PRELIMINARY NOTES ON O-MINIMALITY

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The purpose of these notes is to give a "model theory-lite" introduction to the theory of o-minimality. The definition of o-minimal structures can be phrased succinctly without understanding any model theory, and the aim of these notes is to give a development of the foundations of o-minimality with correspondingly little in the way of formal model theory. This will form the basis of the Spring 2022 seminar on o-minimality and Ax–Schanuel at Harvard University.

1. Structures

Definition 1.1. A *structure* consists of a set M and, for each[1] $n \in \mathbb{N}$, a collection $D_n \subseteq \mathbb{P}(M^n)$ of subsets of M^n. These are required to satisfy the following properties:

1. D_n contains the empty subset $\emptyset \subseteq M^n$, and is closed under finite intersections, unions, complements and the natural action of S_n on $\mathbb{P}(M^n)$ by permutation of coordinates.
2. For any $n \geq 1$ the diagonal $\Delta \subseteq M^n$ is in D_n.
3. If $V_1 \in D_{n_1}$ and $V_2 \in D_{n_2}$ then $V_1 \times V_2 \in D_{n_1+n_2}$.
4. If $V \in D_{n+1}$, then the image $\pi(V)$ of V under any of the projections $\pi\colon M^{n+1} \to M^n$ is in D_n.

Given a structure M, we refer to the elements of D_n as the *basic definable subsets of* M^n.

Example 1.2. Let k be an algebraically closed field. For $n \in \mathbb{N}$, let $D_n \subseteq \mathbb{P}(k^n)$ denote the set of constructible subsets of k^n in the sense of algebraic geometry (i.e. finite unions of locally closed subsets in the Zariski topology). Then $(k, (D_n)_{n\in\mathbb{N}})$ is a structure. The content in this statement is the fact that a projection of a constructible set is constructible: this is a theorem of Chevalley.

Example 1.3. Let $D_n \subseteq \mathbb{P}(\mathbb{R}^n)$ denote the set of semialgebraic subsets of $\mathbb{R}^n$. Then $\mathbb{R}_{\mathrm{alg}} = (\mathbb{R}, (D_n)_{n\in\mathbb{N}})$ is a structure. Again, the content in this statement is the fact that a projection of a semialgebraic set is semialgebraic: this is a theorem of Tarski–Seidenberg.

Example 1.4. Let M be a set and $D_n^0 \subseteq \mathbb{P}(M^n)$ a subset for each $n \in \mathbb{N}$. Then there is a smallest structure $(M, (D_n)_{n\in\mathbb{N}})$ such that $D_n^0 \subseteq D_n$ for all $n \in \mathbb{N}$. We say that $(M, (D_n)_{n\in\mathbb{N}})$ is the structure *generated* by the $(D_n^0)_{n\in\mathbb{N}}$.

Example 1.5. This example won't be necessary for us, but may be useful to anyone who wants to do some further reading into "proper" model theory. Here is how structures are usually developed in model theory, see e.g. [5, §1.1]. Let us fix a *signature* σ, by which we mean two collections of sets $(\mathcal{R}_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_n)_{n\in\mathbb{N}}$

[1] In these notes we follow the convention that $0 \in \mathbb{N}$.

indexed by $n \in \mathbb{N}$. Elements of $\mathcal{R}_n$ are referred to as n-ary relation symbols, and elements of $\mathcal{F}_n$ are referred to as n-ary function symbols. A *σ-structure* consists of a set M together with, for each $R \in \mathcal{R}_n$, a subset $[R]_M \subseteq M^n$ and, for each $f \in \mathcal{F}_n$, a function $f^M : M^n \to M$. The sets $[R]_M$ and functions f^M are referred to as the *interpretations* of the symbols R and f in M. See e.g. [5, §1.1] or any other book on model theory.

Given a σ-structure M for some signature σ, one obtains a structure $(M, (D_n)_{n\in\mathbb{N}})$ in the sense of Definition 1.1 by taking the structure generated by the subsets $[R]_M$ and the graphs of the functions f^M.

All of the preceding examples are special cases of this construction.

- Given a field k, consider the signature σ_k containing two 2-ary function symbols $+$ and $\times$ and a 0-ary function symbol c_λ for each $\lambda \in k$. We can then make k into a σ_k-structure where $+^k$ and $\times^k$ are the addition and multiplication in k, and c_λ^k is the function taking the unique point of k^0 to $\lambda \in k$. If k is algebraically closed, the structure generated by this σ-structure is the one from Example 1.2.
- In the case $k = \mathbb{R}$, we can enlarge the above signature by adding in a single 2-ary relation symbol $<$, and then make $\mathbb{R}$ into a $\sigma_{\mathbb{R}}^+$-structure by setting $[<]_{\mathbb{R}} = \{(x, y) \in \mathbb{R}^2 \,:\, x < y\}$. The structure generated by this $\sigma_{\mathbb{R}}^+$-structure is the one from Example 1.3.
- In fact, the structure in Example 1.3 is *also* the structure generated by the $\sigma_{\mathbb{R}}$-structure on $\mathbb{R}$. Indeed, the set $\{(x, y, z) \in \mathbb{R}^3 \,:\, x = y + z^2\}$ is basic definable in $\sigma_{\mathbb{R}}$, and its projection to $\mathbb{R}^2$ is the set $\{(x, y) \in \mathbb{R}^2 \,:\, x \geq y\}$. So $[<]_{\mathbb{R}}$ is already basic definable in the structure generated by $+, \times$ and the singletons in $\mathbb{R}$.

The final two examples illustrate a particular phenomenon called *definitional expansion* [5, §2.6]: given a σ-structure M it is often possible to enlarge the signature σ while keeping the same basic definable sets in M. The theory of o-minimality is insensitive to definitional expansions, which is why we prefer here to phrase all our results in the language of *structures* as opposed to *σ-structures* for particular signatures σ.

We will need to work with a notion of "definability" for subsets which is slightly more flexible than basic definability. To motivate this, consider the structure on, say, $\mathbb{R}$ generated by the graphs of addition and multiplication in $\mathbb{R}^3$. In this structure, sets like

$$P_0 := \{(x, y) \in \mathbb{R}^2 \,:\, y = x^2\}\,,\; P_1 := \{(x, y) \in \mathbb{R}^2 \,:\, y = x^2 + 1\}$$

are all basic definable (in $\mathbb{R}^2$), but the set

$$P_\pi := \{(x, y) \in \mathbb{R}^2 \,:\, y = x^2 + \pi\}$$

is not. In fact, even the singleton set $\{\pi\} \subseteq \mathbb{R}$ is not basic definable. We will use a slightly more general notion of definability, ("definability with parameters" in the language of model theory) which also encompasses sets like P_π which can be described in terms of basic definable sets and individual elements of the structure.

For this, we fix the following notation. If $V \subseteq M^{m+n} = M^m \times M^n$ is a subset and $y \in M^m$ is a point, we write $V_y \subseteq M^n$ for the fibre of V over y (viewed as a subset of M^n by projection onto the second factor).

