O-MINIMALITY AND AX–SCHANUEL

The topic of the seminar this semester will be the Ax–Schanuel Theorem, a result in functional transcendence theory which appears, for example, in the recent work of Dimitrov–Gao–Habegger on uniform Mordell that we studied last semester. There are many avatars of the Ax–Schanuel Theorem in varying contexts; for this seminar we will focus on the following simple case.

Theorem 1 (Ax–Schanuel)**.** *Let $f_1, \dots, f_n \in \mathbb{C}[[t_1, \dots, t_m]]$ be power series over $\mathbb{C}$ in m variables which are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$. Then the transcendence degree of the field extension*

$$\mathbb{C}(f_1, \dots, f_n, \exp(f_1), \dots, \exp(f_n))$$

over $\mathbb{C}$ is at least

$$n + \mathrm{rk}(J)$$

where J is the Jacobian matrix: $J_{jk} = \frac{\partial f_j}{\partial t_k}$.

The proof we will follow is the short proof by Tsimerman in [5]. The key technical input in Tsimerman's argument is the Counting Theorem of Pila–Wilkie [7], a result which is of independent but related interest in the theory of unlikely intersections. Informally speaking, the Pila–Wilkie Theorem says that "most" rational points in a "nice" subset $X \subseteq \mathbb{R}^n$ lie in semialgebraic subsets[1] of X of positive dimension. To make this precise, let us write $X^{\mathrm{alg}} \subseteq X$ for the union of all positive-dimensional semialgebraic subsets of X, and $X^{\mathrm{tr}} := X \setminus X^{\mathrm{alg}}$ for the remainder (the "transcendental part"). For $T > 0$ we write

$$X^{\mathrm{tr}}(\mathbb{Q}, T) = \left\{ x \in X^{\mathrm{tr}} \cap \mathbb{Q}^n \ : \ h(x) \le T \right\}$$

for the set of rational points in X^{tr} of naive height $\le T$. Then the Pila–Wilkie Theorem says the following.

Theorem 2 (Pila–Wilkie)**.** *Suppose that $X \subseteq \mathbb{R}^n$ is definable in an o-minimal expansion of $\mathbb{R}$. Then for any $\epsilon > 0$ there exists a constant c such that*

$$\# X^{\mathrm{tr}}(\mathbb{Q}, T) \le cT^{\epsilon}$$

for all $T > 0$.

Already the statement—and certainly the proof—of the Pila–Wilkie Theorem involves the theory of o-minimality, a powerful tool from model theory with recent applications to the study of period maps in complex Hodge theory. The most part of this seminar series will be devoted to developing enough of the theory of o-minimality to prove the Pila–Wilkie Counting Theorem.

Here, we will take a slightly unorthodox approach inspired by the syntax-free exposition of o-minimality in [4, Chapter 2]. We will use a minimal amount of model-theoretic language, avoiding talking about things like signatures, languages,

---

Date: Spring 2022.

[1] A *semialgebraic subset* of $\mathbb{R}^n$ is a subset cut out by polynomial inequalities, or a finite union of such. For example, the set $\{x, y \in \mathbb{R}^2 \ : \ x^2 \le y < x^3\}$ is semialgebraic in $\mathbb{R}^2$.

theories and interpretations; instead we will develop all the model theory we need phrased in terms of the basic calculus of definable sets and functions. As such, while we will largely follow the exposition of [2] and [3], we will also refer to the self-contained appendix of this programme in some of the more foundational talks. Hopefully by de-emphasising the syntactic aspects of model theory, the concepts we need will be more readily accessible to number theorists.

1. Functional Transcendence

Alex will give an introductory talk giving an overview of the course.

2. O-minimal structures

The aim of this talk is to introduce the basic language of definable sets and functions, which will be used continually through the rest of the talks. In this language, we then formulate the notion of an *o-minimal structure* and give some examples. The talk should follow [1, §1–3], and should include at least the following:

- [§1] Define a structure on a set M. Give some examples of structures. Discuss structures generated by some basic sets.
- [§1] Define definable subsets. State that definable subsets are closed under finite unions, intersections, complements, products, projections.
- [§2] Define definable functions. State their basic properties: images and preimages of definable sets by definable functions are definable; composites of definable functions are definable; a definable bijection has definable inverse. Prove one or two of these properties.
- [§3] Define an o-minimal structure on an ordered field[2] R. Give examples: $\mathbb{R}_{\mathrm{alg}}$; $\mathbb{R}_{\mathrm{exp}}$, $\mathbb{R}_{\mathrm{an}}$, $\mathbb{R}_{\mathrm{an,exp}}$.
- Give some examples and non-examples of definable sets/functions in o-minimal structures: $x \mapsto \sqrt{x}$ and $(x, y) \mapsto \max\{x, y\}$ are definable in $\mathbb{R}_{\mathrm{alg}}$; $x \mapsto \sin(x)$ is not definable in any o-minimal structure on $\mathbb{R}$.
- [§3] State and prove the theorem of definable choice.

