=0} a_i E_2 p^K_D(\Theta_{c_i^\perp/c_i})$$

is a holomorphic modular form of weight $1 + \frac{n}{2}$ *and representation* $\overline{\rho}_K$.

Proof. We first prove the convergence of Θ^+: by Lemma 2.11, there exists an euclidean norm $||\cdot||$ such that in the region $U \cap K^\vee$ where $\lambda \mapsto p(\lambda)$ does not vanish, we have $-Q(\lambda) \geq ||\lambda||$. So for every $m \in \frac{1}{D_K}\mathbb{Z}$, the number of vectors $\lambda \in K^\vee$ such that $|Q(\lambda)| \leq m$ is finite and upper bounded by a polynomial in m. This implies the uniform absolute convergence of Θ^+ on compact subsets of $\mathbb{H}$ to a holomorphic function on $\mathbb{H}$.

To obtain the mixed mock modularity, we approach Θ^+ with a sequence of mixed mock modular theta series. Let $\epsilon > 0$ be small enough so that all the vectors $c_i(\epsilon)$ are in the positive cone C_K.

Consider the holomorphic function on $\mathbb{H}$:

$$\Theta_\epsilon(\tau) = \sum_{\lambda \in K^\vee} p^+_\epsilon(\lambda) q^{-Q(\lambda)} v_{[\lambda]} \in \mathbb{C}[K^\vee/K][[q^{\frac{1}{D_K}}]]$$

where $p^+_\epsilon(\lambda) := \sum_i a_i |\lambda \cdot c_i(\epsilon)|$. The above series is absolutely convergent and uniformly convergent on compact subsets of $\mathbb{H}$. Notice also that, as $\epsilon \to 0$, $p_\epsilon(\lambda) \to p(\lambda)$ for every $\lambda \in K^\vee$.

Lemma 2.14. *As* $\epsilon \to 0$, $\Theta^+_\epsilon(\tau) \to \Theta^+(\tau)$ *uniformly in* τ *in compact regions of* $\mathbb{H}$.

Proof. Note that in the region where $(\lambda \mapsto \lambda \cdot c_i)_{i \in S}$ all have the same sign, the functions p^+ and p_ϵ both vanish. Indeed, if all the linear forms $\lambda \mapsto \lambda \cdot c_i$ have the same sign, then $\lambda \mapsto \lambda \cdot c$ also has the same sign as them, and hence for any $\epsilon > 0$, the linear forms $\lambda \mapsto \lambda \cdot c_i(\epsilon)$ have the same sign. Hence the function p^+_ϵ vanishes. Let U be the complement of the above region. Then, using Lemma 2.11, we have for any $\tau \in \mathbb{H}$

with $\mathrm{Im}(y) > B$:

$$\left|\left|\Theta_\epsilon^+(\tau) - \Theta^+(\tau)\right|\right| \leq \sum_{\lambda \in K^\vee \cap U} \sum_{i \in S} |a_i|(|\lambda \cdot c_i| - |\lambda \cdot c_i(\epsilon)|)e^{2\pi y Q(\lambda)}$$

$$\leq \epsilon \sum_{\lambda \in K^\vee \cap U} \sum_{i \in S} |a_i||\lambda \cdot c|e^{-2\pi B||\lambda||}.$$

Hence the result. $\qquad\qquad\square$

Let p_ϵ be the completion of p_ϵ^+ as in Equation (2.6.2). Then by Proposition 2.10 and Theorem 2.8, the function:

$$\widehat{\Theta}_\epsilon(\tau) = \frac{1}{\sqrt{y}} \sum_{\lambda \in K^\vee} p(\sqrt{\lambda})q^{-Q(\lambda)}v_{[\lambda]}$$

transforms like a non-holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_K$. As $\epsilon \to 0$, its holomorphic part converges to Θ^+ by Lemma 2.14 and we will analyze the convergence of its non-holomorphic part. For $\epsilon > 0$, the non-holomorphic part is given as the sum over i of terms of the form:

$$\Theta_K^{c_i(\epsilon)}(\tau) = \sqrt{\frac{N_i(\epsilon)}{y}} \sum_{\lambda \in K^\vee} \beta\left(\frac{y(\lambda \cdot c_i(\epsilon))^2}{N_i(\epsilon)}\right) q^{-Q(\lambda)}v_{[\lambda]}.$$

If c_i is not isotropic, then, as $\epsilon \to 0$, $N_i(\epsilon) \to N_i \neq 0$ and the quadratic form $\lambda \mapsto \frac{(\lambda \cdot c_i(\epsilon))^2}{N_i(\epsilon)} - 2Q(\lambda)$ limits to the positive definite quadratic form $2Q_{c_1}$, hence there exists $A > 0$ depending only on c and c_1 such that for $\epsilon \leq A$, we have for every $\lambda \in K_\mathbb{R}$ the inequality

$$\frac{(\lambda \cdot c_i(\epsilon))^2}{N_i(\epsilon)} - 2Q(\lambda) \geq Q_{c_1}(\lambda) \geq 0.$$

Hence, using the estimate $\beta(t) \ll e^{-\pi t}$, we get

$$\left|\beta\left(\frac{y(\lambda \cdot c_i(\epsilon))^2}{N_i(\epsilon)}\right) q^{-Q(\lambda)}\right| \ll \frac{1}{2\pi} e^{-\pi y\left(\frac{(\lambda \cdot c_i(\epsilon))^2}{N_i(\epsilon)} - 2Q(\lambda)\right)}$$

$$\ll e^{-\pi y Q_{c_1}(\lambda)}.$$

This implies the uniform convergence and hence we can interchange the limit as $\epsilon \to 0$ with the sum over $\lambda \in K^\vee$ yielding the convergence of the function $\Theta_K^{c_i(\epsilon)}$ to $\Theta_K^{c_i}$ introduced in the previous section.

We now identify the limit of $\Theta_K^{c_i(\epsilon)}$ when c_i is an isotropic vector. To simplify the notations and the computations hereafter, assume that $i = 1$ and up to rescaling ϵ, that c is primitive integral in K and $c_1(\epsilon) = c_1 + \epsilon c$. Let M be the orthogonal of $U = \mathbb{Z}c_1 + \mathbb{Z}c$ in K. Denote the intersection matrix of U by

$$\begin{pmatrix} 0 & a \\ a & 2b \end{pmatrix}.$$

then the dual lattice $U^\vee$ is generated by $a^{-1}c_1$ and $a^{-1}\widetilde{c}$ where $\widetilde{c} = c - 2\frac{b}{a}c_1$. The lattice M maps injectively to a finite index sub-lattice of $M_1 = c_1^\perp/c_1$ and we have the chain of inclusions

$$U \oplus M \subseteq K \subseteq K^\vee \subseteq U^\vee \oplus M^\vee.$$

Hence every element of $K^\vee$ has a decomposition $a^{-1}(n_1c_1 + n_2\widetilde{c}) + \gamma$ where $\gamma \in M_1^\vee$, $n_1, n_2 \in \mathbb{Z}$. We denote by $[n_1, n_2] \in (\mathbb{Z}/a\mathbb{Z})^2$ the image of $a^{-1}(n_1c_1 + n_2\widetilde{c})$ in $U^\vee/U$. We have that $\Theta_K^{c_1(\epsilon)}(\tau)$ equals

$$\sqrt{\frac{N_1(\epsilon)}{y}} p_{U\oplus M}^K \left(\sum_{\substack{n_1,n_2\in\mathbb{Z} \\ \gamma\in M^\vee}} \beta\left(\frac{y(n_2+\epsilon n_1)^2}{\epsilon a+\epsilon^2 b}\right) q^{-\frac{n_1 n_2}{a}+\frac{n_2^2 b}{a^2}-Q(\gamma)} v_{[n_1,n_2]}\otimes v_{[\gamma]} \right)$$

$$= \sqrt{\frac{N_1(\epsilon)}{y}} p_{U\oplus M}^K \left[\left(\sum_{n_1,n_2\in\mathbb{Z}} \beta\left(\frac{y(n_2+\epsilon n_1)^2}{\epsilon a+\epsilon^2 b}\right) q^{-\frac{n_1 n_2}{a}+\frac{n_2^2 b}{a^2}} v_{[n_1,n_2]} \right) \right.$$

$$\left. \otimes \left(\sum_{\gamma\in M^\vee} q^{-Q(\gamma)} v_{[\gamma]} \right) \right].$$

Notice that because $\beta(t) \ll e^{-\pi t}$, the first sum above is absolutely convergent for ϵ small enough and $\frac{N_1(\epsilon)}{\epsilon} = a + b\epsilon$. Furthermore, we can decompose the first sum according to $n_2 = 0$ or $n_2 \neq 0$ as:

$$\sqrt{\frac{N_1(\epsilon)}{y}} \sum_{n_1,n_2\in\mathbb{Z}} \beta\left(\frac{y(n_2+\epsilon n_1)^2}{\epsilon a+\epsilon^2 b}\right) q^{-\frac{n_1 n_2}{a}+\frac{n_2^2 b}{a^2}} v_{[n_1,n_2]}$$

$$= \left(\frac{\sqrt{N_1(\epsilon)}}{2\pi\sqrt{y}} \sum_{n_1\in\mathbb{Z}} \int_1^\infty u^{-\frac{3}{2}} e^{-\pi y u \frac{\epsilon n_1^2}{a+b\epsilon}} du \right) v_{[n_1,0]}$$

$$+ \sqrt{\frac{N_1(\epsilon)}{y}} \sum_{n_1,n_2\in\mathbb{Z},\, n_2\neq 0} \beta\left(\frac{y(\epsilon n_1+n_2)^2}{a\epsilon+b\epsilon^2}\right) q^{-\frac{n_1 n_2}{a}+\frac{n_2^2 b}{a^2}} v_{[n_1,n_2]}.$$