Definition 1.6. A subset $V \subseteq M^n$ is said to be *definable* just when there exists some $m \in \mathbb{N}$, a basic definable subset $W \subseteq M^{m+n}$ and a point $y \in M^m$ such that $V = W_y$.

Lemma 1.7. *Every basic definable subset of M^n is definable. The collection of definable subsets is closed under finite intersections, unions, complements, permuation of coordinates, products and projections.*

Proof. We prove closure under pairwise intersections; the rest is straightforward. Let $V_1, V_2 \subseteq M^n$ be definable subsets. This means that there are basic definable subsets $W_i \subseteq M^{m_i+n}$ and points $y_i \in M^{m_i}$ such that $V_i = W_{i,y_i}$ for $i = 1, 2$. Define subsets $W_1', W_2' \subseteq M^{m_1+m_2+n} = M^{m_1} \times M^{m_2} \times M^n$ by

$$W_1' = \{(z_1, z_2, x) \in M^{m_1} \times M^{m_2} \times M^n \ : \ (z_1, x) \in W_1\},$$
$$W_2' = \{(z_1, z_2, x) \in M^{m_1} \times M^{m_2} \times M^n \ : \ (z_2, x) \in W_1\}.$$

These subsets are both basic definable: for W_2' this is because $W_2' = M^{m_1} \times W_2$; for W_1' this is because it is obtained from $M^{m_2} \times W_1$ by permutation of the coordinates on $M^{m_1+m_2+n}$. So $W = W_1' \cap W_2' \subseteq M^{m_1+m_2+n}$ is basic definable, and by inspection we see that $W_{(y_1,y_2)} = W_{1,y_1} \cap W_{2,y_2} = V_1 \cap V_2$. So we've shown that $V_1 \cap V_2$ is definable. □

Remark 1.8. If we write $D_n' \subseteq \mathbb{P}(M^n)$ for the collection of definable subsets of M^n, then the above lemma says that $M' = (M, (D_n')_{n \in \mathbb{N}})$ is again a structure. In fact, M' is the structure generated by the basic definable sets in M and the singleton subsets $\{x\} \subseteq M$. In M', the basic definable and definable subsets coincide. So for many purposes one doesn't need to distinguish between basic definable and definable subsets.

From now on, we fix a structure M, and adopt the convenient shorthand that "V is a (basic) definable set" means "V is a (basic) definable subset of M^n for some n". We also adopt the obvious conventions: "$U \subseteq V$ is (basic) definable" means that U is a (basic) definable subset of M^n, and the product $V_1 \times V_2$ of two (basic) definable sets $V_1 \subseteq M^{n_1}$ and $V_2 \subseteq M^{n_2}$ means the product $V_1 \times V_2 \subseteq M^{n_1+n_2}$.

How does one show in practice that a subset is (basic) definable? Let's illustrate this with an example. Suppose that we start with (basic) definable subsets $W \subseteq M^2$ and $U \subseteq M$. Then I claim that the set

$$V := \{v \in M \ : \ v \notin U \text{ and } (\exists w \in M)((v, w) \in W)\}$$

is also (basic) definable. Let's unpack this carefully. First of all, the set

$$V_1 := \{v \in M \ : \ v \notin U\}$$

is (basic) definable: it is the complement of U. Similarly, the set

$$V_2 := \{v \in M \ : \ (\exists w \in M)((v, w) \in W)\}$$

is (basic) definable: it is the image of W under the first projection $\pi \colon R^2 \to R$. So $V = V_1 \cap V_2$ is also (basic) definable.

One can concoct even more convoluted examples of (basic) definable sets using this idea. Informally speaking, if the members of a subset $V \subseteq M^n$ can be described in terms of some initial (basic) definable sets, forming pairs, taking components, the symbol "=" and the words "and", "or", "not" and "there exists", then V is also (basic) definable. Such a description of V describes a way to construct V from the initial (basic) definable sets and diagonals via products, intersections, unions, complements and projections. We won't formulate this principle in a precise way, but it will be appealed to several times later to show that certain subsets are (basic) definable. The reader who wants to check they understand this process should try the following example.

Exercise. Suppose that U and V are (basic) definable sets and $U_1 \subseteq U$ and $W_1, W_2 \subseteq U \times V$ are (basic) definable subsets, then the set

$$U' = \big\{(u_1, u_2) \in U \times U \ : \ u_1 \in U_1 \text{ or } (\exists v \in V)((u_1, v) \in W_1 \text{ and } (u_2, v) \notin W_2)\big\}$$

is a (basic) definable subset of $U \times U$.

As an illustration of this perspective, here is a handy lemma.

Lemma 1.9. *Let U and V be (basic) definable sets and $W \subseteq U \times V$ a (basic) definable subset. Then the set*

$$\big\{u \in U \ : \ (\forall v \in V)((u, v) \in W)\big\}$$

is a (basic) definable subset of U.

Proof. $(\forall v \in V)((u, v) \in W)$ is equivalent to $(\nexists v \in V)((u, v) \notin W)$. □

2. Definable functions

We continue to fix a structure M.

Definition 2.1. Let U and V be (basic) definable sets. A function $f\colon U \to V$ is called *(basic) definable* just when its graph $\Gamma_f \subseteq U \times V$ is (basic) definable.

We record here a few basic facts about the behaviour of definable functions. The proofs are mostly left to the reader.

Lemma 2.2 ("Definable sets and functions form a category")**.**

(1) *For any (basic) definable set V, the identity function $1_V : V \to V$ is (basic) definable.*
(2) *If $f\colon U \to V$ and $g\colon V \to W$ are (basic) definable functions, then the composite $g \circ f\colon U \to W$ is (basic) definable.*

Proof. The statement regarding identities is easy. For compositions, consider the set

$$\{(u, v, w) \in U \times V \times W : v = f(u) \text{ and } w = g(v)\} = (\Gamma_f \times W) \cap (U \times \Gamma_g) \subseteq U \times V \times W\,.$$

This is a (basic) definable subset of $U \times V \times W$, and its projection to $U \times W$ is the graph $\Gamma_{g\circ f}$ of $g \circ f$. So $g \circ f$ is (basic) definable. □

Lemma 2.3 ("Images and preimages of definable subsets under definable functions are definable")**.** *If $f\colon U \to V$ is a (basic) definable function and $U_1 \subseteq U$ is (basic) definable, then the image $f(U_1) \subseteq V$ is also (basic) definable. Similarly, if $V_1 \subseteq V$ is (basic) definable, then the preimage $f^{-1}(V_1) \subseteq U$ is also (basic) definable.*

Lemma 2.4 ("Definability is componentwise")**.** *If U, V_1 and V_2 are (basic) definable sets, then a function $f = (f_1, f_2)\colon U \to V_1 \times V_2$ is (basic) definable if and only if the two components f_1 and f_2 of f are (basic) definable.*