Remark. This talk and the next form a related whole, and you may find it useful to work together with the following speaker. In particular, if the material above seems like too much for a single talk, you could consider moving the discussion of definable choice to the next talk.

3. Calculus in an o-minimal structure

In this talk we further develop the theory of o-minimal structures on a general ordered field R. The punchline here is that any o-minimal structure R looks a lot like $\mathbb{R}$: one can make sense of continuous and differentiable functions, and staples from calculus such as the intermediate value theorem hold (if one restricts attention to definable functions). The talk should follow [1, §4] ([2, §3.2] is also a good reference), and should include the following:

- Define the topology on definable sets induced from the order topology on R. Prove that interiors, closures and boundaries of definable subsets are definable [1, Lemma 4.1].

[2]Although we are only ultimately interested in o-minimal structures $\mathbb{R}$, in order to apply the compactness theorem at one point it will be necessary to work over a general ordered field R.

- State (and optionally prove) that for a definable function $f\colon U \to V$, the locus in U where f is continuous is definable.
- Define definably connected sets [1, Definition 4.2]. Prove that the image of a definably connected set under a definable continuous function is again definably connected [1, Lemma 4.4].
- State and prove the definable intermediate value theorem [1, Proposition 4.5]. State that every definable continuous function on a closed interval attains a maximum [1, Proposition 4.7].
- Define an r-fold differentiable function $U \to R^m$ where $U \subseteq R^n$ (you can just do the case $n = m = 1$ if it makes the definition easier to state). Prove/observe that if $f\colon U \to R^m$ is a definable function with $U \subseteq R^n$ definable, then the locus in U where f is r-fold differentiable is definable, and all of the r-fold partial derivatives of f are definable.
- State and prove Rolle's Theorem [1, Proposition 4.10].

Remark. One consequence of the intermediate value theorem is that any ordered field R admitting an o-minimal structure is automatically real-closed. That is, every odd-degree polynomial with coefficients in R has a root in R, and every positive element of R has a square root. We will not use this fact in the seminar, but could be worth mentioning here for context.

4. Cell decomposition

This talk proves the cell decomposition theorem following [2, §§4.1–4.4] or [3, Chapter 3], whichever you find best.

- State the monotonicity theorem [2, Theorem 4.1].
- Prove the uniform finiteness in R^2 following [2, Proposition 4.7, Corollary 4.8].
- Define cells [2, §4.3], their properties and definition of cell decompositions.
- State Theorem 4.12 and Proposition 4.17. Sketch the proof of Proposition 4.17.
- Sketch the proof of the cell decomposition theorem following [2, §4.4].
- If time permits, say few words about the $C^{(k)}$ version of the Cell Decomposition theorem [2, Theorem 4.36].

This section is quite long, and it may not be possible to treat this all within a single talk. On the other hand, the following talk is relatively short, so you could black-box e.g. the monotonicity theorem and ask the next speaker to cover the proof.

5. Dimension theory

In this talk introduces the notion of the dimension of a definable set, following [2, §4.4], and should cover the following topics:

- Definition of the dimension of a definable set [2, Def. 4,19].
- Statement and proof of the basic properties of dimension [2, Proposition 4.20].
- Using the trivialization theorem of definable maps [3, Chapter 9, Theorem 1.2] as a black box, prove that if $f\colon X \to Y$ is a definable function, then the set of points in Y whose fibre has dimension k is definable for all $k \in \mathbb{N}$.

6. Elementary extensions and the compactness theorem

This talk contains the only "true" model theory we need for the Pila–Wilkie Counting Theorem: the compactness theorem[3]. For us, the importance of the compactness theorem is that it makes certain results in the theory of o-minimality automatically uniform in families. For example, it is a theorem that (in an o-minimal structure) every definable set X of dimension 0 is finite. Compactness allows us to upgrade this immediately to the following relative result: for any definable function $f\colon X \to Y$ of fibre dimension 0 there is a positive integer N such that every fibre of f has size $\leq N$.

