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Sur certains aspects géométriques et arithmétiques des variétés de Shimura orthogonales

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Chapitre 1

Introduction

Cette thèse a pour objet l'étude de quelques propriétés arithmétiques et géométriques des variétés de Shimura orthogonales. Ces variétés apparaissent naturellement comme espaces de modules de structures de Hodge de type K3. Dans certains cas, elles paramètrent des objets géométriques tels que les surfaces K3 et leurs analogues en dimensions supérieures, les variétés hyperkählleriennes. Ce point de vue modulaire sera notre fil conducteur tout au long de ce mémoire.

Ainsi, dans la première partie, à paraître dans le journal de Crelle [105], on démontre un résultat d'équirépartition du lieu de Hodge dans les variations de structures de Hodge de type K3 au dessus d'une courbe complexe quasi-projective. Dans la deuxième partie, on étudie des analogues arithmétiques du résultat précédent. Un exemple d'énoncés qu'on obtient est le suivant : étant donnée une surface K3 définie sur un corps de nombres et ayant partout bonne réduction, alors sous certaine hypothèse d'approximation, il existe une spécialisation en laquelle le nombre de Picard géométrique croît strictement.

Dans les deux cas, nos méthodes utilisent de manière cruciale les riches structures géométrique, arithmétique et automorphe des variétés de Shimura orthogonales ainsi que la construction de Kuga-Satake qui permet de les relier à des espaces de modules de variétés abéliennes. Dans la troisième partie, on relie la question du saut du nombre de Picard dans les familles de surfaces K3 à celle de la construction de courbes rationnelles sur ces surfaces. Enfin, on explique le contenu de [106] qui étend un résultat de Bogomolov et Tschinkel. On montre notamment que toute surface K3 définie sur un corps algébriquement clos de caractéristique quelconque et admettant une fibration elliptique non-isotriviale contient une infinité de courbes rationnelles.

1.1 Structures de Hodge de type K3 et variétés de Shimura orthogonales

1.1.1 Structures de Hodge

Il est bien connu depuis les travaux fondateurs de Hodge [61] que la cohomologie de de Rham en degré n d'une variété complexe projective et lisse X admet une décomposition de la forme

$$H^n_{dR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X)$$

qui vérifie $H^{p,q}(X) = \overline{H^{q,p}(X)}$, la conjugaison étant donnée par la structure réelle de la cohomologie de de Rham.

Réinterprétant les travaux de Hodge et Griffths, Deligne a défini dans [37] la notion de *structure de Hodge réelle de poids* n *dans* $\mathbb{Z}$ comme la donnée d'un $\mathbb{R}$-espace vectoriel V de dimension finie et d'une décomposition sur $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ de la forme

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}, \quad \text{qui vérifie} \quad V^{p,q} = \overline{V^{q,p}},$$

la conjugaison étant prise par rapport à la structure réelle de V.

De manière équivalente, c'est la donnée d'un espace vectoriel réel V et d'une filtration décroissante, dite *filtration de Hodge* $(F^i V_{\mathbb{C}})_{i \in \mathbb{Z}}$ sur $V_{\mathbb{C}}$ qui vérifie

$$V_{\mathbb{C}} = F^p V_{\mathbb{C}} \oplus \overline{F^{n+1-p} V_{\mathbb{C}}} \quad \text{pour tout} \quad p.$$

La filtration de Hodge et la bigraduation se déterminent mutuellement en posant $F^i V_{\mathbb{C}} = \oplus_{p \geq i} V^{p,q}$ dans un sens et dans l'autre sens en posant $V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^{n-p} V_{\mathbb{C}}}$.

On peut également donner une troisième description d'une structure de Hodge réelle de poids n : notons $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ le *tore de Deligne*. C'est un groupe algébrique sur $\mathbb{R}$ et on a $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. La donnée d'une structure de Hodge de poids n est équivalente à la donnée d'un morphisme de groupes algébriques sur $\mathbb{R}$:

$$\rho : \mathbb{S} \to \mathrm{GL}(V)$$

tel que pour tout r dans $\mathbb{R}^\times$, $\rho(r)$ est la multiplication par r^{-n}. L'équivalence avec la première définition est obtenue en posant

$$V^{p,q} = \{x \in V_{\mathbb{C}}, \forall z \in \mathbb{C}^\times \rho(z).x = z^{-p}\overline{z}^{-q}\}.$$

Définition 1.1.1. Une structure de Hodge entière V de poids n consiste en

(i) un $\mathbb{Z}$-module libre de type fini $V_{\mathbb{Z}}$ (*le réseau entier*) ;

(ii) une structure de Hodge réelle de poids n sur $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

Un morphisme de structures de Hodge $f : V \to V'$ est un morphisme de $\mathbb{Z}$-modules $f : V_{\mathbb{Z}} \to V'_{\mathbb{Z}}$ tel que $f_{\mathbb{R}} : V_{\mathbb{R}} \to V'_{\mathbb{R}}$ soit équivariant par rapport à l'action du groupe $\mathbb{S}$. De manière équivalente, $f_{\mathbb{C}}$ est compatible à la filtration de Hodge.

