ISOMORPHISMS BETWEEN MODULI SPACES

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1. Introduction

Today, we will be studying various connections between three moduli spaces of objects in non-abelian Hodge theory.

Fix a smooth projective variety X. We use the following notation:

(1) We use $R_{\mathrm{B}}(X, r)$ denotes the moduli space of r-dimensional representations $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_r(\mathbb{C}))$.
(2) We use $M_{\mathrm{B}}(X, r, \zeta)$ for the quotient (which is a scheme) of this by the conjugation action $\mathrm{GL}_r(\mathbb{C})$ on these representations, with basepoint $\zeta \in X$, whose points are in bijection with semisimple representations.
(3) We use $R_{\mathrm{dr}}(X, r, \zeta)$ for the rigidified moduli space of flat bundles (V, ∇) of rank r, together with a chosen isomorphism of a fiber $V_\zeta \simeq \mathbb{C}^r$, for $\zeta \in X$ a point.
(4) We use $M_{\mathrm{dr}}(X, r)$ for the moduli space of flat bundles (V, ∇) of rank r.
(5) We use $R_{\mathrm{Dol}}(X, r)$ for the rigidified moduli space of Higgs bundles $(E, \theta : E \to E \otimes \Omega^1_X)$ of rank r, together with an isomorphism of a fiber $E_x \simeq \mathbb{C}^r$.
(6) We use $M_{\mathrm{Dol}}(X, r)$ for the quotient of $R_{\mathrm{Dol}}(X, r)$ by the action of GL_r altering the chosen isomorphism. This quotient has geometric points in bijection with polystable Higgs bundles of rank r.

Remark 1.1. The spaces R_{B}, R_{dr}, R_{Dol} are actual moduli spaces with universal families. The quotients are schemes, which do not have universal families, but their geometric points are in bijection with the above prescribed objects.

The main results of today are about comparing these different moduli spaces.

Theorem 1.2. *There is an analytic, but not algebraic, isomorphism $R_{\mathrm{dr}}(X, r, \zeta) \simeq R_{\mathrm{B}}(X, r)$. This is compatible with the $\mathrm{GL}_r(\mathbb{C})$ action and so induces an analytic isomorphism $M_{\mathrm{dr}}(X, r) \simeq M_{\mathrm{B}}(X, r)$.*

Theorem 1.3. *There is a bijection of the underlying sets of geometric points $R_{\mathrm{dr}}(X, r) \to R_{\mathrm{Dol}}(X, r)$ which induces a homeomorphism $M_{\mathrm{dr}}(X, r) \simeq M_{\mathrm{Dol}}(X, r)$.*

However, the bijection $R_{\mathrm{dr}}(X,r) \to R_{\mathrm{Dol}}(X,r)$ is not in general continuous with respect to the analytic topology.

Remark 1.4. The above isomorphisms also hold for smooth projective maps $X \to S$, over a base scheme S.

Example 1.5. Some parts of the above theorems aren't too difficult to see. For example, let's see why the analytic isomorphism $R_{\mathrm{dr}}(X,r) \simeq R_{\mathrm{B}}(X,r)$ cannot be algebraic. Let's focus on the case X is a curve of genus g and $r = 1$. The point is that $R_{\mathrm{B}} \simeq (\mathbb{C}^\times)^{2g}$ is affine as it corresponds to specifying a tuple of $2g$ nonzero complex numbers, where each generator of $\pi_1(X)$ lands. On the other hand, in this case, there is a surjective projection map $M_{\mathrm{dr}}(X,r) \to \mathrm{Pic}^0_X$ which sends $(\mathscr{L}, \nabla) \mapsto \mathscr{L}$. Since Pic^0_X is not affine, the analytic isomorphism cannot be algebraic. Alternatively, one can recover X from Pic^0_X, so the algebraic structure of $M_{\mathrm{dr}}(X,1)$ depends on X, and even lets you recover X. However, the algebraic structure of $M_{\mathrm{B}}(X,r)$ is independent of X.