Write $v_{[n_1,0]} = v_{[n_1]}$. The first sum can be rewritten using dominated convergence and Poisson's summation formula as follows:

$$\sum_{n_1\in\mathbb{Z}} \int_1^\infty u^{-\frac{3}{2}} e^{-\pi y u \frac{\epsilon n_1^2}{a+b\epsilon}} du\, v_{[n_1]} = \int_1^\infty u^{-\frac{3}{2}} \sum_{n_1\in\mathbb{Z}} e^{-\pi y u \frac{\epsilon n_1^2}{a+b\epsilon}} du\, v_{[n_1]}$$

$$= \int_1^\infty u^{-\frac{3}{2}} \frac{\sqrt{a+\epsilon b}}{a\sqrt{yu\epsilon}} \sum_{n_1\in\mathbb{Z}} e^{-\frac{\pi n_1^2(a+b\epsilon)}{yu\epsilon}} du\, v_{[n_1]} .$$

The last sum, multiplied by $\frac{1}{2\pi}\sqrt{\frac{N_1(\epsilon)}{y}}$, converges uniformly as $\epsilon \to 0$ to its constant term which is $\frac{1}{2\pi y}\int_1^\infty u^{-2} = \frac{1}{2\pi y}$. As for the second sum,

we can write it using the estimate $\beta(t) \ll e^{-\pi t}$ as:

$$\left\| \sqrt{\frac{N_1(\epsilon)}{y}} \sum_{n_1, n_2 \in \mathbb{Z}, n_2 \neq 0} \beta\left(\frac{y(n_2 + \epsilon n_1)^2}{\epsilon a + b\epsilon^2}\right) q^{-\frac{n_1 n_2}{a} + \frac{n_2^2 b}{a^2}} v_{[n_2]} \right\|$$

$$\ll \sqrt{\frac{N_1(\epsilon)}{y}} \sum_{n_1, n_2 \in \mathbb{Z}, n_2 \neq 0} e^{-\pi y\left(n_2^2\left(\frac{2b}{a^2} + \frac{1}{\epsilon(a+b\epsilon)}\right) - \frac{2b\epsilon n_1 n_2}{a(a+b\epsilon)} + \epsilon\frac{n_1^2}{a+b\epsilon}\right)}.$$

By computing the discriminant of the quadratic form in the exponent above, we see that there exists A_1 such that for $\epsilon \leq A_1$, the following inequality holds for every $n_1, n_2 \in \mathbb{Z}$:

$$n_2^2\left(\frac{2b}{a^2} + \frac{1}{\epsilon a + b\epsilon}\right) - \frac{2b\epsilon n_1 n_2}{a(a + b\epsilon)} + \epsilon\frac{n_1^2}{a + b\epsilon} \geq \frac{1}{2a}\left(\frac{n_2^2}{\epsilon} + \epsilon n_1^2\right).$$

Hence combining the previous inequalities and Poisson summation formula we get:

$$\left\| \sqrt{\frac{N_1(\epsilon)}{y}} \sum_{n_1, n_2 \in \mathbb{Z}, n_2 \neq 0} \beta\left(\frac{y(n_2 + \epsilon n_1)^2}{\epsilon a + b\epsilon^2}\right) q^{-\frac{n_1 n_2}{a} + \frac{n_2^2 b}{a^2}} v_{[n_2]} \right\|$$

$$\ll \sqrt{\frac{N_1(\epsilon)}{y}} \sum_{n_1, n_2 \in \mathbb{Z}, n_2 \neq 0} e^{-\frac{\pi y}{2a}\left(\frac{n_2^2}{\epsilon} + \epsilon n_1^2\right)}$$

$$\ll \left(\sqrt{\frac{\epsilon}{y}} \sum_{n_1 \in \mathbb{Z}} e^{-\frac{\pi y}{2a}\epsilon n_1^2}\right)\left(\sum_{n_2 \in \mathbb{Z}, n_2 \neq 0} e^{-\frac{\pi y}{2a}\frac{n_2^2}{\epsilon}}\right)$$

$$\ll \left(\frac{\sqrt{2aN_1(\epsilon)}}{y\sqrt{\epsilon}} \sum_{n_1 \in \mathbb{Z}} e^{-2\pi a\frac{n_1^2}{y\epsilon}}\right)\left(\sum_{n_2 \in \mathbb{Z}, n_2 \neq 0} e^{-\frac{\pi y}{2a}\frac{n_2^2}{\epsilon}}\right),$$

and by normal convergence, the last product converges to 0 as $\epsilon \to 0$. We conclude that the limit of $\Theta_K^{c_1(\epsilon)}$ as $\epsilon \to 0$ is equal to $\frac{1}{2\pi y}p_M^K(\Theta_M) = \frac{1}{2\pi y}p_{M_1}^K(\Theta_{M_1})$ where $M_1 = c_1^{\perp}/c_1$, which is the non-holomorphic part of the mixed mock modular form $-\frac{1}{6}E_2 \cdot p_{M_1}^K(\Theta_{M_1})$, where $E_2(\tau) = 1 - 24\sum_{n \geq 1}\sigma_1(n)q^n$. Let $E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi y}$ be the completion of E_2 to a non-holomorphic modular form.

Notice that

$$\widehat{\Theta}_\epsilon - \sum_{Q(c_i) > 0} a_i p_{c_i^{\perp} \oplus \mathbb{Z} c_i}^K(\Theta_{c_i^{\perp}} \otimes F_{N_i}) + \sum_{Q(c_i) = 0} a_i p_{M_1}^K(\Theta_{M_1}) \otimes \frac{1}{6}E_2^*$$

transforms like a holomorphic modular form of weight $1 + \frac{n}{2}$ with respect to $\bar{\rho}_K$. Hence by letting $\epsilon \to 0$, it still transforms appropriately and the previous computations show that its non-holomorphic part vanishes, whereas its holomorphic part is equal to

$$\Theta^+ - \sum_{Q(c_i) > 0} a_i p_{K_{c_i} \oplus \mathbb{Z} c_i}^K(\Theta_{c_i^{\perp}} \otimes F_{N_i}^+) + \sum_{Q(c_i) = 0} \frac{a_i}{6}E_2 p_{M_i}^K(\Theta_{M_i})$$

which concludes the proof of Theorem 2.13.   □

3. Orthogonal Shimura varieties

In this section, we give an explicit description of the special divisors in toroidal compactifications of orthogonal Shimura varieties and we derive the main intersection formula, see Theorem 3.13, of special divisors with curves contained in the boundary. For general references on toroidal compactifications, the reader may consult [AMRT10] or [BZ22, Fio16, Loo03] for the orthogonal case.

3.1. **Baily-Borel compactification.** Let $(L, \cdot)$ be an even lattice of signature $(2, n)$, let $L^\vee$ be its dual lattice and let $\mathbb{D}$ be the associated Hermitian symmetric domain, a connected component of

$$\{\mathbb{C}x \in \mathbb{P}(L_\mathbb{C}) \,\big|\, x \cdot x = 0,\, x \cdot \overline{x} > 0\}.$$

It embeds into its compact dual

$$\mathbb{D}^c := \{\mathbb{C}x \in \mathbb{P}(L_\mathbb{C}) \,\big|\, x \cdot x = 0\}$$

which is a quadric hypersurface in the projective space $\mathbb{P}(L_\mathbb{C}) \simeq \mathbb{P}^{n+1}$.

Let $\Gamma \subset \mathrm{O}^+(L)$ be a subgroup of the stable orthogonal group, defined as the kernel of the reduction $\mathrm{O}^+(L) \to \mathrm{O}(L^\vee/L)$. Then Γ acts on $\mathbb{D}$ and the quotient has the structure of a complex orbifold, it is an *orthogonal Shimura variety*. The restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}^{n+1}}(-1)$ to $\mathbb{D}$ is naturally Γ-equivariant and hence descends to an orbi-line bundle $\mathcal{L}$ on $X_\Gamma := \Gamma\backslash\mathbb{D}$, called the *Hodge bundle*. For any $\beta \in L^\vee/L$ and $m \in -Q(\beta) + \mathbb{Z}$, we define the special divisor of discriminant (β, m) in X_Γ to be:

$$Z^o(\beta, m) = \bigcup_{\substack{\lambda \in \beta + L \\ Q(\lambda) = -m}} \{x \in \mathbb{D} \,\big|\, x \cdot \lambda = 0\} \bmod \Gamma. \qquad (3.1.1)$$

By the Baily-Borel theorem [BB66], the line bundle $\mathcal{L}$ is ample and sections of its powers determine an embedding of the quotient X_Γ into a projective space. The closure of the image is a projective variety, and is by definition the *Baily-Borel compactification*: $X_\Gamma \hookrightarrow X_\Gamma^{\mathrm{BB}}$.

This compactification can be described more explicitly using the rational boundary components of $\mathbb{D}$ in its topological closure $\overline{\mathbb{D}} \subset \mathbb{D}^c$.