Définition 1.1.2. Soit V une structure de Hodge entière de poids pair $n = 2k$. Le groupe des classes de Hodge de V est l'intersection $V_{\mathbb{Z}} \cap V^{k,k}$.

La catégorie des structures de Hodge de poids n est une catégorie abélienne. Si V et V' sont deux structures de Hodge de poids n et n', on peut former les structures de Hodge $V^\vee$, $V \otimes V'$, $\mathrm{Hom}(V, V')$ de poids $-n$, $n+n'$, $n'-n$ respectivement. Si V et V' ont le même poids, les morphismes de structures de Hodge sont alors exactement les classes de Hodge dans la structure de Hodge $\mathrm{Hom}(V, V')$ de poids 0.

Exemples 1.1.3. 1. On note $\mathbb{Z}(1)$ la structure de Hodge de Tate de poids -2, purement de bidegré $(-1,-1)$ et de $\mathbb{Z}$-module sous-jacent $2i\pi\mathbb{Z} \subset \mathbb{C}$. Pour n dans $\mathbb{Z}$, on définit $\mathbb{Z}(n)$ comme étant la $n^{\text{iéme}}$ puissance tensorielle de $\mathbb{Z}(1)$. C'est alors une structure de Hodge de poids $-2n$.

2. Si X est une variété complexe projective, ou plus généralement une variété kählerienne compacte, alors les groupes de cohomologie singulière $H^n(X,\mathbb{Z})$ sont munis d'une structure de Hodge de poids n telle que le bigradué (p,q) s'identifie à l'espace vectoriel des formes harmoniques de type (p,q).

Remarque 1.1.4. Plus généralement, si A est un sous-anneau de $\mathbb{R}$ tel que $A \otimes \mathbb{Q}$ soit un corps, une *A-structure de Hodge* est la donnée d'un A-module de type fini V muni d'une structure de Hodge sur $V \otimes_A \mathbb{R}$.

Définition 1.1.5. Une polarisation sur une structure de Hodge V de poids n est la donnée d'un morphisme de structures de Hodge

$$\psi : V \otimes V \to \mathbb{Z}(-n),$$

tel que la forme bilinéaire $(x,y) \mapsto (2i\pi)^n \psi_{\mathbb{R}}(x, \rho(i).y)$ soit symétrique et définie positive sur $V_{\mathbb{R}}$.

Introduisons maintenant les principales actrices de ce mémoire qui sont les surfaces K3 et leurs avatars en théorie de Hodge : les structures de Hodge de type K3. Pour plus de détails, on pourra consulter les trois premiers chapitres de [64].

Définition 1.1.6. Une surface K3 sur un corps k est un schéma projectif de dimension 2, géométriquement intègre, lisse et propre sur k tel que

$$\Omega^2_{X/k} \simeq \mathcal{O}_X \quad \text{et} \quad H^1(X, \mathcal{O}_X) = 0.$$

En particulier lorsque $k = \mathbb{C}$ dans la définition précédente, l'espace analytique associé à une surface K3 sur $\mathbb{C}$ est une surface complexe projective et le groupe de cohomologie singulière $H^2(X,\mathbb{Z})$ est muni d'une structure de Hodge de poids 2 de nombres de Hodge $h^{2,0} = h^{0,2} = 1$ et $h^{1,1} = 20$. La forme d'intersection de Poincaré munit $H^2(X,\mathbb{Z})$ d'une structure de réseau unimodulaire pair de signature $(3,19)$ (voir [64, Chap.1, Prop.3.5]), donc abstraitement isomorphe au *réseau K3*

$$\Lambda_{K_3} := U^3 \oplus E_8(-1).$$

La suite exacte longue en cohomologie associée à la suite exacte courte exponentielle fournit une application *classe de Chern*

$$c_1 : \operatorname{Pic}(X) \to H^2(X,\mathbb{Z})$$

compatible au produit d'intersection sur $\operatorname{Pic}(X)$. Cela fait de $\operatorname{Pic}(X)$ un sous-réseau primitif de $H^2(X,\mathbb{Z})$ et l'identifie au groupe des classes de Hodge par le théorème des classes $(1,1)$ de Lefschetz. On note $T(X)$ le réseau orthogonal à $\operatorname{Pic}(X)$, dit *réseau transcendant* de X. Il est muni d'une structure de Hodge de poids 2 telle que $T(X)^{2,0}$ est de dimension 1 sur $\mathbb{C}$ et c'est un réseau minimal pour cette propriété. De plus, la forme d'intersection de Poincaré en fait une structure de Hodge polarisée. Cela motive la définition suivante.