Further, in this case the moduli space of Higgs bundles can be identified with $\mathrm{Pic}^0_X \times \mathbb{A}^g$ corresponding to a line bundle $\mathscr{L}$ and a Higgs field $\mathscr{L} \to \mathscr{L} \otimes \Omega^1_X$, where we can identify the Higgs field with an element of the g-dimensional vector space $H^0(X, \Omega^1_X)$. There is a homeomorphism $(\mathbb{C}^\times)^{2g} \simeq \mathrm{Pic}^0_X \times \mathbb{A}^g$ by writing both as $\mathbb{R}^{2g} \times (S^1)^{2g}$ but this isomorphism is quite non-algebraic.

2. The analytic isomorphism between de Rham and Betti moduli spaces

The idea of this isomorphism is to show that both spaces, $R_{\mathrm{dr}}(X,r)$ and $R_{\mathrm{B}}(X,r)$ represent the same functor on the category of analytic spaces. Once we have this, we will have a natural identification of these analytic spaces. Namely, we need to prove the following two results which describe the functors associated to R_{B} and R_{dr}.

Lemma 2.1. *$R_{\mathrm{B}}(X,r,\zeta)$ represents the functor which assigns to S the set of isomorphism classes of pairs $(\mathscr{F}, \beta)$ where $\mathscr{F}$ is a locally free sheaf of $f^{-1}(\mathscr{O}_S)$ rank r modules, for $f : X \times S \to S$ the structure map, and $\beta : \mathscr{F}|_\zeta \simeq \mathscr{O}_S^r$.*

Proof. Affineness of R_{B} implies that maps $S \to R_{\mathrm{B}}$ are identified with $\mathrm{Hom}(\pi_1(X,\zeta), \mathrm{GL}_r(H^0(S, \mathscr{O}_S$ The usual correspondence between representations and local systems with a framing, obtained by passing to the universal cover and looking at the monodromy action of the fundamental group then applies to give the functorial bijection. □

Lemma 2.2. *$R_{\mathrm{dr}}(X, r, \zeta)$ represents the functor which assigns to S the set of isomorphism classes of pairs $(\mathscr{F}, \beta)$ where $\mathscr{F}$ is a locally free sheaf of $f^{-1}(\mathscr{O}_S)$ rank r modules, for $f : X \times S \to S$ the structure map, and $\beta : \mathscr{F}|_\zeta \simeq \mathscr{O}_S^r$.*

Proof. The definition of R_{dr} identifies this with the functor assigning to S the set of (E, ∇, α) where E is a holomorphic vector bundle on X, $\nabla : E \to E \otimes \Omega^1_{X\times S/S}$ is a flat connection, and $\alpha : E|_\zeta \simeq \mathscr{O}_S^n$ is a frame. We want to identify this with the functor in the statement of the lemma.

The bijection is given by sending (E, ∇, α) to the locus of flat sections $(\ker \nabla, \alpha)$. Conversely, let $f : X \times S \to S$ be the structure map. Given a local system $\mathscr{F}$, one obtains E by taking $E = \mathscr{F} \otimes_{f^{-1}(\mathscr{O}_S)} \mathscr{O}_{X\times S}$ and taking the connection $\nabla = 1 \otimes d_{X\times S/S}$.

The key issue is to verify this is a bijection. By passing to an open covering, one can reduce to the case that X is a small open disc. We can similarly reduce to the case that $S \subset V$, for V a small open disc. In the case S is a point, this is the classical Riemann-Hilbert correspondence. A similar proof works when S is any smooth base. However, we need to verify this for general S which may fail to be smooth. One can deduce from this the case that S is an Artinian analytic space (corresponding to a local Artinian ring). In the case $\dim X = 1$, one can extend the trivial bundle E to a trivial bundle on $U \times V$. Here, flatness holds automatically because we are in relative dimension 1, so $\Omega^2 = 0$. The result then holds in this case because $U \times V$ is smooth, and one can use this to restrict the bijection to $U \times S$. For the general case, one can induct on the dimension of U, using the fact that the result holds on smooth lower dimensional objects, as well as on all infinitesimal thickenings of lower dimensional objects. We omit the details. □

3. The map on representation varieties is not continuous

Recall R_{dr} parameterizes flat bundles and R_{Dol} parameterizes polystable Higgs bundles with a framing, i.e. we specify an isomorphism to a fixed vector space at a point. This has the effect of rigidfying the moduli problem so that these actually have universal families over them, i.e., they are the moduli stacks of flat bundles/polystable Higgs bundles with a framing

When we forget the framing and take the quotient by GL_n parameterizing these isomorphisms to a fixed vector space, we obtain that the bijection $M_{\mathrm{dr}} \simeq R_{\mathrm{dr}}$ we have previously seen induces a homeomorphism between these spaces. However, although this bijection lifts to $R_{\mathrm{dr}} \to R_{\mathrm{Dol}}$ it is not continuous at the semistable points.