Definition 3.1. The *rational boundary components* $\mathbb{D}_I$ of $\mathbb{D}$ are in bijection with primitive isotropic sublattices $I \subset L$ as follows:

$$\mathbb{D}_I := \{\mathbb{C}x \in \overline{\mathbb{D}} \subset \mathbb{D}^c \,\big|\, \mathrm{span}\{\mathrm{Re}\,x, \mathrm{Im}\,x\} = I_\mathbb{R}\}.$$

Notation 3.2. The possible ranks of an isotropic sublattice of L are 1 and 2. To distinguish them, we henceforth use the letter I when the rank is 1 and the letter J when the rank is 2.

Definition 3.3. For a rank 1 isotropic lattice, $\mathbb{D}_I$ is a point. For a rank 2 isotropic lattice, $\mathbb{D}_J$ is a copy of the upper half-plane $\mathbb{H}$. We call these boundary components *Type III* and *Type II* respectively.

Remark 3.4. The terminology of Type II and III comes from the classification [Kul77, PP81] of one-parameter degenerations of K3 surfaces.

One defines the *rational closure* of $\mathbb{D}$ to be

$$\mathbb{D}^+ := \mathbb{D} \cup_J \mathbb{D}_J \cup_I \mathbb{D}_I \subset \overline{\mathbb{D}},$$

endowed with a horoball topology near the boundary components $\mathbb{D}_I$ and $\mathbb{D}_J$. An explicit topological description of the Baily-Borel compactification is then $X_\Gamma^{\mathrm{BB}} = (\Gamma \backslash \mathbb{D})^{\mathrm{BB}} = \Gamma \backslash \mathbb{D}^+$. Thus, boundary components of $(\Gamma \backslash \mathbb{D})^{\mathrm{BB}}$ are the

(1) Type III components, which are points, in bijection with Γ-orbits of isotropic lines $I \subset L$, and
(2) Type II components, which are modular curves $\mathrm{Stab}_\Gamma(J) \backslash \mathbb{H}$, in bijection with Γ-orbits of isotropic planes $J \subset L$.

The closures of Type II modular curves in $(\Gamma \backslash \mathbb{D})^{\mathrm{BB}}$ contain Type III boundary points, which form the cusps of the modular curves. Such incidences correspond to Γ-orbits of inclusions $I \subset J \subset L$.

3.2. Special divisors at the Type III boundary.

We now describe the geometry of special divisors of $\mathbb{D}$ near a cusp. Let $I = \mathbb{Z}\delta \subset L$ be a primitive isotropic sublattice. Given any point $\mathbb{C}x \in \mathbb{D}$, there is a unique representative $x \in L_\mathbb{C}$ for which $x \cdot \delta = 1$—we cannot have $x \cdot \delta = 0$ for signature reasons. This defines an embedding of $\mathbb{D}$ into the affine hyperplane $\{x \in L_\mathbb{C} \mid x \cdot \delta = 1\}$.

Let $U_I \subset \mathrm{Stab}_\Gamma(I)$ be the unipotent radical of the integral parabolic stabilizer of I. Consider an integral (but non-orthogonal) decomposition $L^\vee = \mathbb{Z}\delta' + I_{L^\vee}^\perp$ with $\delta' \cdot \delta = 1$. Then we can write

$$x = \delta' + x_0 + c\delta$$

where x_0 lies in $\{\delta, \delta'\}_\mathbb{C}^\perp \simeq (I^\perp/I)_\mathbb{C}$ and $c \in \mathbb{C}$ is uniquely determined by the property that $x \cdot x = 0$. Let $K := I^\perp/I$. The action of some $\gamma \in U_I$ necessarily takes the form:

$$\delta \mapsto \delta$$
$$v \mapsto v - (v \cdot w)\delta \text{ for } v \in I^\perp$$
$$\delta' \mapsto \delta' + w - (\delta' \cdot w)\delta - \tfrac{1}{2}(w \cdot w)\delta \text{ for some } w \in K.$$

This formula above doesn't depend on how w is lifted into $I^\perp$.

On the coordinate $x_0 \in K_\mathbb{C}$, the unipotent radical U_I acts by a finite index subgroup of the translation group K. In the full stable orthogonal group, we get the full translation group K. Thus,

$$U_I \backslash \mathbb{D} \hookrightarrow U_I \backslash (U_I \otimes \mathbb{C}) \simeq (\mathbb{C}^*)^n$$

admits a torus embedding. This torus $U_I \otimes \mathbb{C}^* = U_I \backslash (U_I)_\mathbb{C}$ is isogenous to the torus $K \backslash K_\mathbb{C} = K \otimes \mathbb{C}^*$.

We analyze the special divisors in the above torus embedding. It suffices to restrict to $U_I = K$—in the more general case where $U_I \subsetneq K$, we just take the inverse images of the special divisors under the isogeny $U_I \backslash K_\mathbb{C} \to K \backslash K_\mathbb{C}$. Consider

$$Z^o(\widetilde{\lambda}) := \{x \in \mathbb{D} \,|\, x \cdot \widetilde{\lambda} = 0\} \bmod \Gamma$$

for some primitive $\widetilde{\lambda} \in L^\vee$ which passes through the cusp of $(\Gamma \backslash \mathbb{D})^{\mathrm{BB}}$ corresponding to I. Choose a representative of the Γ-orbit for which $\widetilde{\lambda} \in I^\perp_{L^\vee}$. Let λ be the image of $\widetilde{\lambda}$ under reduction to $I^\perp_{L^\vee} / I^{\mathrm{sat}}_{L^\vee} = K^\vee$, where $I^{\mathrm{sat}}_{L^\vee}$ is the saturation of I in $L^\vee$.

Note that the vector $\lambda \in K^\vee$ may not be primitive. Write $\lambda = u\lambda^{\mathrm{prim}}$ where $\lambda^{\mathrm{prim}} \in K^\vee$ is primitive. The lifts of λ to $\widetilde{\lambda} \in I^\perp_{L^\vee}$ are the set

$$\{u\widetilde{\lambda}^{\mathrm{prim}} + \tfrac{k}{d}\delta \,|\, k \in \mathbb{Z}\}$$

where d is the imprimitivity of $\delta \in L^\vee$ and $\widetilde{\lambda}^{\mathrm{prim}}$ is a lift of λ^{prim}. Such a lift is primitive in $L^\vee$ if and only if $\gcd(k, u) = 1$. Furthermore, there exists some $\gamma \in K$ for which $\gamma(\widetilde{\lambda}^{\mathrm{prim}}) = \widetilde{\lambda}^{\mathrm{prim}} + \delta$. So a collection of coset representatives of the K-action on $\{u\widetilde{\lambda}^{\mathrm{prim}} + \tfrac{k}{d}\delta \,|\, k \in \mathbb{Z}\}$ is provided by taking all $0 \le k < du$.

The equation of $Z^o(\widetilde{\lambda})$ in coordinates $x_0 \in K_\mathbb{C}$ is the affine hyperplane

$$x_0 \cdot \lambda = -\delta' \cdot \widetilde{\lambda} \in \mathbb{Q}.$$

Under the exponentiation isomorphism $K \backslash K_\mathbb{C} \to K \otimes \mathbb{C}^*$, the equation of $Z^o(\widetilde{\lambda})$ becomes a torsion translate of a character hypersurface

$$\chi_{\lambda^{\mathrm{prim}}}(z) = z^{\lambda^{\mathrm{prim}}} = \exp(-\tfrac{2\pi i k}{du} - 2\pi i \delta' \cdot \widetilde{\lambda}^{\mathrm{prim}})$$

where $z \in (\mathbb{C}^*)^n$ is the coordinate-wise exponential of $x_0 \in K_\mathbb{C}$.

Proposition 3.5. *Let $\lambda \in K^\vee$ have imprimitivity u. In the torus embedding $U_I \backslash \mathbb{D} \hookrightarrow U_I \otimes \mathbb{C}^*$ there are $d\varphi(u)$ special divisors associated to lifts of λ to a primitive vector $\widetilde{\lambda} \in I^\perp_{L^\vee}$, each one being a torsion translate of the hypersurface $\chi_{\lambda^{\mathrm{prim}}}(z) = 1$.*

Proof. When $U_I = K$, the proposition follows from the above computation of the K-coset representatives of lifts of λ. When $U_I \subsetneq K$, the hypersurface $\chi_{\lambda^{\mathrm{prim}}}(z) = 1$ may break into multiple character hypersurfaces in $U_I \otimes \mathbb{C}^*$—this occurs exactly when $\lambda^{\mathrm{prim}} \in K^\vee$ becomes imprimitive in the character group $U_I^\vee \supset K^\vee$. But the equations of the special divisors are still pulled back from $K \otimes \mathbb{C}^*$ along the isogeny $U_I \otimes \mathbb{C}^* \to K \otimes \mathbb{C}^*$, reducing the proposition to the $U_I = K$ case. $\square$

Corollary 3.6. *Let $Z^o_I(\beta, m) \subset Z^o(\beta, m)$ be the union of the components of $Z^o(\beta, m)$, see 3.1.1, whose Zariski closure in $(\Gamma \backslash \mathbb{D})^{\mathrm{BB}}$ contains the Type III cusp associated to I. The pullback of $Z^o_I(\beta, m)$ to*

$U_I \backslash \mathbb{D} \subset U_I \otimes \mathbb{C}^*$ *is*

$$\bigcup_{\substack{\lambda \in p_L^K(\beta)+K \\ Q(\lambda)=-m}} \{u \text{ translates of } \chi_{\lambda^{\mathrm{prim}}}(z) = 1\}$$

where $u = u(\lambda)$ *is the imprimitivity of* λ.