Définition 1.1.7. Une structure de Hodge entière V est dite de type K3 si V est de poids 2, polarisée et vérifie

$$\dim V_{\mathbb{C}}^{2,0} = 1, \quad \text{et} \quad V^{p,q} = 0 \quad \text{si} \quad |p-q| > 2.$$

En particulier, le réseau transcendant d'une surface K3 sur $\mathbb{C}$ est une structure de Hodge de type K3. Remarquons aussi que la signature de la polarisation sous-jacente à une structure de Hodge de type $K3$ est de la forme $(2, b)$ pour un certain $b \geq 0$.

Fixons dorénavant un réseau pair (L, Q) de forme bilinéaire associée définie par

$$(x.y) = Q(x + y) - Q(x) - Q(y),$$

pour tous $x, y \in L$. La forme Q est supposée non-dégénérée et de signature $(2, b)$. On voit alors que la donnée d'une structure de Hodge de type K3 sur (L, Q) est équivalente à la donnée d'une droite $L^{2,0} = \mathbb{C}w \subset L_{\mathbb{C}}$ telle que $(w.w) = 0$ et $(w.\overline{w}) > 0$. Dans ce cas, $V^{1,1} = (V^{2,0} \oplus V^{0,2})^{\perp}$. Autrement dit, c'est la donnée d'un point de l'espace topologique

$$D_L = \{\omega \in \mathbb{P}(V_{\mathbb{C}}),\, (\overline{\omega}.\omega) > 0, (\omega.\omega) = 0\},$$

dit *domaine des périodes*. Ses composantes connexe sont naturellement munies d'une structure complexe qui en fait des domaines symétriques hermitiens.

1.1.2 Variations de structure de Hodge

Nous voulons maintenant décrire comment varie la structure de Hodge dans les familles de surfaces K3. Griffiths fut le premier à s'intéresser à cette question dans le cadre général des familles projectives de variétés algébriques et s'en inspirera pour introduire dans [53] la notion plus vaste de *variation de structure de Hodge* que nous rappelons maintenant. Pour plus de détails à ce sujet, on réfère à [92, Part IV].

Définition 1.1.8. Soit S une variété complexe. Une *variation de structure de Hodge* entière de poids n sur S est la donnée d'un système local $\mathbb{V}_{\mathbb{Z}}$ de $\mathbb{Z}$-modules libres, d'une filtration décroissante $(F^l\mathcal{V}))_{0 \leq l \leq n}$ par des sous-fibrés holomorphes sur le fibré vectoriel holomorphe $\mathcal{V} = \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{O}_S$ muni de la connexion intégrable ∇ dont le système des sections plates est égal à $\mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ et qui vérifie la condition suivante, dite de *transversalité de Griffiths*

$$\nabla(F^p\mathcal{V}) \subset F^{p-1}\mathcal{V} \otimes \Omega^1_{S/\mathbb{C}}.$$

Il existe également une notion de variation de structure de Hodge polarisée que nous omettons de définir et pour laquelle nous renvoyons à [92, Def. 10.8]. Un exemple important est donné par les variations dites *d'origine géométrique* : étant donné un morphisme projectif et lisse $f : X \to S$ entre variétés complexes, alors pour $k \geq 0$, $R^k f_* \underline{\mathbb{Z}}_X$ est un système local sur S de fibre en un point s de S égale à la cohomologie singulière $H^k(X_s, \mathbb{Z})$. De plus, le groupe d'hypercohomologie du complexe de de Rham relatif $R^k f_* \Omega^{\bullet}_{X/S}$ s'identifie à $R^k f_* \underline{\mathbb{Z}}_X \otimes \mathcal{O}_S$. La filtration bête sur le complexe $\Omega^{\bullet}_{X/S}$ induit alors une filtration de Hodge sur $R^k f_* \Omega^{\bullet}_{X/S}$, voir [92, Section 10.4] pour plus de détails.