Lemma 2.5 ("Gluing definable functions")**.** *If U is a (basic) definable set covered by two (basic) definable subsets U_1, U_2, then a function $f\colon U \to V$ is (basic) definable if and only if $f|_{U_1}$ and $f|_{U_2}$ are (basic) definable.*

Lemma 2.6 ("Definable bijections are definable isomorphisms")**.** *Let $f\colon U \xrightarrow{\sim} V$ be a (basic) definable function which is bijective. Then $f^{-1}\colon V \to U$ is also (basic) definable.*

Proof. Let $\tau\colon U \times V \xrightarrow{\sim} V \times U$ be the isomorphism given by interchanging the two factors. Then τ is (basic) definable: its graph in $U \times V \times V \times U$ is, up to permutation of coordinates, the diagonal in $(U \times V) \times (U \times V)$. The graph of f^{-1} is the subset $\tau(\Gamma_f) \subseteq V \times U$. This subset is (basic) definable by Lemma 2.3, so f^{-1} is a (basic) definable function. □

Lemma 2.7. *Let $f\colon U \to V$ be a definable function and $k \in \mathbb{N}$. Then the set*

$$V_k := \{y \in V \;:\; \#U_y = k\}$$

of points in V with exactly k preimages in U is definable.

Proof. For simplicity, we just prove the case $k = 1$. Note that $y \in V_1$ if and only if y satisfies

$$(\exists u \in U)(v = f(u)) \text{ and } (\nexists u_1, u_2 \in U)(u_1 \neq u_2 \text{ and } v = f(u_1) = f(u_2))\,.$$

So $V_1 \subseteq V$ definable (note that $v = f(u)$ is equivalent to $(u, v) \in \Gamma_f$). □

3. O-minimal structures

Now we come to the main definition in these notes. These are *o-minimal structures*, which are structures whose underlying sets are ordered fields[2], such that the structure obeys a certain compatibility property with respect to the ordering.

Definition 3.1. Let R be an ordered field (e.g. the real numbers $\mathbb{R}$). A structure on R is said to be compatible with the ordered field operations if:

(1) the addition and multiplication functions $R^2 \to R$ are basic definable; and
(2) the set
$$[<]_R := \{(x, y) \in R^2 \;:\; x < y\} \subseteq R^2$$
is basic definable.

Lemma 3.2.

- *The singleton sets $\{0\}, \{1\}$ are basic definable (in R).*
- *The negation map $[-1]\colon R \to R$ is basic definable.*
- *The multiplicative inverse map $(-)^{-1}\colon R\setminus\{0\} \to R\setminus\{0\}$ is basic definable.*
- *Every polynomial map $R^n \to R^m$ is definable.*
- *For every $a < b \in R \cup \{\pm\infty\}$ the set*
$$]a, b[:= \{x \in R \;:\; a < x < b\} \subseteq R$$
is definable.

Proof. Easy. □

[2]An ordered field is a field R equipped with a total ordering on its underlying set such that the maps $x \mapsto x + a$ and $x \mapsto bx$ are order-preserving for all $a, b \in R$ with $b > 0$.

Sets of the form $]a,b[$ are called *open intervals* (in R). We avoid the more usual notation (a,b) to avoid confusion with the ordered pair $(a,b) \in R^2$.

In what follows, it will be convenient to be able to discuss e.g. definable subsets of $\bar{R} := R \cup \{\pm\infty\}$. In order to do this rigorously, we identify $\bar{R}$ with the basic definable subset

$$R \times \{0\} \cup \{(0,\pm 1)\} \subseteq R^2$$

in the obvious way. Note that the total ordering $<$ on $\bar{R}$ is basic definable, meaning that the subset

$$[<]_{\bar{R}} := \big\{(x,y) \in \bar{R}^2 \,:\, x < y\big\} \subseteq \bar{R}^2$$

is basic definable.

Definition 3.3. The structure on R is said to be *o-minimal* just when every definable subset of R is a finite union of points and open intervals.

Example 3.4. The following structures on $\mathbb{R}$ are all o-minimal (these are all hard theorems):

- the structure $\mathbb{R}_{\mathrm{alg}}$ generated by (the graphs of) $+, \times, <$ and the singletons in $\mathbb{R}$;
- the structure $\mathbb{R}_{\mathrm{exp}}$ generated by $+, \times, <$, the singletons in $\mathbb{R}$ and the graph of the real exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$;
- the structure $\mathbb{R}_{\mathrm{an}}$ generated by $+, \times, <$, the singletons in $\mathbb{R}$ and the graph of every function $[0,1]^n \to \mathbb{R}^m$ which is the restriction of a real-analytic function on a neighbourhood of $[0,1]^m$; and
- the structure $\mathbb{R}_{\mathrm{an,exp}}$ generated by $+, \times, <$, the singletons in $\mathbb{R}$, the graph of the real exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$, and the graph of every function $[0,1]^n \to \mathbb{R}^m$ which is the restriction of a real-analytic function on a neighbourhood of $[0,1]^m$.

From now on, we fix an o-minimal structure R. We note for later use that suprema and infima in o-minimal structures are well-behaved.

Lemma 3.5.

(1) *Any definable subset $V \subseteq \bar{R}$ has a supremum $\sup(V) \in \bar{R}$ (and also an infimum $\inf(V) \in \bar{R}$).*
(2) *If V is a definable set and $U \subseteq V \times \bar{R}$ is a definable subset, then the function $f\colon V \to \bar{R}$ given by $f(y) = \sup(U_y)$ is definable.*

Proof. The first part follows directly from the o-minimal criterion. For the second, we note that the set

$$U^+ = \{(y,c) \in V \times \bar{R} \,:\, c \text{ is an upper bound on } U_y\}$$

is definable, since it can be written as

$$U^+ = \{(y,c) \in V \times \bar{R} \,:\, (\nexists c' \in \bar{R})(c' > c \text{ and } (y,c') \in U)\}\,.$$

of fibrewise upper bounds on U is definable. Hence the set

$$U' = \{(y,c) \in V \times \bar{R} \,:\, c \text{ is a least upper bound on } U_y\}$$

is also definable. But U' is the graph of the function $f\colon y \mapsto \sup(U_y)$, so f is definable as desired. □

The first non-trivial result in the theory of o-minimality is the theorem of definable choice.

Proposition 3.6 (Definable choice)**.** *Let $f\colon U \twoheadrightarrow V$ be a definable surjection. Then f has a definable splitting.*

The proof requires a lemma, describing how definable subsets of R can vary in families.