We now want to explain this through an example.

Remark 3.1. It is actually continuous if one works restricts to stable Higgs bundles and irreducible flat bundles.

Consider X a genus 1 curve over $\mathbb{C}$ and $x \in X$ a point. Let $E = \mathscr{O}_X \oplus \mathscr{O}_X$ the trivial rank 2 bundle together with a a fixed isomorphism $E_x \simeq \mathbb{C}^2$. For any fixed $t \in \mathbb{R}$, define the Higgs field

$$\theta_{t,a} := \begin{pmatrix} 0 & dz \\ 0 & at\, dz \end{pmatrix}$$

By looking at the matrix, we see $\lim_{t \to 0} \theta_{t,a} = 0$. In particular this limit is independent of a. Let $\Psi : R_{\mathrm{Dol}} \to R_{\mathrm{dr}}$ denote the bijection between Higgs bundles and flat bundles. We will see that $\lim_{t \to 0} \Psi(\theta_{t,a})$ depends on a. In particular, the map Ψ cannot be continuous, as the image of a limit point is not the limit point of the image. (The relevant spaces are separated/Hausdorff so limit points are unique.)

Now, let's compute where these Higgs fields are sent under Ψ. Let K be the constant Hermitian metric so that for e, f local sections of E we have $e, f = e_1 \overline{f_1} + e_2 \overline{f}_2$.

Lemma 3.2. *The connection associated to the Higgs field θ with the above constant metric is $d + (\theta + \overline{\theta})$.*

Proof. Recall how we constructed the associated connection. We write $D = \overline{\partial} + \theta$. Then, we let $\overline{\theta}$ be the adjoint of θ and let $\overline{\partial}$ be the adjoint of ∂. This means

$$\begin{aligned} \overline{\partial}(e, f) &= (\overline{\partial} e, f) + (e, \partial_K f) \\ (\overline{\theta} e, f) &= (e, \theta_K f). \end{aligned}$$

In this sense $\overline{\theta}_K$ is the adjoint matrix of θ. To compute ∂_K, we may observe that

$$\overline{\partial} \sum_i e_i \overline{f_i} = \sum_i \overline{\partial}(e_i) \overline{f}_i + \sum_i e_i \overline{\partial}(\overline{f}_i).$$

Therefore, it is enough to find ∂_K which satisfies

$$\overline{\partial_K f_i} = \overline{\partial}(\overline{f}_i)$$

We can simply take $\partial_K f = \partial f$.

Then, we define

$$\begin{aligned} D'' &= \overline{\partial} + \theta \\ D'_K &= \partial_K + \overline{\theta}_K = \partial + \overline{\theta}. \end{aligned}$$

We then take the associated connection to be $D = D'' + D'_K = \overline{\partial} + \partial + \theta + \overline{\theta} = d + (\theta + \overline{\theta})$. This shows that if we start with a connection θ, the resulting connection is $d + \theta + \overline{\theta}$. $\square$

Let's use the above to compute $\Psi(\theta_{t,a})$.

Proposition 3.3. *$\Psi(\theta_{t,a})$ depends on a.*

Proof. Note that θ is not flat with respect to the constant connection because if we set $D = d + A$ where $A = \theta + \overline{\theta}$, we do not have $D^2 = 0$, i.e., we do not have $dA = A \wedge A$, as we find $dA = 0$ but $A \wedge A$ has nonzero upper left hand entry. However, set

$$g_{a,t} = \begin{pmatrix} 1 & 1/at \\ 0 & 1 \end{pmatrix}$$

We find

$$\phi_{a,t} := g_{a,t}^{-1} \theta_{a,t} g_{a,t} = \begin{pmatrix} 0 & 0 \\ 0 & ad\, dz. \end{pmatrix}$$