On fixe comme avant un réseau (L, Q), la forme Q étant toujours supposée non-dégénérée et de signature $(2, b)$ avec b plus grand que 3, et on se donne une courbe complexe S quasi-projective, connexe et $\{\mathbb{V}_{\mathbb{Z}}, F^{\bullet}\mathcal{V}, Q\}$ une variation de structure de Hodge de type K3 au dessus de S, i.e pour tout s dans S la structure de Hodge induite sur chaque $\mathbb{V}_{\mathbb{Z},s}$ est de type K3. On suppose qu'en tout point s dans S, le réseau $(\mathbb{V}_{\mathbb{Z},s}, Q)$ est isomorphe au réseau (L, Q). Pour $s \in S$, soit $\rho(\mathbb{V}_{\mathbb{Z},s})$ le *nombre de Picard* de $\mathbb{V}_{\mathbb{Z},s}$, i.e le rang du groupe des classes de Hodge dans $\mathbb{V}_{\mathbb{Z},s}$. Soit M la valeur minimale des entiers $\rho(\mathbb{V}_{\mathbb{Z},s})$ pour $s \in S$. Un résultat classique de Green [109, Prop. 17.20] et Oguiso [90] affirme que le lieu de Noether-Lefschetz

$$\mathrm{NL}(\mathbb{V}_{\mathbb{Z}}) = \{s \in S,\ \rho(\mathbb{V}_{\mathbb{Z},s}) > M\}$$

est dénombrable et dense dans S pour la toplogie analytique, quand la variation de structure de Hodge est non-triviale. Un résultat plus faible obtenu dans [18] suppose que la base S est projective. Il est donc intéressant de savoir comment l'ensemble des points où le nombre de Picard saute se distribue dans S. Notre premier résultat est un énoncé quantitatif sur la distribution du lieu de Noether-Lefschetz.

On dit que la variation de structure de Hodge $\{\mathbb{V}_{\mathbb{Z}}, F^{\bullet}\mathcal{V}, Q\}$ est simple si elle n'admet pas de sous-variation $\{\mathbb{V}'_{\mathbb{Z}}, F^{\bullet}\mathcal{V}', Q\}$ telle que le système local $\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}}$ soit non-nul et pas de torsion. En fait, partant d'une variation de structure de Hodge de type K3 arbitraire $\{\mathbb{V}_{\mathbb{Z}}, F^{\bullet}\mathcal{V}, Q\}$ la sous-variation minimale $\{\mathbb{V}'_{\mathbb{Z}}, F^{\bullet}\mathcal{V}', Q\}$ pour laquelle $\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}}$ n'a pas de torsion est simple. Son orthogonal par rapport à la forme de polarisation Q est une sous-variation de structure de Hodge entière par un théorème de semi-simplicité de Deligne et Schmid [39, 95] et qui est purement de type $(1,1)$. À revêtement fini étale de S près, c'est une variation triviale.

On dit aussi que la variation $\{\mathbb{V}_{\mathbb{Z}}, F^{\bullet}\mathcal{V}, Q\}$ est non-triviale si le fibré en droites $F^2\mathcal{V}$ n'est pas plat. Par un résultat de Griffiths [52, Chapter II], la première classe de Chern de $F^2\mathcal{V}$ est définie positive et l'intégration par rapport à celle-ci définit une mesure finie μ sur S.

Notre premier résultat est le théorème suivant. Nous en donnons dans le corps du texte une version raffinée, voir theorem 2.1.1.

Théorème 1.1.9. *Soit $\{\mathbb{V}_{\mathbb{Z}}, F^{\bullet}\mathcal{V}, Q\}$ une variation simple non-triviale de structure de Hodge de type K3 au dessus d'une courbe complexe quasi-projective S. Soit μ la mesure définie par intégration de la première classe de Chern de $F^2\mathcal{V}$ et soit $A = \{-Q(x), x \in L\}$. Alors*

(i) Pour n dans $\mathbb{Q}_{>0}$, le nombre $N(n)$ des points $s \in S$ pondérés par le nombre de classes $(1,1)$ x dans $\mathbb{V}_{\mathbb{Z},s}$ telles que $(x.x) = -2n$, est égal à 0 si $n \notin A$, et sinon vérifie

$$N(n) \sim \mu(S) \frac{(2\pi)^{1+\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|L^{\vee}/L|}\Gamma(1+\frac{b}{2})}.\prod_p \mu_p(n, L)$$

quand n tend vers l'infini le long de A, où

$$\prod_{p<\infty} \mu_p(n, L) \asymp 1.$$

Si S est projective, alors le terme d'erreur est $O_{\epsilon}(n^{\frac{2+b}{4}+\epsilon})$ pour tout $\epsilon > 0$.