Lemma 3.7. *Let V be a definable set, and let $U \subseteq V \times R$ be a definable subset.*

(1) *Let $V_1 \subseteq V$ denote the set of points $y \in V$ such that the fibre $U_y \subseteq R$ has a least element. For $y \in V_1$ write $c(y)$ for the least element of U_y. Then $V_1 \subseteq V$ is a definable subset, and $c\colon V_1 \to R$ is a definable function.*
(2) *Let $V_2 \subseteq V$ denote the set of points $y \in V$ such that the fibre $U_y \subseteq R$ contains an open interval. For $y \in V_2$ write $]a(y), b(y)[$ for the first maximal open interval in U_y. Then $V_2 \subseteq V$ is a definable subset, and $a, b\colon V_2 \to \bar{R}$ are definable functions.*

Proof. We prove the first part in slightly greater generality, allowing $U \subseteq V \times \bar{R}$. Consider the subset $W_1 \subseteq V \times \bar{R}$ defined by

$$W_1 = \left\{(y, c) \in V_1 \times \bar{R} \ :\ c \text{ is the minimal element of } U_y\right\}.$$

This subset is definable, since it consists of all pairs (y, c) satisfying

$$(y, c) \in U \text{ and } (\nexists c' \in \bar{R})(c' < c \text{ and } (y, c') \in U).$$

Now V_1 is the image of the projection $W_1 \to V$, so is a definable subset of V. Moreover, W_1 is the graph of $c\colon V_1 \to \bar{R}$, so c is a definable function.

For the second part, consider the subset $W \subseteq V \times \bar{R}^2$ by

$$W_2 = \left\{(y, a, b) \in V \times \bar{R}^2 \ :\ a < b \text{ and }]a, b[\subseteq U_y\right\}.$$

This subset is definable, since it consists of all triples (y, a, b) satisfying

$$a < b \text{ and } (\nexists c \in \bar{R})(a < c < b \text{ and } (y, c) \notin U).$$

Then $V_2 \subseteq V$ is the image of the projection $W_2 \to V$, so is definable.

Now let $W_2' \subseteq V \times \bar{R}$ be the image of W_2 under the projection $(y, a, b) \mapsto (y, a)$, i.e. W_2' is the set of pairs $(y, a) \in V \times \bar{R}$ such that U_y contains an open interval of the form $]a, b[$ for some $b \in \bar{R}$. It follows from this description (and the definition of o-minimality) that the fibres of the projection $W_2' \to V_2$ all have least elements, so the function $a\colon V_2 \to \bar{R}$ sending y to the least element of $W'_{2,y}$ is definable by the first part. Similarly, the fibres of the projection $W_2 \to W_2'$ all have greatest elements, so the function $b'\colon W_2' \to \bar{R}$ sending w to the least element of $W_{2,w}$ is also definable. It follows by definition that $]a(y), b(y)[$ is the first maximal open interval in U_y for all $y \in V_2$, where $b(y) := b'(y, a(y))$, and we are done. □

Proof of Proposition 3.6. Replacing U by the graph of f, it suffices to prove the result when $U \subseteq V \times R^n$ and $f\colon U \to V$ is the second projection. Proceeding inductively, if suffices moreover to deal with the case $n = 1$, where we will describe a splitting explicitly.

Let us write $V_2 \subseteq V$ for the set of points $y \in V$ such that U_y contains an interval, and $V_1 = V \setminus V_2$ for the complement. For $y \in V_1$, the fibre U_y is a finite union of points, so has a least element. We let $a, b\colon V_2 \to \bar{R}$ and $c\colon V_1 \to R$ be as in

Lemma 3.7. We define a function $s\colon V \to R$ by

$$s(y) = \begin{cases} c(y) & \text{if } y \in V_1, \\ \frac{1}{2}(a(y)+b(y)) & \text{if } y \in V_2,\ a(y) > -\infty,\ b(y) < \infty, \\ b(y)-1 & \text{if } y \in V_2,\ a(y) = -\infty,\ b(y) < \infty, \\ a(y)+1 & \text{if } y \in V_2,\ a(y) > -\infty,\ b(y) = \infty, \\ 0 & \text{if } y \in V_2,\ a(y) = -\infty,\ b(y) = \infty. \end{cases}$$

Lemma 3.7 implies that s is definable, and that $s(y) \in U_y$ for every $y \in V$. Hence the map $y \mapsto (y, s(y))$ is a definable splitting of f. □

4. Topology and calculus in an o-minimal structure

4.1. **Topology.** See [4, §3.2]. We continue to fix an o-minimal structure R. We topologise R by giving it the coarsest topology for which the open intervals are open, topologise R^n with the product topology, and endow any definable subset $V \subseteq R^n$ with the subspace topology. It turns out that definability interacts nicely with this topology, as follows.

Lemma 4.1 (See [4, Lemma 3.8])**.**

(1) *Let V be a definable set and $U \subseteq V$ be definable. Then the interior* $\mathrm{int}(U)$*, closure* $\mathrm{cl}(U)$ *and boundary* $\partial(U) = \mathrm{cl}(U) \setminus \mathrm{int}(U)$ *are all definable.*
(2) *Let $f\colon U \to V$ be a definable function. Then the set of points $x \in U$ such that f is continuous*[3] *at x is definable.*

Definable continuous functions turn out to behave particularly well, and many theorems for calculus over $\mathbb{R}$ also hold in the general o-minimal structure R if one sprinkles the statements with the word "definable" everywhere. To show this, it is helpful to introduce a notion of *definable connectedness*, which agrees with connectedness in the case $R = \mathbb{R}$ but is stricter in general.

Definition 4.2. A non-empty definable set V is said to be *definably connected* just when V is not the disjoint union of two non-empty definable open subsets.

Example 4.3. In R, the definably connected subsets are the points and the intervals: open, half-open or closed. (This is an immediate consequence of the definition of o-minimality.)

Lemma 4.4. *If $f\colon V \to U$ is a definable continuous function and V is definably connected, then the image of f is definably connected.*

Proof. The usual proof works. □

Let us give some consequences.

Proposition 4.5 (Definable intermediate value theorem)**.** *Let $f\colon [a,b] \to R$ be a definable continuous function such that $f(a) < 0 < f(b)$. Then there exists $c \in]a,b[$ such that $f(c) = 0$.*

[3]A function $f\colon U \to V$ between topological spaces is said to be *continuous at* $x \in U$ just when, for any open neighbourhood V_0 of $f(x)$ in V there is an open neighbourhood U_0 of x in U such that $U_0 \subseteq f^{-1}V_0$.

Proof. By Lemma 4.4, the image $f([a,b])$ is a definably connected subset of R, so is either a point or an interval. Since by assumption it contains both a positive and a negative element, it must contain 0. □

Remark 4.6. One consequence of the definable intermediate value theorem is that any o-minimal structure R is always a real-closed field, i.e. it is an ordered field in which any odd-degree polynomial has a root, and any positive element has a square root.