In order to apply the compactness theorem, we have to work not just in a single o-minimal structure, but also in so-called *elementary extensions*. This talk should follow [1, §§7–8], and should cover the following topics:

- Define elementary extensions. Discuss extending basic definable subsets and functions.
- Introduce the definition of ultrafilters. Describe how to construct elementary extensions via ultrapowers [1, Definition 7.5].
- State and prove the compactness theorem [1, Theorem 7.7].
- State and prove that elementary extensions of o-minimal structures are again o-minimal [1, Theorem 8.2].
- (Optional): re-proof of the uniform boundedness theorem [2, Proposition 4.7] using the compactness theorem[4].

Remark. The compactness theorem says something about coverings $(V_i)_{i\in I}$ indexed by any set I. You may find it simpler expositionally to just prove the special case that $I = \mathbb{N}$ and the sets $V_0 \subseteq V_1 \subseteq \dots$ are nested. (Though we will need the full version.)

7. O-minimal geometry

This talk is rather orthogonal to the rest of the seminar series: the one ingredient we will need is the o-minimal Chow Lemma. The talk should cover the following points:

- Recall briefly without proof the classical Chow lemma. Then, as a warm-up, prove the affine version of o-minimal Chow following [9, Theorem 1.3.1], specifically the second proof on page 8.
- For the general version, introduce the notion of a definable topological spaces [9, Section 2], see also [3, Chapter 10]. Introduce the definabilization functor [9, Definition 2.1.5] from algebraic varieties over $\mathbb{C}$ to definable topological spaces.

[3]If you've taken a course in first-order logic, you will have already met a compactness theorem there, which is a close cousin of the result we will use.

[4]The logic is circular here, of course: the uniform boundedness theorem is required to prove cell decomposition, which is required in turn to prove that elementary extensions of o-minimal structures are again o-minimal. So proving uniform boundedness this way is insufficient for developing the theory *ab initio*. But this does at least provide an example of how one can apply the compactness theorem in practice.

- Introduce definable analytic spaces: start with basic definable analytic spaces [9, Section 2.2] then definable analytic spaces [9, Section 2.3]. Introduce the analytification functor. You can skip the whole discussion around sheaves.
- State the general version of o-minimal Chow [9, Corollary 3.4.4] and prove it using a finite definable affine cover.

8. The Pila–Wilkie Counting Theorem

In this talk, we will state and prove the Pila–Wilkie Counting Theorem, following e.g. [7]. The focus of this talk is the arithmetic content of the proof, so we will treat the material in §§2–5 of [7] as a black box. The talk should contain the following:

- State the Pila–Wilkie Counting Theorem [7, Theorem 1.8], as well as the uniform version [7, Theorem 1.9].
- State the Main Lemma [7, Proposition 6.2].
- Prove uniform Pila–Wilkie Counting assuming the Main Lemma [7, §7].
- If time, indicate some of the proof of the Main Lemma assuming uniform reparametrisation [7, Corollary 5.2] as a black box.

Remarks.

(1) Pila and Wilkie give a third statement of their counting theorem [7, Theorem 1.10], which is stronger even than [7, Theorem 1.9]. It is up to you which version you prove: the proof in [7] is of the stronger result, but the argument is easily adapted to prove the weaker result, and this may be simpler expositionally.

(2) I personally think that the structure of the proof becomes clearer if you introduce an intermediate step, essentially a souped-up version of the Main Lemma. For this, let's define a *k-cylinder*[5] *of degree d* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ of the form
$$\bigcap_{\sigma \in S} \pi_\sigma^{-1} H_\sigma$$
where S is the set of $k+1$-element subsets of $\{1, 2, \ldots, n\}$ (empty if $k \geq n$), $\pi_\sigma \colon \mathbb{R}^n \to \mathbb{R}^{k+1}$ is the corresponding projection, and $H_\sigma \subseteq \mathbb{R}^{k+1}$ is a hypersurface of degree d. A k-cylinder is an algebraic variety of dimension at most k.