Proof. Note that $Z^o(\beta, m)$ ranges over both primitive and imprimitive vectors. So it contains the divisors $Z^o(\widetilde{\mu})$ associated to primitive lifts of $u' \lambda^{\mathrm{prim}}$ ranging over the divisors $u' \mid u$. By the formula $\sum_{u' \mid u} \varphi(u') = u$ and Proposition 3.5, there are du special divisors associated to lifts of λ, whose classes may vary in the discriminant group $L^\vee/L$. Observe that the classes $[\widetilde{\lambda}]$ of lifts of λ equidistribute in the classes $\beta \in L^\vee/L$ which push to the class $[\lambda] = p_L^K(\beta) \in K^\vee/K$. Hence there are exactly u translates of the hypersurface $\chi_{\lambda^{\mathrm{prim}}}(z) = 1$ appearing in $Z_I^o(\beta, m)$. $\quad\square$

3.3. Toroidal compactification.
We now describe the toroidal compactification in a neighborhood of the Type III cusps. As above, let $I \subset L$ denote an isotropic line, and let $K := I^\perp/I$. Define a finite index subgroup $\Gamma_I \subset O^+(K)$ by the exact sequence

$$0 \to U_I \to \mathrm{Stab}_\Gamma(I) \to \Gamma_I \to 0.$$

Let $C_K \subset K_{\mathbb{R}}$ denote the *positive cone*—a connected component of the positive norm vectors in $K_{\mathbb{R}} \simeq \mathbb{R}^{1,n-1}$ and let C_K^+ denote its *rational closure*: The union of C_K with all rational isotropic rays at its boundary.

Definition 3.7. A Γ-*admissible collection of fans* (or simply *fan*) is a collection $\Sigma = \{\Sigma_I\}$ of polyhedral decompositions $\Sigma_I = \{\sigma\}$ of C_K^+ ranging over all primitive isotropic $I \subset L$, such that:
  (1) the cones $\sigma \subset C_K^+$ are strictly convex, rational polyhedral cones, and Σ_I is closed under taking intersections and faces, and
  (2) the collection $\Sigma = \{\Sigma_I\}$ is Γ-invariant with only finitely many Γ-orbits of cones $\sigma \in \Sigma_I$.

Equivalently, we can choose one polyhedral decomposition Σ_I for each Γ-orbit of isotropic line I, and require that it be Γ_I-invariant, with finitely many orbits of cones. A useful visualization is to consider $\{\mathbb{P}\sigma\}_{\sigma \in \Sigma_I}$ which is a Γ_I-periodic tiling of (rationally cusped) hyperbolic $(n-1)$-space $\mathbb{P}C_K^+$.

We extend the algebraic torus $U_I \otimes \mathbb{C}^*$ by the infinite-type toric variety $X(\Sigma_I)$ whose fan is Σ_I. It admits an action of $U_I \otimes \mathbb{C}^*$ whose orbits are in bijection with the cones $\sigma \in \Sigma_I$. The orbit corresponding to σ is isomorphic to $(U_I/\mathrm{span}(\sigma)) \otimes \mathbb{C}^* \simeq (\mathbb{C}^*)^{\mathrm{codim}(\sigma)}$.

Let V_I be a Γ_I-invariant, analytic tubular neighborhood of the *Type III strata* of $X(\Sigma_I)$: These are the torus orbits corresponding to cones σ satisfying $\sigma \cap C_K \neq \emptyset$—exactly the strata corresponding to cones σ

which are neither the origin $0 \in K$ nor a rational isotropic ray. Such cones σ only have finite stabilizers in Γ_I, so we may assume that the action of Γ_I on V_I is properly discontinuous. Let

$$A_I := V_I \cap (U_I \otimes \mathbb{C}^*)$$

be the intersection with the open torus orbit.

Choosing V_I sufficiently small, we can ensure that $\Gamma_I \backslash A_I$ embeds into $\Gamma \backslash \mathbb{D}$. This follows from the fact that a punctured neighborhood of the Type III boundary component associated to I is locally modeled as the quotient of some analytic open subset of $\mathbb{D}$ by the parabolic stabilizer $\mathrm{Stab}_\Gamma(I)$. Then, we define the *Type III extension* of $\Gamma \backslash \mathbb{D}$ to be the glued analytic space

$$(\Gamma \backslash \mathbb{D})^{\Sigma,\mathrm{III}} := (\Gamma \backslash \mathbb{D}) \cup_{\Gamma_I \backslash A_I} (\Gamma_I \backslash V_I)$$

with the union taken over all I.

Taking the further union with the Type II extensions $\Gamma_J \backslash V_J$ defined in the following subsection gives the *toroidal compactification*:

$$X_\Gamma^\Sigma = (\Gamma \backslash \mathbb{D})^\Sigma := (\Gamma \backslash \mathbb{D})^{\Sigma,\mathrm{III}} \cup_{\Gamma_J \backslash A_J} (\Gamma_J \backslash V_J).$$

For any Σ, it admits a morphism $(\Gamma \backslash \mathbb{D})^\Sigma \to (\Gamma \backslash \mathbb{D})^{\mathrm{BB}}$.

Definition 3.8. We say that Γ is *neat* if the eigenvalues of every element $\gamma \in \Gamma$ acting on $L_\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}^*$.

Definition 3.9. We say that Σ is *regular* if every cone σ is a standard affine cone ($\sigma \cap U_I$ is isomorphic to $\mathbb{N}^{\dim(\sigma)}$ as a monoid). More generally, Σ is *simplicial* if the generators of any cone $\sigma \in \Sigma_I$ are linearly independent.

If Γ is neat and Σ is regular, then the toroidal compactification $(\Gamma \backslash \mathbb{D})^\Sigma$ is smooth with reduced normal crossings boundary, because $\Gamma, \Gamma_I, \Gamma_J$ act freely on $\mathbb{D}$ and the respective toroidal extensions. Since the toric variety associated to a simplicial cone σ has only finite quotient singularities, it is a local $\mathbb{Q}$-Poincaré duality space. For the same reason, X_Γ^Σ is a $\mathbb{Q}$-Poincaré duality space (regardless of whether Γ is neat) for any simplicial Σ.

Remark 3.10. We can and will assume that Γ is neat and Σ is regular. One can reduce the main result of the paper for simplicial fans Σ and arbitrary Γ to this case: By passing to a finite index subgroup $\Gamma' \subset \Gamma$ we can make the action neat, and by refining $\Sigma \prec \Sigma'$, we can make the fan regular. We get a morphism

$$\pi \colon X_{\Gamma'}^{\Sigma'} \to X_{\Gamma'}^\Sigma \to X_\Gamma^\Sigma$$

which is the composition of a birational morphism and a finite cover. Then $\pi^* \colon H^2(X_\Gamma^\Sigma, \mathbb{Q}) \to H^2(X_{\Gamma'}^{\Sigma'}, \mathbb{Q})$ is an embedding.

The main theorem is proved by intersecting the Zariski closures $Z(\beta, m) = \overline{Z^o(\beta, m)}$ with curve classes $C \in H_2(X_\Gamma^\Sigma, \mathbb{Q})$. Such a class

C can be expressed as $[\Gamma : \Gamma']C \equiv \pi_* D$ for some $D \in H_2(X_{\Gamma'}^{\Sigma'}, \mathbb{Z})$ which intersects the exceptional divisors of π to be zero: This is achieved by using a wrong-way map $D := (\mathrm{PD} \circ \pi^* \circ \mathrm{PD})(C)$. By push-pull,

$$Z(\beta, m) \cdot_{X_\Gamma^\Sigma} C = \frac{1}{[\Gamma : \Gamma']} \pi^* Z(\beta, m) \cdot_{X_{\Gamma'}^{\Sigma'}} D$$

$$= \frac{1}{[\Gamma : \Gamma']} Z(\beta, m) \cdot_{X_{\Gamma'}^{\Sigma'}} D.$$

The last equality holds because D intersects the exceptional divisors to be zero and the strict transforms of the $Z(\beta, m)$'s for Γ, Σ are the $Z(\beta, m)$'s for Γ', Σ'. The same equality holds for any boundary divisor in X_Γ^Σ associated to a ray in Σ. This implies that the pairing with D of the generating series from Theorem 1.2 applied to $X_{\Gamma'}^{\Sigma'}$ is equal (up to the factor $[\Gamma : \Gamma']$) to the pairing of C with the analogous series for X_Γ^Σ. This proves that the validity of Theorem 1.2 for $X_{\Gamma'}^{\Sigma'}$ implies its validity for X_Γ^Σ.

3.4. **The Type II boundary.** We now roughly sketch the structure of the Type II boundary component associated to the Γ-orbit of an isotropic plane $J \subset L$.