Proposition 4.7. *Let $f\colon [a,b] \to R$ be a definable continuous function from a closed interval to R. Then f attains a maximum (and a minimum).*

Proof. Since the image of f is definably connected, the closure of the image of f inside $\bar{R}$ (for the order topology) is some closed interval $[c,d]$ for $c \leq d \in \bar{R}$. We want to show that the image of f contains d. For any $s \in]c,d[$, the set $f^{-1}[s,d]$ is non-empty, closed and definable in $[a,b]$, so has a minimum $g(s)$. The function $g\colon]c,d[\to [a,b]$ is definable, and is weakly increasing by definition. Let $x \in [a,b]$ be the supremum of the image of g. We have that $g(s) \to x$ as $s \to d$ (in the order topology). Continuity of f implies that $fg(s) \to f(x)$ as $s \to d$. But $fg(s) \geq s$ by definition, so $f(x) \geq d$. Thus $f(x) = d$ and we are done. □

4.2. **Calculus.** One can even go further and make sense of differentiable functions in the o-minimal structure R. To state this, it is convenient to introduce the functions

$$|\cdot|\colon R^n \to R$$

defined as follows. For $n = 1$ we have

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0, \end{cases}$$

and for $n \geq 1$ we define

$$|x| = \max_i \{|x_i|\}$$

for $x = (x_1, \dots, x_n) \in R^n$.

Definition 4.8. Let $U \subseteq R^n$ be an open subset and $f\colon U \to R^m$ a function. We say that f is *differentiable at* $x = (x_1, \dots, x_n) \in U$ with partial derivatives $y = (y_1, \dots, y_n) \in R^{nm}$ just when, for every $\epsilon > 0$ in R there exists $\delta > 0$ in R such that we have

$$\big|f(x') - f(x) - \sum_{i=1}^{n} y_n(x'_n - x_n)\big| \leq \epsilon|x' - x|$$

for all $x' = (x'_1, \dots, x'_n) \in U$ with $|x' - x| < \delta$.

We say that f is *continuously differentiable* at x if it is differentiable on an open neighbourhood of x and the partial derivatives of f are continuous. More generally, we say that f is *k-times continuously differentiable at x* just when it is differentiable on an open neighbourhood of x and the partial derivatives of f are $k-1$-times continuously differentiable.

Lemma 4.9. *Let $U \subseteq R^n$ be a definable open subset and $f\colon U \to R^m$ be a definable function. Fix $k \in \mathbb{N}$. Then the set $U(k)$ of points $x \in U$ at which f is k times continuously differentiable is a definable subset of U, and the k-fold partial derivatives of f are all definable functions $U^{(k)} \to R^m$.*

Proof. We will just prove the case $k = 1$, the general case following by an easy induction. It is clear from Definition 4.8 that the set

$$W = \{(x, y) \in U \times R^{nm} \ : \ f \text{ is differentiable at } x \text{ with partial derivatives } y\}$$

is a definable subset of $U \times R^{nm}$. The image of the projection $W \to U$ is the set U' of points at which f is differentiable, so this set is definable. Moreover, W is the graph of the partial derivatives of f, so the partial derivatives of f are all definable functions on U'. The set $U^{(1)}$ is just the interior (in U) of the subset of U' on which all of these partial derivatives are continuous, so $U^{(1)}$ is also definable (and the partial derivatives are definable on it). □

Proposition 4.10 (Rolle's Theorem)**.** *Let $f \colon [a, b] \to R$ be definable and continuous, and also differentiable on $]a, b[$. Then there exists a point $c \in]a, b[$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. It suffices to prove the result when $f(a) = f(b) = 0$, in which case we can take c to be a point where f attains a maximum or minimum. □

4.3. **The monotonicity Theorem.** One remarkable fact about o-minimal structures is that a general definable function is always piecewise continuous in a certain sense. We will give a basic statement of this below. For this, we say that a function f from an interval I to R is *strictly monotone* if f is either strictly increasing, strictly decreasing or constant on I.

Proposition 4.11 (Monotonicity Theorem)**.** *Let $I =]a, b[$ be an interval and $f \colon I \to R$ a definable function. Then there is a finite sequence*

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

such that $f|_{]a_i, a_{i+1}[}$ is continuous and strictly monotone for $0 \leq i < n$.

Before we prove the Monotonicity Theorem, let us unpack a few consequences.

Corollary 4.12. *Let $f \colon]a, b[\to R$ be a definable function. Then for any $x \in [a, b[$ (resp. $y \in]a, b]$), the limit $\lim_{z \to x^+} f(z)$ (resp. $\lim_{z \to y^-} f(z)$) exists in $\bar{R}$.*

Proof. In the first case, we know that f is strictly monotone on $]x, x'[$ for some x', in which case the result is obvious (e.g. if f is strictly decreasing, then $\lim_{z \to x^+} f(z) = \sup\{f(z) \ : \ z \in]x, x'[\}$). The second case is identical. □

Corollary 4.13. *Let $f \colon I \to R$ be a definable function on an interval I. Then for every $k \in \mathbb{N}$, there is a finite subset $S \subseteq I$ such that f is continuously k-times differentiable on $I \setminus S$.*

Proof. It suffices to prove that f is differentiable outside a finite set, the general case following by the monotonicity theorem and induction. It also suffices by the monotonicity theorem to restrict ourselves to the case that f is continuous. We know by the previous corollary that for every $x \in I$ the limits

$$f'_+(x) := \lim_{z \to x^+} \frac{f(z) - f(x)}{z - x} \quad \text{and} \quad f'_-(x) := \lim_{z \to x^-} \frac{f(z) - f(x)}{z - x}$$

exist in $\bar{R}$, and define definable functions $f'_\pm \colon I \to \bar{R}$. Moreover f is differentiable at x if and only if $f'_+(x) = f'_-(x) \in R$. We will show that this happens outside a finite set.

First, let us show that $f'_+(x) \neq +\infty$ outside a finite set. We know that the set

$$I_{++} := \{x \in I \ : \ f'_+(x) = +\infty\}$$

is definable, so is either finite or contains an interval. Suppose for contradiction that I_{++} contains an interval I', and consider for any $\lambda \in R$ the function $g\colon x \mapsto f(x) - \lambda x$. By the monotonicity theorem, I' can be broken up into subintervals on which g is strictly monotone. But since $g'_+(x) = +\infty > 0$ on I', it certainly cannot be constant or strictly decreasing on any subinterval of I'. So we deduce that g is strictly increasing on I'. This means that for any $x < y$ in I' we have

$$f(y) - f(x) > \lambda(y - x)\,.$$

But this holds for any λ, giving a contradiction. So $f'_+(x) \neq +\infty$ outside a finite set.

A similar argument shows that $f'_\pm(x) \neq \pm\infty$ outside a finite set for all values of $\pm$. It remains to prove that $f'_+(x) = f'_-(x)$ outside a finite set, for which we consider the set

$$I_{\neq} := \{x \in I \ : \ f'_+(x) \neq f'_-(x)\}\,.$$

Again, this set is definable, so it suffices to prove that it does not contain an interval. If it did contain an interval I', then by shrinking I' if necessary we could assume without loss of generality that f'_+ and f'_- were both continuous and not infinity on I'. Shrinking I' further, we could assume that either $f'_+(x) > f'_-(x)$ for all $x \in I'$, or $f'_+(x) < f'_-(x)$ for all $x \in I'$. We will deal with the former case, the latter following by a similar argument.