One then has the following generalisation of the Main Lemma (which is the special case $k = n - 1$):

Proposition (Main Lemma, II)**.** *Let $Y \subseteq \mathbb{R}^m$ be definable, and let $X \subseteq \mathbb{R}^n \times Y$. Suppose that the fibres of $X \to Y$ all have dimension $\leq k$. Then for every $\epsilon > 0$ there exists a positive integer d and a positive constant c such that*
$$X_y(\mathbb{Q}, T)$$
is contained in the union of at most cT^ϵ k-cylinders in $\mathbb{R}^n$ of degree $\leq d$ for all $y \in Y$.

[5] I made this terminology up.

The key observation used in the proof is that if $X \subseteq \mathbb{R}^n$ is definable of dimension at most k, then every component of its intersection with a k-cylinder Σ is either semialgebraic (so contained in X^{alg}) or of dimension $\leq k-1$. So in order to bound the rational points on X^{tr} of height $\leq T$ it suffices to bound the rational points on $(X \cap \Sigma)^{\mathrm{tr}}$ *uniformly* in Σ of a given degree, and this can be done by induction.

9. Reparametrisation of definable sets

This talk provides the missing ingredient from the proof of Pila–Wilkie in the existence of strong parametrisations of definable subsets of $]0,1[^n$. The talk should follow [2, §§5.2–5.3] or the original material in [7], and include the following:

- Define strong r-parametrisations and strong r-reparametrisations [2, Definition 5.6].
- Outline the inductive structure of the proof [2, Remark 5.16]. (We will not cover the proof in full detail in the seminar, but the next three points will indicate to the audience how some steps of the induction look like.)
- Prove that every definable map $F\colon]0,1[\to]0,1[$ admits a strong r-reparametrisation for any r [2, Proposition 5.18].
- Prove that every definable map $F\colon]0,1[\to]0,1[^2$ admits a strong r-reparametrisation for any r [2, Lemma 5.19].
- Prove that every open cell $X \subseteq]0,1[^2$ admits a strong r-reparametrisation for any r [2, Lemma 5.20].
- Prove the existence of *uniform* strong r-parametrisations of definable families of k-dimensional subsets of $]0,1[^n$ [2, Corollary 5.15] or see below. This contains real model-theoretic input in the form of the compactness theorem, so make sure to emphasise this in your presentation.

Remark. Because of our different approach to model theory in this seminar, the argument required to deduce uniform strong r-parametrisations in families looks a bit different to the exposition in [2, Corollary 5.15]. Here is one way to phrase the argument in our language. Consider a definable subset $X \subseteq]0,1[^n \times Y$ of fibre dimension $\leq k$ over Y. We may assume that X and Y are 0-definable. For every $N \in \mathbb{N}$, every 0-definable set Z and every N-tuple of 0-definable functions $\phi_1, \dots, \phi_N\colon]0,1[^k \times Z \times Y \to X$ over Y, let's write $Y_{Z,\phi_1,\dots,\phi_N} \subseteq Y$ for the set of points $y \in Y$ for which there exists a point $z \in Z$ such that the maps $\phi_{1,z,y}, \dots, \phi_{N,z,y}\colon]0,1[^k \to X_y$ are a strong r-parametrisation of X_y.

The existence of strong r-parametrisations tells us that the subsets $Y_{Z,\phi_1,\dots,\phi_N}$ cover Y. In fact, since we proved the existence of strong r-parametrisations for any o-minimal structure, if R^* is any elementary extension of R then the sets $Y^*_{Z^*,\phi_1^*,\dots,\phi_N^*} = (Y_{Z,\phi_1,\dots,\phi_N})^*$ cover Y^*. So by the compactness theorem, Y is in fact covered by a finite number of the sets $Y_{Z,\phi_1,\dots,\phi_N}$. This gives the existence of uniform strong r-parametrisation for $X \subseteq]0,1[^n \times Y$.

10. Proof of Ax–Schanuel

This talk finally proves the Ax–Schanuel Theorem following [5].

- Explain carefully the equivalence between the different formulations of Ax–Schanuel [5, Thm. 1.1, 1.2, 1.3]. State and prove Corollary 1.4.

- Prove [5, Theorem 1.3] following [5, §2]. Highlight the role of o-minimality in the proof (point counting and o-minimal Chow).

11. Ax–Schanuel for Shimura varieties

This talk introduces and sketches the proof of Ax–Schanuel for pure Shimura varieties following [16, Part I].