If we take $A := \phi_{a,t} + \overline{\phi}_{a,t}$ We find $0 = dA = A^2$ which shows that $d + A$ is a flat connection. Using Lemma 3.2, if we take $\phi_{a,t}$ and K to be the constant metric, the associated bundle is flat, and so gives the corresponding flat bundle. Transforming back to our original basis, we find the flat bundle corresponding to the Higgs field θ is

$$g_{a,t} \begin{pmatrix} 0 & 0 \\ 0 & at\, dz + \overline{at} d\overline{z} \end{pmatrix} g_{a,t}^{-1} = \begin{pmatrix} 0 & dz + \frac{\overline{at}}{at} d\overline{z} \\ 0 & ad\, dz + \overline{at} d\overline{z}. \end{pmatrix}$$

Now, we take the limit as $t \to 0$. For the Higgs bundle, this approaches

$$\begin{pmatrix} 0 & dz \\ 0 & 0 \end{pmatrix}$$

and is independent of a. However, for the flat bundle, this is

$$\begin{pmatrix} 0 & dz + \frac{\overline{a}}{a} d\overline{z} \\ 0 & 0 \end{pmatrix}$$

and depends on a. □

Corollary 3.4. *The bijection $R_{\mathrm{dr}} \simeq R_{\mathrm{Dol}}$ is not continuous.*

Proof. Since the Higgs bundle is independent of a but the flat bundle depends on a, the correspondence cannot be continuous since the limit of the Higgs bundles cannot be sent under Ψ to all of these simultaneous distinct limits of flat bundles. □

4. THE HOMEOMORPHISM BETWEEN BETTI AND DOLBEAULT MODULI SPACES

The main goal of this section is to explain the basic idea of the proof of the following.

Theorem 4.1. *The bijection on geometric points of $M_{\mathrm{dr}} \to M_{\mathrm{Dol}}$ in fact induces a homeomorphism.*

This is the last part of Theorem 1.3 we have yet to explain.

To explain the relation, we need to recall the basic bijection between M_{dr} and M_{Dol}. If we start with (E, D) there is a way to decompose the connection into $D = D' + D''$ with $D'_K = \partial + \overline{\theta}$ and $D''_K = \overline{\partial} + \theta$, where θ is a Higgs field. These operators depend on a choice of metric K, and one can choose a special harmonic metric (making $\Lambda(D'')^2 = 0$) so that the flat bundle yields a Higgs bundle. Conversely, given a Higgs bundle, one can find a Harmonic metric K and define a corresponding operator D_K yielding the structure of a flat bundle. Altogether, this Harmonic bundle and Harmonic metric interpolates between flat bundles and Higgs bundles.

the rest of this section morally describes the idea, but I didn't carefully go through all the details, and so what I say may be slightly off

The main difficult input is the following:

Proposition 4.2. *Choose a sequence of harmonic bundles V_i on X with Harmonic metric K_i so that the coefficient of the corresponding Higgs field θ_i are uniformly bounded in L^1 norm. Then there is a harmonic bundle V with harmonic metric K and a subsequence $\{i'\}$ of the above bundles so that the V_i converge to V as harmonic bundles, and the same holds for $\partial, \overline{\partial}, \theta, \overline{\theta}$*

Proof. In the case we only wanted to the above property for Higgs bundles, in place of harmonic bundles, this would essentially be properness of the Hitchin map. This properness of the Hitchin map can be proven algebraically.

To get this convergence property for Harmonic metrics (which is essentially a version of sequential compactness, hence properness) we can use Uhlenbeck's weak compactness lemma, which says that if the curvatures of a sequence of bundles are uniformly bounded, the V_i converge to some unitary V with unitary connection $\partial + \overline{\partial}$ so that $\partial + \overline{\partial} - \partial_i + \overline{\partial}_i$ converge to 0 as differential operators on L^p functions.

Since the θ_i are uniformly bounded, one deduces that the coefficients of its characteristic polynomial are uniformly bounded, which one can show implies the $|\theta_i|_{K_i}$ are uniformly bounded. The curvatures $F_{\partial_i + \overline{\partial}_i} = -\phi_i \overline{\phi_i} - \overline{\phi}_i \phi_i$, using flatness of the bundle, which implies the hypothesis for Uhlenbeck's compactness lemma.