Shrinking I' a final time and using continuity of f'_+ and f'_- we may assume that there is some $\lambda \in R$ such that $f'_+(x) > \lambda > f'_-(x)$ for all $x \in I'$. If we set $g(x) = f(x) - \lambda x$, then by the monotonicity theorem we know that there is a subinterval of I' on which g is strictly monotone. But this is impossible: the fact that $g'_+(x) > 0$ means that g cannot be constant or strictly decreasing on any subinterval, while the fact that $g'_-(x) < 0$ means that g cannot be strictly increasing on any subinterval. Either way we obtain a contradiction.

Hence $f'_+(x) = f'_-(x)$ outside a finite set, and we have completed the proof. □

Now let us prove the Monotonicity Theorem. The proof revolves around the following two lemmas.

Lemma 4.14. *If $f\colon I \to R$ is definable, then there is a subinterval $I' \subseteq I$ on which f is strictly monotone.*

Lemma 4.15. *If $f\colon I \to R$ is definable and strictly monotone, then there is a subinterval $I' \subseteq I$ on which f is continuous.*

Assuming these lemmas for the time being, here is the proof.

Proof of the Monotonicity Theorem. We say that f is locally continuous and strictly monotone at $x \in I$ just when it is continuous and strictly monotone on a subinterval containing x. It follows by the above two lemmas that the set of points x at which f is locally continuous and strictly monotone is open and dense in I; since this set is also definable, it is cofinite. Hence, shrinking I if necessary, we may assume that f is continuous and everywhere locally strictly monotone. It remains to prove that f is actually strictly monotone.

For this, the set of points $a' < b'$ in $[a,b]$ such that $f|_{]a',b'[}$ is strictly monotone is definable in $[a,b]^2$. Hence there exists an interval $]a',b'[\subseteq I$ on which f is strictly monotone, maximal with this property. But if $a' > a$ then by the fact that f is strictly monotone on a neighbourhood of a' we see that f is strictly monotone on $]a'-\epsilon, b'[$ for some $\epsilon > 0$ in R, violating maximality of $]a',b'[$. So $a' = a$, and $b' = b$ similarly, and f is strictly monotone on I. □

Now we prove the two lemmas. The second is easy.

Proof of Lemma 4.15. We may suppose that f is strictly increasing, so injective. The image $f(I)$ of I is infinite and definable, so there is an interval $J \subset f(I)$. For any $c < d$ in J we have $f^{-1}(]c,d[) =]f^{-1}(c), f^{-1}(d)[$, which is open in I. Hence the restriction of f to the interval $I' = f^{-1}(J)$ is continuous. □

The first requires a preparatory step.

Lemma 4.16. *Let $f\colon I \to R$ be a definable function such that $f(x) > 0$ for all $x \in I$. Then there is a subinterval $I' \subseteq I$ and some $\epsilon > 0$ in R such that $f(x') \geq \epsilon$ for all $x' \in I'$.*

Proof. Consider the subset $V \subseteq I$ defined by

$$V = \{x \in I \ : \ f(y) < f(x) \text{ for all } y < x\}\,.$$

Since V is definable, it either contains an interval or is finite. If V contains an interval then f is strictly increasing on a subinterval of I and we are certainly done.

If instead V is finite, then by replacing I with a subinterval we may assume $V = \emptyset$. Then by definition we may choose an infinite decreasing sequence $x_0 > x_1 > \dots$ in I such that $f(x_0) < f(x_1) < \dots$. In particular, the set of points $x \in I$ such that $f(x) \geq \epsilon := f(x_0)$ is infinite and definable, so contains an interval I'. So we are done in this case too. □

Proof of Lemma 4.14. For a point $x \in I$, the set of points y such that $f(y) > f(x)$, resp. $f(y) = f(x)$, resp. $f(y) < f(x)$ is definable. Hence one of these three sets must contain an interval of the form $]x,x'[$ for some $x' > x$, i.e. one of three things occurs:

1. there is some $x' > x$ such that $f(y) > f(x)$ for all $y \in]x,x'[$;
2. there is some $x' > x$ such that $f(y) = f(x)$ for all $y \in]x,x'[$; or
3. there is some $x' > x$ such that $f(y) < f(x)$ for all $y \in]x,x'[$.

Since the set of points x for which each of these three things occurs is definable, one of them must contain an interval, so shrinking I if necessary, we may assume that only one of these occurs, say the first.

In this case we define a function $g\colon I \to R$ by

$$g(x) = \inf\{y > x \ : \ f(y) \leq f(x)\}\,.$$

Note that our assumption ensures that $g(x) > x$ for all $x \in I$. Since g is definable, Lemma 4.16 ensures that there is a subinterval $I' \subseteq I$ and some $\epsilon > 0$ in R such that $g(x) \geq x + \epsilon$ for all $x \in I'$. But if we shrink I' further to have length $< \epsilon$, then the inequality $g(x) \geq x + \epsilon$ tells us that f is strictly increasing on I' and we are done. □

5. The cell decomposition theorem

Continue to fix an o-minimal structure R. The definition of o-minimality tells us, by *fiat*, what the definable subsets of R are. It is natural to wonder what the definable subsets of R^n are for $n > 1$. Remarkably, it turns out that there is a good answer to this question, provided by the *cell decomposition theorem*. In order to state this theorem, we need to introduce some notation. If V is a definable set and $f, g\colon V \to R$ are two definable functions with $f < g$ pointwise, then we define

$$]f, g[:= \left\{(v, a) \in V \times R \,:\, f(v) < a < g(v)\right\},$$

which is a definable subset of $V \times R$. We define subsets $]f, \infty[$, $]-\infty, g[$ and $]-\infty, \infty[$ of $V \times R$ analogously.

Using this, we define the notion of a *cell* in R^n recursively as follows.

Definition 5.1 (Cells)**.** For $(i_1, \ldots, i_n) \in \{0, 1\}^n$, we define a $(i_1, \ldots, i_n)$-cell in R^n as follows:

- for $n = 0$, R^0 is the unique ()-cell in R^0;
- for $n > 0$, an $(i_1, \ldots, i_{n-1})$-cell $C \subseteq R^{n-1}$ and a continuous definable function $f\colon C \to R$, the graph of f inside R^n is an $(i_1, \ldots, i_{n-1}, 0)$-cell; and
- for $n > 0$, an $(i_1, \ldots, i_{n-1})$-cell $C \subseteq R^{n-1}$ and continuous definable functions $f, g\colon C \to R$ with $f < g$ pointwise, the sets $]f, g[$, $]f, \infty[$, $]-\infty, g[$ and $]-\infty, \infty[$ inside R^n are all $(i_1, \ldots, i_{n-1}, 1)$-cells.

We say that an $(i_1, \ldots, i_n)$-cell C is a *cell of dimension* $i_1 + \cdots + i_n$; we say that it is an *open cell* if it is a $(1, 1, \ldots, 1)$-cell.

Example 5.2. Inside $R = R^1$, the cells are exactly the singleton sets and the open intervals.