Choose a basis $J = \mathbb{Z}\delta \oplus \mathbb{Z}\lambda$ and as before, represent $\mathbb{C}x \in \mathbb{D}$ uniquely by the $x \in L_\mathbb{C}$ for which $x \cdot \delta = 1$. Choose δ', λ' which generate a lattice $J' = \mathbb{Z}\delta' + \mathbb{Z}\lambda'$ in $L^\vee$ realizing $\mathrm{Hom}(J, \mathbb{Z})$ and write

$$x = (\delta' + \tau\lambda') + x_0 + (c_1\delta + c_2\lambda)$$

where $x_0 \in \{\delta, \delta', \lambda, \lambda'\}_\mathbb{C}^\perp \cong (J^\perp/J)_\mathbb{C}$ and $c_1, c_2 \in \mathbb{C}$. The condition $x \cdot x = 0$ specifies that $(c_1, c_2) \in L \subset \mathbb{C}^2$ lies in an affine line L, and the condition $x \cdot \overline{x} > 0$ further implies that (c_1, c_2) lies in a half-space $\mathbb{H}_{(x_0, \tau)} \subset L$ inside this line depending on $(x_0, \tau) \in (J^\perp/J)_\mathbb{C} \times \mathbb{H}$. The unipotent radical $U_J \subset \mathrm{Stab}_\Gamma(J)$ is an extension of the form

$$0 \to \mathbb{Z} \to U_J \to T_J \to 0$$

where T_J is a finite index translation subgroup

$$T_J \subset (J^\perp/J) \otimes (\mathbb{Z} \oplus \mathbb{Z}\tau) \subset (J^\perp/J)_\mathbb{C}.$$

First taking the quotient by $\mathbb{Z} \subset U_J$ embeds $\mathbb{Z}\backslash\mathbb{D}$ as a holomorphic punctured disk bundle (the punctured disks are $\mathbb{Z}\backslash\mathbb{H}_{(x_0, \tau)}$) over the product $(J^\perp/J)_\mathbb{C} \times \mathbb{H}$. We then quotient by the action of T_J. The punctured disk bundle descends to one over a family of abelian varieties $T_J \otimes \mathcal{E}$. Here $\mathcal{E} \to \mathbb{H}$ is the universal elliptic curve whose fiber of $\tau \in \mathbb{H}$ is $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$.

We define V_J as a tubular neighborhood of the zero section of the extension of $U_J\backslash\mathbb{D}$ from a punctured disk bundle to a disk bundle. Then the *Type II toroidal extension* $(\Gamma\backslash\mathbb{D})^{\mathrm{II}}$ is the result of gluing $\Gamma\backslash\mathbb{D}$ to $\Gamma_J\backslash V_J$ along a punctured tubular neighborhood $\Gamma_J\backslash A_J$ of the zero

section. Here Γ_J is defined by the exact sequence

$$0 \to U_J \to \mathrm{Stab}_\Gamma(J) \to \Gamma_J \to 0$$

and is a finite index subgroup $\Gamma_J \subset \mathrm{SL}(J) \times O(J^\perp/J)$. The Type II extension is notably independent of the choice of a fan Σ; it is the same in every toroidal compactification, and the boundary divisor is always isomorphic to $\Gamma_J \backslash (T_J \otimes \mathcal{E})$.

Finally, we comment on how the Type II and III locus meet. Given a flag of isotropic subspaces $I \subset J \subset L$, the quotient $\Gamma_I \backslash V_I$ contains a tubular neighborhood of the intersection of the closure of the Type II locus $\Gamma_J \backslash V_J$ with the Type III locus. This intersection is a maximal degeneration of the family $T_J \otimes \mathcal{E}$ of abelian varieties over the cusp of the modular curve $\mathrm{Stab}_\Gamma(J) \backslash \mathbb{D}_J$ corresponding to $I \subset J$. More explicitly, it is a Mumford degeneration, whose fan is the quotient fan associated to the isotropic ray $\overline{J} \subset I^\perp/I$, necessarily a cone of Σ_I. This fan is the cone over a periodic polyhedral decomposition of $(J^\perp/J)_\mathbb{R}$ (put at height 1), which is invariant under the action of $\mathrm{Stab}_{\overline{J}}(\Gamma_I)$.

A special divisor $Z(\widetilde{\lambda})$ intersects the Type II extension associated to J if and only if we can represent $\widetilde{\lambda} \in J_{L^\vee}^\perp$. Let $M := J^\perp/J$. In general, if we denote by $J_{L^\vee}^{\mathrm{sat}}$ the saturation of J in $L^\vee$, then the image $\lambda \in J_{L^\vee}^\perp/J_{L^\vee}^{\mathrm{sat}} = M^\vee$ may be imprimitive, and we can write $\lambda = u\lambda^{\mathrm{prim}}$. Essentially, the computations are the same as in the Type III case: The primitive lifts of λ correspond to primitive u-torsion translates of a character hypersurface $\{\chi_{\lambda^{\mathrm{prim}}} = 1\} \subset T_J \otimes \mathcal{E}$, which is a family of codimension 1 abelian subvarieties of $T_J \otimes \mathcal{E}$.

Proposition 3.11. *In any toroidal compactification, the intersection of $Z(\beta, m)$ with the boundary divisor Δ_J associated to J is*

$$\bigcup_{\substack{\lambda \in p_L^M(\beta) + M \\ Q(\lambda) = -m}} \{u^2 \ \text{translates of } \chi_{\lambda^{\mathrm{prim}}}(z) = 1\}$$

There are u^2 translates rather than u translates of the character hypersurface, as the u-torsion on an elliptic curve has size u^2.

3.5. **Intersection formula.** Let $\sigma = \mathrm{span}\{c_1, \ldots, c_{n-1}\} \in \Sigma_I$ be a codimension 1 Type III face of a regular fan and let

$$\sigma^+ = \mathrm{span}(\sigma, c^+)$$
$$\sigma^- = \mathrm{span}(\sigma, c^-)$$

be the two maximal cones containing σ, so that $\sigma^+ \cap \sigma^- = \sigma$. The corresponding open torus orbit is isomorphic to $\mathbb{C}^*$ and the closed stratum is $C_\sigma = \mathbb{P}^1$. A small analytic neighborhood of $C_\sigma \subset (\Gamma \backslash \mathbb{D})^\Sigma$ is analytically-locally modeled by the toric variety with two maximal

cones σ^+, σ^-, which is isomorphic to

$$\operatorname{Spec}\mathbb{C}[(\sigma^+)^\vee] \cup \operatorname{Spec}\mathbb{C}[(\sigma^-)^\vee] \simeq \mathbb{C}^n \cup_{\mathbb{C}^{n-1} \times \mathbb{C}^*} \mathbb{C}^n$$
$$\simeq \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1}))$$

for some integers $a_i \in \mathbb{Z}$.

Since σ^+, σ^- are standard affine cones, $\det(c_1, \ldots, c_{n-1}, c^+) = \pm 1$ and $\det(c_1, \ldots, c_{n-1}, c^-) = \mp 1$. Thus, there is an expression

$$c^+ + c^- + \sum_{i=1}^{n-1} a_i c_i = 0$$

for unique integers $a_i \in \mathbb{Z}$, identical to the integers as above.

Proposition 3.12. *Let λ be primitive in $K^\vee$ and consider $Z(\lambda)$ the Zariski closure of the character hypersurface*

$$Z^o(\lambda) := \{z \in K \otimes \mathbb{C}^* \,|\, \chi_\lambda(z) = 1\}$$

in the toric variety $X(\sigma^+, \sigma^-)$, with $\sigma, \sigma^\pm$ as above. Then

$$Z(\lambda) \cdot C_\sigma = \tfrac{1}{2}\left(|\lambda \cdot c^+| + |\lambda \cdot c^-| + \sum_i a_i |\lambda \cdot c_i|\right).$$

In particular, if $\lambda \cdot c^\pm$ and $\lambda \cdot c_i$ all have the same sign, $Z(\lambda) \cdot C_\sigma = 0$.

Proof. Let $X(\mathfrak{F})$ be a finite type toric variety, and let v_i be primitive integral generators of the one-dimensional rays of $\mathfrak{F}$. There is a bijection $\Delta_{v_i} \leftrightarrow v_i$ with toric boundary divisors. We have a linear equivalence

$$Z(\lambda) \equiv \sum_i \max\{0, \lambda \cdot v_i\} \Delta_{v_i}$$

which can, for instance, be verified by intersecting $Z(\lambda)$ with cocharacters $v_i \otimes \mathbb{C}^* \subset K \otimes \mathbb{C}^*$. Note that replacing λ with $-\lambda$ does not change the answer, since

$$\sum_i \max\{0, \lambda \cdot v_i\} \Delta_{v_i} - \sum_i \max\{0, -\lambda \cdot v_i\} \Delta_{v_i} = \sum_i (\lambda \cdot v_i) \Delta_{v_i} \equiv 0.$$

Applying this formula to the case at hand, we have

$$Z(\lambda) \cdot \mathbb{P}^1 = \max\{0, \lambda \cdot c^+\} \Delta_{c^+} \cdot \mathbb{P}^1 + \max\{0, \lambda \cdot c^-\} \Delta_{c^-} \cdot \mathbb{P}^1$$
$$+ \sum_i \max\{0, \lambda \cdot c_i\} \Delta_{c_i} \cdot \mathbb{P}^1$$

where $\mathbb{P}^1 = C_\sigma$. Re-expressing the function $\max\{0, r\} = \tfrac{1}{2}(|r| + r)$ and noting that $\sum_i (\lambda \cdot c_i) \Delta_{c_i} \equiv 0$, we get

$$Z(\lambda) \cdot \mathbb{P}^1 = \tfrac{1}{2}(|\lambda \cdot c^+|(\Delta_{c^+} \cdot \mathbb{P}^1) + |\lambda \cdot c^-|(\Delta_{c^-} \cdot \mathbb{P}^1) + \sum_i |\lambda \cdot c_i|(\Delta_{c_i} \cdot \mathbb{P}^1)).$$