Once one obtains this convergence of harmonic bundles, one can further massage terms to obtain convergence of the remaining operators. We omit several tricky details. □

Let J denote the standard unitary metric on $\mathbb{C}^r$ and let $R^J_{\mathrm{Dol}}(X, \zeta, r) \subset R_{\mathrm{Dol}}(X, \zeta, r)$ denote the subset of (E, β) for E a Harmonic Higgs bundle on X_s and $\beta : E_\zeta \simeq \mathbb{C}^r$ a frame, so that E has a Harmonic metric K with $K_\zeta = J$. The harmonic metric turns out to be determined by its restriction to a fiber.

Similarly, define $R^J_{\mathrm{dr}}(X,\zeta,r) \subset R_{\mathrm{dr}}(X,\zeta,r)$ as the subset of (E,β) where E is a semisimple vector bundle with flat connection on X and $E_\zeta \simeq \mathbb{C}^r$ is a frame with a harmonic metric K on E with $K_\zeta = J$.

Corollary 4.3. *If we have a sequence $(E_i,\beta_i) \in R^J_{\mathrm{Dol}}(X,\zeta,r)$ in the preimage of a compact set of $M_{\mathrm{Dol}}(X,r)$. After passing to a subsequence, we can find (E,β) so that there are C^∞ isomorphisms $E_i \simeq E$ so that the θ_i and $\overline{\partial}_i$ converge to $(\theta,\overline{\partial})$ associated to E and β_i converge to β. A similar statement holds with flat bundles in place of Higgs bundles.*

Proof. Using the Hitchin map, and the assumption that E_i lie in the preimage of a compact set, we find the eigenvalues of the θ_i are uniformly bounded. Using Proposition 4.2, we can assume there are C^∞ bundle isomorphisms so that the θ_i and $\overline{\partial}_i$ converge to $\theta, \overline{\partial}$. Then, using compactness of the unitary group, we can also assume the β_i converge to β.

The argument for flat bundles is more involved, and we omit it. □

Proposition 4.4. *The bijection of sets $R_{\mathrm{Dol}} \simeq R_{\mathrm{dr}}$ induces a homeomorphism $R^J_{\mathrm{Dol}}(X,\zeta,r) \simeq R^J_{\mathrm{dr}}(X,\zeta,r)$. Quotienting by the unitary group, we obtain the desired homeomorphism $M^J_{\mathrm{Dol}}(X,\zeta,r) \simeq M^J_{\mathrm{dr}}(X,\zeta,r)$.*

Proof. We'll just show the forward direction is continuous. Start with a sequence $(E_i,\beta_i) \in R^J_{\mathrm{Dol}}(X,\zeta,r)$ converging to (E,β). We want to show that the flat bundles corresponding to (E_i,β_i) converge to the flat bundle corresponding to (E,β). We can restrict to a subsequence to assume all points lie over a compact set in $M_{\mathrm{Dol}}(X,\zeta,r)$. By passing to a further subsequence, we can apply Proposition 4.2 to assume the (E_i,β) converge to a harmonic bundle V. The point is now that this V and the V_i let us see both sides of the Dolbeault and de Rham moduli spaces, in order to deduce the desired continuity. By Proposition 4.4 the d''_i and ∇_i associated to the flat bundles V_i converge to d'' and ∇ for V, and the β_i converge to β. Similarly, the Higgs bundles associated to the V_i converge to the Higgs bundles corresponding to V. This is what we wanted to show. □

5. An application to PVHS

Recall there is a $\mathbb{C}^\times$ scaling action on M_{Dol} given by

$$\begin{aligned} \mathbb{C}^\times \times M_{\mathrm{Dol}}(X) &\to M_{\mathrm{Dol}}(X) \\ t,(E,\theta) &\mapsto (E,t\theta)\,. \end{aligned}$$

The fixed points of this action are precisely those Higgs bundles which underly a PVHS. The reason for this is that being a fixed point essentially gives a grading on the bundle, which corresponds to the Hodge structure.

This tells us we can deform any Higgs bundle to a bundle underlying a VHS. As a corollary of the homeomorphism between Higgs bundles and representations, (where one has to think a little bit more carefully about reducible representations) we obtain the following:

Corollary 5.1. *Any irreducible representation* $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$ *can be deformed to a representation* $\rho' : \pi_x(X) \to \mathrm{GL}_r(\mathbb{C})$ *so that the corresponding flat vector bundle underlies a complex VHS.*

References