Remark 5.3. The cells in R^n are not invariant under permutations of the coordinate axes. For example, the set

$$\left\{(x, y) \in R^2 \,:\, y = x^3 - x\right\} \subseteq R^2$$

is a $(1, 0)$-cell, but

$$\left\{(x, y) \in R^2 \,:\, x = y^3 - y\right\} \subseteq R^2$$

is not a cell.

Theorem 5.4 (Cell decomposition theorem, basic version)**.** *Any definable subset $V \subseteq R^n$ can be written as a finite disjoint union of cells.*

For the proof, see [4, §4.4] and the preceding sections.

6. Dimension

See [4, §4.5].

7. Elementary extensions

Now we come to the only real model theory we will need to develop the theory of o-minimality: the notion of *elementary extensions*. Informally, an elementary extension of a structure M is a way of enlarging the ground set without changing the basic definable sets. Here is the precise definition.

Definition 7.1. Let M and M^* be structures, with $M \subseteq M^*$ as sets. We say that $M \subseteq M^*$ is an *elementary extension* just when:

- for any $n \in \mathbb{N}$ and basic definable subset $V \subseteq M^n$, there exists a unique basic definable subset $V^* \subseteq (M^*)^n$ such that $V = V^* \cap M^n$, and every basic definable subset of $(M^*)^n$ arises in this way; and
- the assignment $V \mapsto V^*$ is compatible with projections: if $V \subseteq M^{n+1}$ and $\pi\colon M^{n+1} \to M^n$ is a projection then $\pi(V^*) = (\pi(V))^*$.

We say that V^* is the *extension* of V.

Lemma 7.2. *Let $M \subseteq M^*$ be an elementary extension. Then the function $V \mapsto V^*$ on basic definable sets is compatible with finite unions, intersections, complements, permutation of coordinates and products.*

Proof. Easy. □

As well as extending basic definable subsets, one can also extend basic definable functions.

Lemma 7.3. *Let $f\colon U \to V$ be a basic definable function. Then there is a unique basic definable function $f^*\colon U^* \to V^*$ such that $f^*|_U = f$, and every basic definable function $U^* \to V^*$ arises in this way.*

Proof. Let $\Gamma_f^* \subseteq U^* \times V^*$ denote the extension of the graph of f. We claim that Γ_f^* is the graph of a function $U^* \to V^*$, which is then the desired f. In other words, the projection $\Gamma_f^* \to U^*$ is bijective.

First observe that, since the projection map $\Gamma_f \to U$ is surjective, so too is the projection $\Gamma_f^* \to U^*$ by compatibility of extensions with projections. For injectivity, let

$$\Gamma_f(2) = \Gamma_f \times_U \Gamma_f := \{(u, v_1, v_2) \in U \times V \times V : (u, v_1) \in \Gamma_f \text{ and } (u, v_2) \in \Gamma_f\}.$$

It follows from compatibility of extensions with products and intersections that

$$\Gamma_f(2)^* = \{(u^*, v_1^*, v_2^*) \in U^* \times V^* \times V^* : (u^*, v_1^*) \in \Gamma_f^* \text{ and } (u^*, v_2^*) \in \Gamma_f^*\}.$$

Since Γ_f is the graph of a function, it follows that $\Gamma_f(2) \subseteq U \times \Delta_V$ where $\Delta_V \subseteq V \times V$ is the diagonal. Hence $\Gamma_f(2)^* \subseteq U^* \times \Delta_{V^*}$, and so the projection $\Gamma_f \to U$ is also injective. This completes the proof. □

There is a well-known construction of elementary extensions using ultrafilters. An *ultrafilter* on an index set I is a family $\mathcal{U} \subseteq \mathbb{P}(I)$ of subsets of I with the following properties:

- if $I_1, I_2 \in \mathcal{U}$, then $I_1 \cap I_2 \in \mathcal{U}$;
- if $I_1 \subseteq I_2 \subseteq I$ and $I_1 \in \mathcal{U}$ then $I_2 \in \mathcal{U}$; and
- if $I_1 \subseteq I$, then exactly one of I_1 and $I \setminus I_1$ lies in $\mathcal{U}$.

You can think of an ultrafilter $\mathcal{U}$ as telling you which subsets of I are "big". The axioms then say that the intersection of two big subsets is big, all supersets of big sets are big, and the complement of a big set is small and vice versa.

Here is a simple example of an ultrafilter: for any element $i \in I$ we can define an ultrafilter $\mathcal{U}_i$ on I by setting $I_1 \in \mathcal{U}_i$ if and only if $i \in I_1$. Ultrafilters of this kind are called *principal*. There do exist other ultrafilters, but they cannot be described without the axiom of choice. The following proposition ensures an adequate supply of non-principal ultrafilters.

Proposition 7.4. *Let I be a set, and let $\mathcal{F}_0 \subseteq \mathbb{P}(I)$ be a family of subsets of I with the following property: the intersection of any finite number of elements of $\mathcal{F}_0$ is non-empty. Then there exists an ultrafilter $\mathcal{U}$ on I containing $\mathcal{F}_0$.*

Proof. We define a *filter* on I to be a non-empty family $\mathcal{F} \subseteq \mathbb{P}(I)$ of subsets of I with the following properties:

- if $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$; and
- if $I_1 \subseteq I_2 \subseteq I$ and $I_1 \in \mathcal{F}$ then $I_2 \in \mathcal{F}$.

A filter is called *proper* if $\mathcal{F} \neq \mathbb{P}(I)$, or equivalently $\emptyset \notin \mathcal{F}$. The proof of the theorem goes by the following steps, none of which are very difficult.

(1) if $\mathcal{F}_0 \subseteq \mathbb{P}(I)$ has the finite intersection property, then there exists a proper filter $\mathcal{F}$ containing $\mathcal{F}_0$;
(2) the ultrafilters on I are exactly the maximal proper filters with respect to inclusion;
(3) by Zorn's Lemma, every proper filter is contained in a maximal proper filter.

□

Definition 7.5. Let M be a structure. Let I be a set and $\mathcal{U}$ an ultrafilter on $\mathcal{U}$. Define the *ultrapower* of M by

$$M^* := M^I / \sim_{\mathcal{U}} ,$$

where $\sim_{\mathcal{U}}$ is the equivalence relation on M^I given by

$$(a_i)_{i\in I} \sim_{\mathcal{U}} (b_i)_{i\in I} \Leftrightarrow \{i \in I \,:\, a_i = b_i\} \in \mathcal{U} .$$

M is the subset of M^* consisting of the diagonal elements.

M^* can be made into a structure as follows. For each basic definable $V \subseteq M^n$, define a subset $V^* \subseteq (M^*)^n$ by

$$(a_i)_{i\in I} \in V^* \Leftrightarrow \{i \in I \,:\, a_i \in V\} \in \mathcal{U} .$$

(This is well-defined.) One checks using the axioms of ultrafilters that the collection of such sets V^* makes M^* into a structure which is an elementary extension of M.

Remark 7.6. In the case that $\mathcal{U}$ is a principal ultrafilter, we have $M^* = M$. So we only really interested in the case that $\mathcal{U}$ is non-principal.