We now describe geometrically the toric boundary divisors associated to $c^\pm$, c_i. In the total space interpretation

$$X(\sigma^+, \sigma^-) = \operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1})),$$

the divisors Δ_{c^+} and Δ_{c^-} are the fibers over $0, \infty \in \mathbb{P}^1$ and so $\Delta_{c^\pm} \cdot \mathbb{P}^1 = 1$. The divisor Δ_{c_i} is $\operatorname{Tot}(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus 0 \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1}))$, leaving

out the ith line bundle factor. Hence the normal bundle to Δ_{c_i} is the pullback of $\mathcal{O}_{\mathbb{P}^1}(a_i)$ to $X(\sigma^+, \sigma^-)$ and so its restriction to the zero section $\mathbb{P}^1$ has degree a_i. We conclude

$$Z(\lambda) \cdot C_\sigma = \tfrac{1}{2}(|\lambda \cdot c^+| + |\lambda \cdot c^-| + \sum_i a_i|\lambda \cdot c_i|).$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.13. *Let $C_\sigma \subset X_\Gamma^\Sigma$ be a one-dimensional Type III boundary stratum of the toroidal compactification, corresponding to the cone $\sigma = \sigma^+ \cap \sigma^-$ in the hyperbolic lattice $K = I^\perp/I$. We have:*

$$\sum_{\beta \in L^\vee/L} \sum_{m \in -Q(\beta)+\mathbb{Z}} (Z(\beta, m) \cdot C_\sigma) v_\beta q^m = \sum_{\lambda \in K^\vee} p^+(\lambda) p_K^L(v_{[\lambda]}) q^{-Q(\lambda)}$$

where $[\lambda]$ is the class of λ in $K^\vee/K$, and

$$p^+(\lambda) = \tfrac{1}{2}(|\lambda \cdot c^+| + |\lambda \cdot c^-| + \sum_i a_i|\lambda \cdot c_i|)$$

is the piecewise linear function from Proposition 3.12.

Proof. Since C_σ is contracted in X_Γ^{BB}, we have $Z(0,0) \cdot C_\sigma = 0$ because the Hodge bundle is ample on X_Γ^{BB}. Next, observe that only the components $Z_I(\beta, m) \subset Z(\beta, m)$ entering the Baily-Borel cusp associated to I can possibly intersect $\mathbb{P}^1 = C_\sigma$. We wish to compute the intersection number

$$Z(\beta, m) \cdot_{X_\Gamma^\Sigma} C_\sigma = Z_I(\beta, m) \cdot_{\Gamma_I \backslash X(\Sigma_I)} C_\sigma = Z_I(\beta, m) \cdot_{X(\Sigma_I)} C_\sigma.$$

The first equality holds from the construction of the toroidal compactification, while the second equality holds because the action of Γ_I freely permutes the infinitely many $\mathbb{P}^1$s in the orbit $\Gamma_I \cdot C_\sigma$. Thus, to compute the intersection number with $Z_I(\beta, m)$ in $\Gamma_I \backslash X(\Sigma_I)$, it suffices to pull back $Z_I(\beta, m)$ to the toric variety $X(\Sigma_I)$, then intersect the pullback with a single orbit representative C_σ.

On the toric variety $X(\Sigma_I)$, we have by Corollary 3.6 that

$$Z_I(\beta, m) \sim_{\mathrm{num}} \sum_{\substack{\lambda \in p_L^K(\beta)+K \\ Q(\lambda)=-m}} u(\lambda) Z(\lambda^{\mathrm{prim}})$$

where $Z(\lambda^{\mathrm{prim}})$ is the closure of the character hypersurface and $u(\lambda)$ is the imprimitivity of λ. While this divisor is not finite, it is locally finite in a neighborhood of the Type III toric boundary.

We take the intersection with C_σ using Proposition 3.12. It gives

$$Z_I(\beta, m) \cdot_{X(\Sigma_I)} C_\sigma = \sum_{\substack{\lambda \in p_L^K(\beta)+K \\ Q(\lambda)=-m}} u(\lambda) p^+(\lambda^{\mathrm{prim}}),$$

which is a sum with only finitely many nonzero terms. Finally, observe that $p^+(r\lambda) = r p^+(\lambda)$ for any $r \in \mathbb{R}^+$, so $u(\lambda) p^+(\lambda^{\mathrm{prim}}) = p^+(\lambda)$. The theorem follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Vanishing theorems and splitting criteria

We prove in this section the splitting statement 4.6 which allows us to prove modularity separately in the boundary and in the interior of X_Γ^Σ. We refer to [Gre19, Lem. 26] for a similar application.

4.1. Vanishing from super-rigidity.

The goal of this subsection is to prove that the first Chern class map

$$c_1 : \operatorname{Pic}(X_\Gamma^\Sigma)_\mathbb{Q} \to H^{1,1}(X_\Gamma^\Sigma, \mathbb{Q})$$

is an isomorphism. This implies that our modularity results for special cycles automatically hold at the level of the $\mathbb{Q}$-Picard group, even though the methods are topological. Using the exponential exact sequence and the Hodge decomposition, it suffices to show that

$$H^1(X_\Gamma^\Sigma, \mathbb{Q}) = 0.$$

First, observe that by deforming loops away from the boundary strata that have real codimension 2, we have surjectivity of the pushforward

$$\pi_1(X_\Gamma) \to \pi_1(X_\Gamma^\Sigma).$$

Since H_1 is the abelianization of π_1, it suffices to prove that the interior $X_\Gamma = \Gamma \backslash \mathbb{D}$ satisfies

$$H^1(X_\Gamma, \mathbb{Q}) = 0.$$

Since $\mathbb{D}$ is contractible, this coincides with group cohomology of the arithmetic group Γ. A consequence of Margulis super-rigidity [Mar91, §7.6, Corollary 7.6.17], see also [WM14], is the following.

Theorem 4.1. *Let G be a semisimple Lie group of rank $r \geq 2$ and $\Gamma \subset G$ an irreducible arithmetic subgroup. Then we have $H^1(\Gamma, V) = 0$ for any Γ-module V.*

Since $O(p, q)$ has rank $\min(p, q)$, the desired vanishing statement holds for any $\Gamma \subset O(2, n)$ for $n > 2$ and $V = \mathbb{Q}$ the trivial module. In the case $n = 2$, $O(2, 2)$ is isogenous to $SL(2) \times SL(2)$, so we need to assume that Γ is not commensurable to a product $\Gamma_1 \times \Gamma_2$.

4.2. Splitting homology classes.

Let X be a smooth complex variety, $\Delta = \bigcup_i \Delta_i \subset X$ a simple normal crossings divisor, and $U = X \setminus \Delta$ its open complement. A basic question is: which rational homology classes $\alpha \in H_2(X, \mathbb{Q})$ can be expressed as $\alpha = \alpha_0 + \alpha_1$ where $\alpha_0 \in H_2(U, \mathbb{Q})$ and $\alpha_1 \in H_2(\Delta, \mathbb{Q})$? In this case, we say that α *splits*. Consider the covering of X by two open subsets:

$$X = U \cup N_\Delta$$

where N_Δ is a tubular neighborhood of Δ. Then Δ is a deformation retract of N_Δ. The Mayer-Vietoris sequence for rational homology with this covering reads

$$H_2(U) \oplus H_2(N_\Delta) \to H_2(X) \xrightarrow{\delta} H_1(N_\Delta^*) \to H_1(U) \oplus H_1(N_\Delta)$$

where N_Δ^* denotes $N_\Delta \setminus \Delta$. By exactness, $\alpha \in H_2(X)$ can be split as above if and only if $\delta(\alpha) = 0$. Since $H_1(U)$ always vanishes in our desired application, we will use the following criterion whose proof is immediate from the Mayer-Vietoris sequence.

Proposition 4.2. *If the pushforward map $H_1(N_\Delta^*) \to H_1(N_\Delta)$ is injective, then any class $\alpha \in H_2(X)$ splits.*

In the case where Δ is irreducible and smooth, this criterion is easily checked using a Gysin sequence.

Proposition 4.3. *If Δ is irreducible and smooth, then the pushforward map $H_1(N_\Delta^*) \to H_1(N_\Delta)$ is injective if and only if the normal bundle $N_{\Delta/X}$ has non-trivial Euler class.*

Proof. Since Δ is smooth, the punctured neighborhood N_Δ^* retracts to a circle bundle over Δ. The Gysin sequence for $N_\Delta^* \to \Delta$ reads:

$$H_2(\Delta) \to H_0(\Delta) \to H_1(N_\Delta^*) \to H_1(\Delta) \to 0.$$

The first map in this sequence is given by cap product with $e(N_{\Delta/X})$. Since $H_0(\Delta) \simeq \mathbb{Q}$, this map is surjective if and only if $e(N_{\Delta/X}) \neq 0$. By exactness of the Gysin sequence, this in turn is equivalent to injectivity of the map $H_1(N_\Delta^*) \to H_1(\Delta) \simeq H_1(N_\Delta)$. $\square$

In the case where Δ is reducible, N_Δ is a plumbing of normal bundles to the irreducible components Δ_i. The splitting criterion can fail to be satisfied even when the individual Δ_i have non-trivial normal bundles, e.g. for Δ the triangle of lines in $\mathbb{P}^2$ we get N_Δ^* deformation retracts to T^3 while $H_1(\Delta) \simeq \mathbb{Q}$, so $H_1(N_\Delta^*) \to H_1(\Delta)$ cannot be injective.

To check the criterion of Prop. 4.2 in the context of toric boundary divisors, we introduce the moment map. Let (X, Δ) be a projective toric variety of dimension n defined by a lattice polytope P in the character lattice of X. The Hamiltonian action of $T^n \subset (\mathbb{C}^*)^n$ on X defines a surjective map

$$\mu : X \to P$$

whose fiber at an interior point of P is T^n, canonically identified with the cocharacter lattice quotient. The fiber at a point in the relative interior of a face of codimension k is T^{n-k}. The pre-image of the boundary $\partial P \subset P$ is the toric boundary divisor Δ:

$$\mu^{-1}(\partial P) = \Delta = \bigcup_i \Delta_i$$

with each facet $P_i \subset \partial P$ giving a component $\Delta_i \subset \Delta$. From this description, the punctured neighborhood N_Δ^* retracts to $\mu^{-1}(S^{n-1})$ which is diffeomorphic to $S^{n-1} \times T^n$.