- Introduce Shimura varieties [15, §2.4] and their weakly special subvarieties [15, Theorem 3.5].
- Illustrate the above (and the below) with the case of $\mathcal{A}_g$, the moduli space of principally polarized complex abelian varieties following for example [14].
- State Ax–Schanuel [16, Theorem 1.1, Theorem 1.2] and follow [16, Part I] to sketch the proof. Introduce Siegel sets [14, Section 3.3] for $\mathcal{A}_g$ and discuss briefly Siegel sets for general Shimura varieties following [13, Section 3.1].
- State the definability of the uniformization map restricted to a Siegel set [14, Theorem 1.1] and [13, Theorem 1.9]. Don't give a proof but only ideas if time permits.
- Follow [16, §4] to give a careful proof of Ax–Schanuel for Shimura varieties.

12. Addendum: Application of Ax–Schanuel to Betti maps

In last semester's seminar on the work of Dimitrov–Gao–Habegger, we black-boxed one result on the non-degeneracy of Betti maps [10, Theorem 6.2]. In this final talk of this seminar, we make the connection back to last semester by explaining how this result is proved, following [11]. A crucial step in the proof uses a version of the Ax–Schanuel Theorem for universal abelian varieties, which is a relative of the Ax–Schanuel Theorem we have studied in this seminar. The talk should include the following:

- Recap the statement of [11, Theorem 1.2'] needed in the work of Dimitrov–Gao–Habegger, including any necessary background on the Betti map (in outline).
- State [11, Theorem 1.3], and use it to deduce [11, Theorem 1.2'].
- Introduce the degeneracy loci [11, Definition 1.6]. State their key properties: [11, Theorems 1.6 & 1.7].
- State (but do not prove) mixed Ax–Schanuel for the universal abelian variety [12, Theorem 1.1] [11, Theorem 5.5].
- Prove [11, Theorem 1.7] for $X \subseteq \mathfrak{A}_g$, following [11, §6], making sure to emphasise the use of Ax–Schanuel in the proof.
- If time permits, give the statement of Theorem [11, Theorem 1.1] (which is a consequence of [11, Theorems 1.7 & 1.8]), and discuss how to use this to prove [11, Theorem 1.3], following [11, §10].

References

[1] L.A. Betts, *Preliminary notes on o-minimality*
See the Dropbox folder for this seminar
[2] A. Forey, *O-minimality and diophantine applications*
See the Dropbox folder
[3] Lou van den Dries, *Tame topology and o-minimal structures*
London Mathematical Society Lecture Note Series, 1998
[4] G.O. Jones and A.J. Wilkie (eds), *O-Minimality and Diophantine Geometry*
Cambridge University Press, 2005

[5] J. Tsimerman, *Ax–Schanuel and o-minimality*
http://www.math.toronto.edu/ jacobt/ASsketch.pdf
[6] B. Bakker and J. Tsimerman, *The Ax-Schanuel conjecture for variations of Hodge structures*
arXiv:1712.05088
[7] J. Pila and A. J. Wilkie, *The rational points of a definable set*
http://eprints.maths.manchester.ac.uk/942/1/The_rational_points.pdf
[8] W. Hodges, *A shorter model theory*
[9] B. Bakker, *Hodge theory and o-minimality*
Notes from the Felix Klein lecture series in Bonn, May 2019, available at his webpage.
[10] V. Dimitrov, Z. Gao and P. Habegger, *Uniformity in Mordell–Lang for curves*
arXiv:2001.10276 (to appear in Ann. Math.)
[11] Z. Gao, *Generic rank of Betti map and unlikely intersections*
arXiv:1810.12929
[12] Z. Gao, *Mixed Ax–Schanuel for the universal abelian varieties and some applications*
arXiv:1806.01408
[13] B. Klingler, E. Ullmo, A. Yafaev *The hyperbolic Ax-Lindemann-Weirstrass conjecture*
Available in Ullmo's webpage.
[14] Y. Peterzil, S. Starchenko *Definability of restricted Theta functions and families of abelian varieties*
Duke Math. J. 162(4): 731-765 (Also available on ArXiv)
[15] B. Klingler *Bi-algebraic geometry and the André–Oort conjecture*
Available in his webpage.
[16] N. Mok, J. Pila, J. Tsimerman *Ax-Schanuel for Shimura varieties*
Annals of Mathematics 189 (2019), 945–978