Theorem 7.7 (Compactness Theorem)**.** *Fix a structure M. Let V be a definable set, and let $(V_i)_{i\in I}$ be a collection of definable subsets of V indexed by some set I. Suppose that for every elementary extension $M \subseteq M^*$ we have*

$$V^* = \bigcup_{i\in I} V_i^* .$$

Then there exists a finite subset $I_0 \subseteq I$ such that

$$V = \bigcup_{i\in I_0} V_i .$$

Proof. We prove the contrapositive. Let J denote the set of finite subsets $I_0 \subseteq I$, and for $i \in I$ let $J_i = \{I_0 \in J \,:\, I_0 \ni i\}$. Any finite intersection of the sets J_i is non-empty, so there exists an ultrafilter $\mathcal{U}$ on J such that $J_i \in \mathcal{U}$ for all i. Write M^* for the ultrapower of M with respect to this ultrafilter.

Now suppose for contradiction that $V \neq \bigcup_{i\in I_0} V_i$ for any finite subset $I_0 \subseteq I$. Choose in this case an element $v_{I_0} \in V \setminus \bigcup_{i\in I_0} V_i$ for each I_0, and let $v^* \in M^*$ be the element defined by the sequence $(v_{I_0})_{I_0\in J}$. We certainly have $v^* \in V^*$, since $v_{I_0} \in V$ for all $I_0 \in J$. But we also have $v^* \notin V_i^*$ for any i, since $v_{I_0} \notin V_i$ for any $I_0 \in J_i$. So we have

$$v^* \in V^* \setminus \bigcup_{i\in I} V_i^* \,,$$

contrary to our assumption. □

Remark 7.8. The discussion of ultrapowers above is just a bowdlerised version of the more general theory of *ultraproducts*, which one needs to adopt proper model-theoretic language to describe. Similarly, the compactness theorem as we've stated it is just a special case of a more general compactness theorem, which can also be proved using ultraproducts (but is more usually proved via first-order logic). See [5, Chapter 8.5] for a discussion of ultraproducts done properly.

8. Elementary extensions of o-minimal structures

We will be interested in elementary extensions of o-minimal structures R. Given such an elementary extension R^*, let $+^*, \times^* \colon (R^*)^2 \to R^*$ denote the extensions of the addition and multiplication functions, and let $<^*$ denote the binary relation on R^* given by

$$x^* <^* y^* \Leftrightarrow (x^*, y^*) \in ([<]_R)^* \,.$$

It is easy to check the following.

Lemma 8.1. $(R^*, +^*, \times^*, <^*)$ *is an ordered field. (We therefore usually write* $+$*,* $\times$ *and* $<$ *instead of* $+^*$*,* $\times^*$ *and* $<^*$*.)*

In fact, it is even true that R^* is an o-minimal structure, but this is more complicated and involves the cell decomposition theorem for o-minimal structures. All of the concepts discussed above behave well with respect to elementary extensions, as follows.

Theorem 8.2. *Let R be an o-minimal structure, and let $R \subseteq R^*$ be an elementary extension. Then R^* is again o-minimal.*

Proof. We prove the result under the simplifying assumption that all singletons in R are basic definable. Let $V^* \subseteq R^*$ be a definable subset; we want to prove that V^* is a finite union of points and open intervals. So there exists a basic definable subset $W^* \subseteq (R^*)^{n+1}$ and a point $y^* \in (R^*)^n$ for some n such that $V^* = W^*_{y^*}$. The set W^* is the extension of a basic definable subset $W \subseteq R^{n+1}$. By the cell decomposition theorem 5.4, W can be written as a finite disjoint union of cells C, so it suffices to prove that each $C^*_{y^*}$ is a finite union of points and open intervals. Consider the case that $C =]f, g[$ for two definable continuous functions $f < g \colon D \to R$ where $D \subseteq R^n$ is a cell. Then $C^* =]f^*, g^*[$, so we have $C^*_{y^*} =]f^*(y^*), g^*(y)[$ if $y^* \in D^*$, and $C^*_{y^*} = \emptyset$ otherwise. So $C^*_{y^*}$ is either empty or an open interval. The case that C is any other kind of open interval can be handled similarly. □

Lemma 8.3. *Let $R \subseteq R^*$ be an elementary extension of o-minimal structures.*

1. *Let V be a definable set and $U \subseteq V$ a definable subset. Then U is open in V if and only if U^* is open in V^*.*
2. *Let V be a definable set. Then* $\dim(V) = \dim(V^*)$.

(3) *Let* $f\colon U \to V$ *be a definable function. Then* f *is continuous if and only if* f^* *is continuous.*
(4) *Let* $f\colon U \to R^m$ *be a definable function where* $U \subseteq R^n$ *is a definable open subset. Then* f *is* k*-times continuously differentiable if and only if* f^* *is* k*-times continuously differentiable, and the* k*-fold partial derivatives of* f^* *are the extensions of the corresponding partial derivatives of* f*.*
(5) *Let* $f\colon U \to V$ *be a definable function. Then all fibres of* f *have dimension* k *if and only if all fibres of* f^* *have dimension* k*.*

Proof. We give a careful explanation of the first part; the others follow by variations on the same argument. Suppose that $V \subseteq R^m$, so the interior of U is the set

$$\mathrm{int}(U) = \big\{x \in U \ :\ (\exists \epsilon \in R)(\epsilon > 0 \text{ and } (\forall z \in V)(|x - z| \geq \epsilon \text{ or } z \in U))\big\}\,.$$

By compatibility of extensions with Boolean operations, products and projections we see that the extension of $\mathrm{int}(U)$ is given by

$$\mathrm{int}(U)^* = \big\{x^* \in U^* : (\exists \epsilon^* \in R^*)(\epsilon^* > 0 \text{ and } (\forall z^* \in V^*)(|x^* - z^*| \geq \epsilon^* \text{ or } z^* \in U^*))\big\}\,.$$

In other words, $\mathrm{int}(U)^* = \mathrm{int}(U^*)$. Hence $U = \mathrm{int}(U)$ (i.e. U is open in V) if and only if $U^* = \mathrm{int}(U^*)$ (i.e. U^* is open in V^*). □

Exercise. Using the compactness theorem, show that there exists an elementary extension $\mathbb{R}^* \supseteq \mathbb{R}$ containing an element $\epsilon > 0$ in $\mathbb{R}^*$ such that $\epsilon < 1/n$ for all positive integers $n > 0$.

References

[1] J. Tsimerman, *Ax–Schanuel and o-minimality*
http://www.math.toronto.edu/ jacobt/ASsketch.pdf
[2] B. Bakker and J. Tsimerman, *The Ax-Schanuel conjecture for variations of Hodge structures*
arXiv:1712.05088
[3] J. Pila and A. J. Wilkie, *The rational points of a definable set*
http://eprints.maths.manchester.ac.uk/942/1/The_rational_points.pdf
[4] A. Forey, *O-minimality and diophantine applications*
[5] W. Hodges, *A shorter model theory*