Consider now the Type III boundary divisors $\Delta_I = \cup_c \Delta_{I,c}$ of a regular toroidal compactification X_Γ^Σ lying over a Type III cusp of X_Γ^{BB}.

Here $\mathbb{R}_{\geq 0}c \in \Sigma_I$ are the one-dimensional cones of the fan Σ_I supported on the rational closure of the positive cone $C_K^+ \subset K_{\mathbb{R}}$, $K = I^{\perp}/I$.

To get a polytope from Σ_I requires a convex $\mathbb{Z}$-piecewise linear function $f \colon C_K \to \mathbb{R}$ whose bending locus is Σ_I and for which $\gamma^* f - f$ is linear for all $\gamma \in \Gamma_I$. Such a function may not exist, as X_{Γ}^{Σ} need not be projective for some Σ. But, there is a combinatorial model for what the polytope would be: the dual polyhedral complex $(\Sigma_I)^{\vee}$ which has a cell $\mathbb{R}_+^{\mathrm{codim}(\sigma)}$ for each cone $\sigma \in \Sigma_I$. There is still a fibration

$$\mu \colon X(\Sigma_I) \to (\Sigma_I)^{\vee}$$

whose fiber over the relative interior of a codimension k stratum of $(\Sigma_I)^{\vee}$ is T^{n-k}.

Consider an analytic neighborhood of Type III toric boundary in the unipotent quotient $U_I \backslash \mathbb{D}$, called $V_I \subset X(\Sigma_I)$ in Section 3.3. We have a restricted fibration

$$\mu \colon V_I \to P_I \subset (\Sigma_I)^{\vee}.$$

The action of the group $\Gamma_I = \mathrm{Stab}_{\Gamma}(I)/U_I$ can be made equivariant with respect to the fibration μ, as it acts torically.

The punctured neighborhood $A_I = V_I \cap (U_I \otimes \mathbb{C}^*)$ of the toric boundary fibers over the open cell $O = \mathbb{R}_+^n \cap P_I$ and is thus a trivial fiber bundle $T^n \times O$. The fibers of μ are, in the toric coordinates $(z_1, \ldots, z_n) \in U_I \otimes \mathbb{C}^*$ given by $|z_i| = r_i$ for $r_i \in \mathbb{R}_+$. So the action of Γ_I on $T^n \times O$ is, on the T_n factor, induced by the action of Γ_I on $I^{\perp}/I \supset U_I = H_1(T_n, \mathbb{Z})$.

Let $\Delta = X_{\Gamma}^{\Sigma} \setminus X_{\Gamma}$. The inclusion $N_{\Delta}^* \to N_{\Delta}$ is locally modeled by

$$\Gamma_I \backslash A_I \to \Gamma_I \backslash V_I$$

in a neighborhood of the cusp I. Note that $\Delta_I \subset \Gamma_I \backslash (V_I \setminus A_I)$ and the latter retracts to the former, by Γ_I-equivariantly retracting the Type II components of $V_I \backslash A_I$ to their intersection with the Type III locus.

Let $\widetilde{\Delta}_I = V_I \setminus A_I$ be the Γ_I-cover of Δ_I. The deformation retraction $V_I \to \widetilde{\Delta}_I$ can be made Γ_I-equivariant.

Proposition 4.4. *The morphism $N_{\Delta_I}^* \to \Delta_I$ induces an isomorphism on H_1. Both can be identified with the group homology $H_1(\Gamma_I, \mathbb{Q})$.*

Proof. The Leray-Cartan spectral sequence computes the homology of an orbifold quotient X/G as follows:

$$H_p(G, H_q(X, \mathbb{Q})) \Rightarrow H_{p+q}(X/G, \mathbb{Q}).$$

The E_2 page yields an exact sequence for computing H_1:

$$H_0(G, H_1(X, \mathbb{Q})) \to H_1(X/G, \mathbb{Q}) \to H_1(G, H_0(X, \mathbb{Q})) \to 0.$$

The spectral sequence is functorial for G-equivariant morphisms, so applying this to the Γ_I-maps $A_I \to V_I \to \widetilde{\Delta}_I$ we get a morphism of

exact sequences:

$$
\begin{array}{ccccccc}
H_0(\Gamma_I, H_1(T^n, \mathbb{Q})) & \longrightarrow & H_1(N^*_{\Delta_I}, \mathbb{Q}) & \longrightarrow & H_1(\Gamma_I, \mathbb{Q}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_0(\Gamma_I, H_1(\widetilde{\Delta}_I, \mathbb{Q})) & \longrightarrow & H_1(\Delta_I, \mathbb{Q}) & \longrightarrow & H_1(\Gamma_I, \mathbb{Q}) & \longrightarrow & 0
\end{array}
$$

We claim both groups on the far left vanish. First, $H_1(\widetilde{\Delta}_I, \mathbb{Q}) = 0$ because $\widetilde{\Delta}_I$ is a union of toric varieties along toric overlaps. The upper left group is $H_0(\Gamma_I, (U_I)_{\mathbb{Q}}) \simeq H^0(\Gamma_I, K_{\mathbb{Q}})^*$, the invariant subspace of the standard representation. As $\Gamma_I \subset O^+(K)$ is finite index, the Borel Density Theorem implies it acts without fixed vectors on K. $\square$

Theorem 4.5. *Let $X = X_\Gamma^\Sigma$, $U = X_\Gamma$, and $\Delta = X_\Gamma^\Sigma \setminus X_\Gamma$. Then every homology class $\alpha \in H_2(X, \mathbb{Q})$ splits.*

Proof. Let $\Delta_{\mathrm{II}} := \bigcup_J \Delta_J$ and $\Delta_{\mathrm{III}} := \bigcup_I \Delta_I$ be the unions of the Type II and III boundary components, respectively. We have $\Delta = \Delta_{\mathrm{II}} \cup \Delta_{\mathrm{III}}$.

Assuming Σ is taken fine enough, $\Delta_{\mathrm{II}} = \coprod_J \Delta_J$ is a disjoint union of smooth divisors, in bijection with orbits of isotropic planes $J \subset L$. On the other hand, the union $\Delta_{\mathrm{III}} = \coprod_I \bigcup_c \Delta_{I,c}$ is a disjoint union of SNC divisors, with one connected SNC divisor for each orbit of isotropic line $I \subset L$. The dual complex of this divisor $\bigcup_c \Delta_{I,c}$ is the hyperbolic orbifold $\Gamma_I \backslash \mathbb{P}C_K$.

Applying Proposition 4.3 to the Type II components, we can split α relative to the boundary divisor Δ_{II}. The normal bundle of each component is anti-ample on the fibers of the contraction map to X_Γ^{BB}, so it has non-trivial Euler class, as required.

Next, we apply Proposition 4.2 to the pair $(X \setminus \Delta_{\mathrm{II}}, \Delta_{\mathrm{III}} \setminus \Delta_{\mathrm{II}})$. The map $H_1(N^*_\Delta) \to H_1(N_\Delta)$ is an isomorphism using Proposition 4.4. $\square$

We have represented every class in $H_2(X_\Gamma^\Sigma, \mathbb{Q})$ as a linear combination of classes supported on either X_Γ or Δ. We require a further strengthening of this:

Theorem 4.6. *Every homology class $\alpha \in H_2(X_\Gamma^\Sigma, \mathbb{Q})$ can be represented as the sum of a cycle supported on the interior X_Γ and an algebraic cycle contracted by the map to X_Γ^{BB}.*

Proof. By Theorem 4.5, we may express α as the sum of a class in $H_2(X_\Gamma, \mathbb{Q})$ and a class supported in the boundary Δ. We will treat Type II and III boundary divisors separately. Recall from Section 3.4 that $X_\Gamma^\Sigma \to X_\Gamma^{\mathrm{BB}}$ induces a fibration $\pi_J \colon \Delta_J \to B_J$ (in finite quotients of abelian varieties) of each Type II toroidal boundary divisor over the Type II Baily-Borel modular curve, and it contracts each Type III boundary divisor Δ_I to a point.

In Type II, suppose that $\alpha_J \in H_2(\Delta_J, \mathbb{Q})$, and let $\mathcal{L}_J$ be the normal bundle of the smooth divisor $\Delta_J \subset X_\Gamma^\Sigma$. For any algebraic curve C_J

contained in a fiber of π_J, we have $\mathcal{L}_J \cdot C_J < 0$ because C is contracted by the morphism to Baily-Borel. Thus, we can choose $r \in \mathbb{Q}$ for which

$$\mathcal{L}_J \cdot (\alpha_J + rC_J) = 0.$$

After clearing denominators, a well-known theorem of Thom says that $N(\alpha_J + rC_J) \in H_2(\Delta_J, \mathbb{Z})$ can be represented by the fundamental class of an immersed (real) oriented surface, S. The pullback of $\mathcal{L}_J$ to S is a degree 0 complex line bundle, so it is smoothly trivial. By flowing along a nowhere vanishing normal vector field, S is homologous to a surface S' in $X_\Gamma^\Sigma \setminus \Delta_J$. We can then express

$$\alpha_J = \frac{1}{N}[S'] - r[C_J].$$

In Type III, suppose that $\alpha_I \in H_2(\Delta_I, \mathbb{Q})$. Recall that Δ_I has simple normal crossings with dual complex homeomorphic to the hyperbolic manifold $\Gamma_I \backslash \mathbb{P}C_K$. From the E_1-page of the Mayer-Vietoris spectral sequence associated to the covering $\mathfrak{U}$ of Δ_I via neighborhoods of the irreducible toric components, we obtain the following short exact sequence for $H_2(\Delta_I, \mathbb{Q})$, using that H_1 vanishes for all toric varieties:

$$H_2(\mathfrak{U}_1, \mathbb{Q}) \to H_2(\mathfrak{U}_0, \mathbb{Q}) \to H_2(\Delta_I, \mathbb{Q}) \to H_2(\Gamma_I, \mathbb{Q}) \to 0.$$

Recall that the map $N_{\Delta_I}^* \to \Delta_I$ is the quotient of a Γ_I-equivariant map $V_I \to \widetilde{\Delta}_I$. Now, we appeal to the same functoriality of the Leray-Cartan spectral sequence as in Proposition 4.4, this time for H_2 on the E_2-page, to obtain

$$\begin{array}{ccccccc}
H_1(\Gamma_I, H_1(T^n, \mathbb{Q})) & \longrightarrow & H_2(N_{\Delta_I}^*, \mathbb{Q}) & \longrightarrow & H_2(\Gamma_I, \mathbb{Q}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_0(\Gamma_I, H_2(\widetilde{\Delta}_I, \mathbb{Q})) & \longrightarrow & H_2(\Delta_I, \mathbb{Q}) & \longrightarrow & H_2(\Gamma_I, \mathbb{Q}) & \longrightarrow & 0.
\end{array}$$

Since $\widetilde{\Delta}_I$ is a union of toric varieties, the Γ_I-coinvariants of $H_2(\widetilde{\Delta}_I, \mathbb{Q})$ are precisely the algebraic curve classes in Δ_I. By a diagram chase, α_I can be expressed as: $\alpha_I = T + C_I$ where C_I is in the image of $H_0(\Gamma_I, H_2(\widetilde{\Delta}_I, \mathbb{Q}))$ and T is in the image of $H_2(N_{\Delta_I}^*, \mathbb{Q})$, so is supported on the interior. There is a surjection

$$\bigoplus_c H_2(\Delta_{I,c}, \mathbb{Q}) \to H_0(\Gamma_I, H_2(\widetilde{\Delta}_I, \mathbb{Q}))$$

and since $\Delta_{I,c}$ is a toric variety, $H_2(\Delta_{I,c}, \mathbb{Q})$ is generated by algebraic curve classes. $\qquad\square$

Proposition 4.7. *Every algebraic curve class which pushes forward to zero along $X_\Gamma^\Sigma \to X_\Gamma^{\mathrm{BB}}$ is homologically equivalent to a linear combination of one-dimensional Type III toric boundary strata.*

Proof. This is shown in [AE21, Thm. 7.18]. Any such curve in Type II can be degenerated into the Type III locus by limiting the image point $j \in B_J$ to a cusp of the modular curve. This reduces to the Type III

case. One applies the torus action on a Type III toric boundary divisor to further break any curve into one-dimensional torus orbits. $\qquad\square$

5. Mixed mock modularity of special divisors

In this section, we assemble all the ingredients from previous sections to prove the main theorems of the paper.

Let Σ be an admissible fan and assume that Σ is regular and Γ is neat—the general case reduces to this one by Remark 3.10. Let $I \subset L$ be an isotropic line and let Σ_I be the Γ_I-invariant polyhedral cone decomposition of the cone C_K^+ inside the hyperbolic lattice $K = I^\perp/I$.

To prove Theorem 1.2 (which in turn implies Theorem 1.1), it suffices, by Poincaré duality, to intersect the generating series of special divisors with each class $\alpha \in H_2(X_\Gamma^\Sigma, \mathbb{Q})$. By Theorem 4.6 and Proposition 4.7, any class α can be expressed as

$$\alpha = \alpha' + \sum_\sigma d_\sigma [C_\sigma], \quad d_\sigma \in \mathbb{Q},$$

where $\alpha' \in H_2(X_\Gamma, \mathbb{Q})$ and $C_\sigma \simeq \mathbb{P}^1$ are the Type III toric boundary strata associated to codimension 1 cones $\sigma \in \Sigma_I$ ranging over all I.

Since the pairing of the generating series of special divisors with the class α' is a holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_L$ by [KM90], we can assume that $\alpha' = 0$ and that $\alpha = [C]$ is the class of a single toric curve determined by a cone σ.

By Theorem 3.13, the $\mathbb{C}[L^\vee/L]$-valued generating series of intersecting $[C]$ with special cycles is equal to

$$\sum_{\lambda \in K^\vee} p^+(\lambda) q^{-Q(\lambda)} p_K^L(v_{[\lambda]}).$$

Here $p^+(\lambda) = \frac{1}{2}(|\lambda \cdot c^+| + |\lambda \cdot c^-| + \sum_i a_i |\lambda \cdot c_i|)$ and $\{c_i\}_{i=1}^{n-1}$ are generators of the rays of σ, which, with c^+, c^-, respectively generate the maximal cones σ^+, σ^- containing σ. To make the notation compatible with Section 2, we declare $c_n := c^+$, $c_{n+1} := c^-$, and $a_n = a_{n+1} = 1$ so that $\{c_i\}_{i=1}^{n+1}$ are a collection of integral vectors in C_K^+ for which $\sum a_i c_i = 0$. We have $\Delta_{I,c_i} \cdot C = a_i$ for all $i = 1, \ldots, n+1$. This holds even when c_i is isotropic, in which case $\Delta_{J_i} \cdot C = a_i$ for the corresponding rank 2 isotropic lattice $J = I \oplus \mathbb{Z}c_i$.

For any non-isotropic ray $\mathbb{R}_{\geq 0}c$ in Σ_I, let $N_c = Q(c) > 0$ and let $c^\perp$ be the orthogonal complement of c in K. It is a negative definite lattice of rank $n-1$. Its vector-valued theta series

$$\Theta_{c^\perp} = \sum_{\delta \in (c^\perp)^\vee} q^{-Q(\delta)} v_{[\delta]} \in \mathbb{C}[[q^{\frac{1}{D_{c^\perp}}}]] \otimes \mathbb{C}[(c^\perp)^\vee/c^\perp]$$

is a holomorphic modular form of weight $\frac{n-1}{2}$ with respect to the Weil representation $\overline{\rho}_{c^\perp}$. Similarly, for any isotropic ray $\mathbb{R}_{\geq 0}c$, the vector-valued theta series

$$\Theta_M = \sum_{\gamma \in M^\vee} q^{-Q(\gamma)} v_{[\gamma]} \in \mathbb{C}[[q^{\frac{1}{D_M}}]] \otimes \mathbb{C}[M^\vee/M]$$

is a holomorphic modular form of weight $\frac{n}{2} - 1$ with respect to $\overline{\rho}_M$, where $M = c^\perp/c$. By Theorem 2.13, the function Θ_σ defined by

$$\sum_{\lambda \in K^\vee} p^+(\lambda) q^{-Q(\lambda)} v_{[\lambda]} - \sum_{Q(c_i)>0} \frac{a_i}{2} p_{c_i^\perp \oplus \mathbb{Z}c_i}^K (\Theta_{c_i^\perp} \otimes F_{N_i}^+) + \sum_{Q(c_i)=0} \frac{a_i}{12} E_2 p_{M_i}^K(\Theta_{M_i})$$

is a holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_K$. On the other hand,

$$\left[\sum_{\beta \in L^\vee/L} \sum_{m \in -Q(\beta)+\mathbb{Z}} [Z(\beta,m)] \otimes v_\beta q^m - \right.$$

$$\sum_{(I,c)} \frac{1}{2} p_{c^\perp \oplus \mathbb{Z}c}^L (\Theta_{c^\perp} \otimes F_N^+) \otimes \Delta_{I,c} + \sum_J \frac{1}{12} E_2 p_M^L(\Theta_M) \otimes \Delta_J \left. \right] \cdot C =$$

$$\sum_{\lambda \in K^\vee} p^+(\lambda) q^{-Q(\lambda)} p_K^L(v_{[\lambda]}) -$$

$$\sum_{Q(c_i)>0} \frac{a_i}{2} p_{c_i^\perp \oplus \mathbb{Z}c_i}^L (\Theta_{c_i^\perp} \otimes F_{N_i}^+) + \sum_{Q(c_i)=0} \frac{a_i}{12} E_2 p_{M_i}^L(\Theta_{M_i}) = p_K^L(\Theta_\sigma)$$

is holomorphic modular form of weight $1 + \frac{n}{2}$ and representation $\overline{\rho}_L$. Here i ranges only over $1, \ldots, n+1$ because all other components of the boundary Δ intersect C to be zero. Hence the pairing of the series in Theorem 1.2 with $[C]$ is a holomorphic modular form, which concludes the proof.

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Department of Mathematics, University of Georgia, Boyd Hall, Athens, GA 30602

Email address: philip.engel@uga.edu

Department of Mathematics, Michigan State University, 619 Red Cedar Rd, East Lansing, MI 48824

Email address: greerfra@msu.edu

Department of Mathematics, Harvard University, 1 Oxford St, Cambridge, MA 02138, USA

Email address: tayou@math.harvard.edu