(ii) L'ensemble des points pondérés comme dans [(i)] s'équirépartit dans S par rapport à la mesure μ.

On renvoie à l'exemple 2.2.3 dans le corps du texte pour une définition des facteurs $\mu_p(n, L)$ ainsi qu'au début du Chapitre 2 pour les notations. Le produit $\prod_{p<\infty} \mu_p(n, L)$ est classiquement appelé *série singulière*. Le lieu de Hodge a une structure schématique, voir [109, Chapitre 17].

Le terme principal dans le théorème 1.1.9 apparaîtra comme le nombre d'intersection global de S avec certains diviseurs spéciaux, *les diviseurs de Heegner*, dans la variété de Shimura associée au réseau (L, Q) et que nous introduirons au prochain paragraphe. Mentionnons à ce propos le résultat de Clozel et Ullmo [32] qui montre que ces diviseurs spéciaux sont équirépartis par rapport à la mesure induite par la métrique de Bergman [59, Chap VIII]. Cette dernière est donnée, à un facteur près, par l'intégration de la puissance extérieure maximale du fibré de Hodge. Notre résultat montre qu'en fait cette équirépartiton reste valide même quand on se restreint à une courbe quasi-projective arbitraire et suffisamment générique.

Donnons maintenant quelques corollaires du théorème précédent. Le premier concerne les familles de surfaces K3. Pour la définition de la norme d'une fibration elliptique qui apparaît ci-après, on renvoie à la Définition 2.4.4 dans le texte.

Corollaire 1.1.10. *Soit $d \geq 1$ un entier naturel et soit $(\mathcal{X}, \mathcal{L}_{2d}) \xrightarrow{\pi} S$ une famille non-isotriviale de surfaces K3 polarisées de degré $2d$ et de rang de Picard générique égal à 1 au-dessus d'une courbe complexe quasi-projective S. Notons μ la mesure sur S donnée par intégration de la première classe de Chern du fibré de Hodge.*

(i) Pour $n \geq 1$, le nombre $N(n)$ de points $s \in S$ (comptés avec multiplicité) pour lesquels $\mathcal{X}_s$ admet un fibré en droites L tel que $(L, L) = 0$ et $(L, \mathcal{L}_{2d,s}) = 2dn$ satisfait

$$N(n) \sim \mu(S) \frac{(2\pi)^{\frac{21}{2}} . (\sqrt{d}n)^{19}}{\sqrt{2d}.\Gamma(\frac{21}{2})} . \prod_{p<\infty} \mu_p(n, 2d)$$

quand n tend vers l'infini. En particulier $N(n) \asymp n^{19}$.

(ii) L'ensemble des tels éléments est équiréparti dans S par rapport à μ.

(iii) Si d est impair et sans facteur carré, alors l'ensemble des points s dans S (comptés avec multiplicités) pour lesquels $\mathcal{X}_s$ admet une fibration elliptique de norme plus petite que n est équiréparti par rapport à μ quand n tend vers l'infini. Le cardinal $N_e(n)$ de ces points satisfait

$$N_e(n) \asymp n^{20},$$

quand n tend vers l'infini.

Signalons qu'un résultat analogue à (i) dans le corollaire précédent a été obtenu par Simion Filip pour les familles génériques de twisteurs dans [48] avec une estimée sur le terme d'erreur donnée par Bergeron et Matheus dans [13]. Le terme principal dans la formule qu'ils donnent croît comme n^{20} et cela est dû au fait que Filip n'impose pas de condition de polarisation. Son résultat était le point de départ de notre investigation, mais notre méthode est différente de la sienne. On pourra par ailleurs noter que le coefficient du terme dominant dans notre cas comme dans le cas de Filip s'exprime comme des volumes d'espaces homogènes. En effet, par la formule de masse de Siegel (voir [47]), le produit $\prod_{p<\infty} \mu_p(n, 2d)$ peut être exprimé comme une somme de volumes d'espaces homogènes, rendant notre formule comparable à celle de Filip (3.1.6 dans [48]).

Le corollaire précédent se généralise mutatis mutandis aux familles de variétés hyperkähleriennes au-dessus d'une courbe complexe quasi-projective. Cela est discuté à la fin de la section 2.4.3.

Mentionnons également un second corollaire qui n'est pas impliqué stricto sensu par le théorème 1.1.9 puisque la signature du réseau sous-jacent est $(2,2)$, mais il est facile de voir que la méthode utilisée s'adapte sans trop de difficultés.

Corollaire 1.1.11. *Soit $\mathcal{E}_1 \to S$, $\mathcal{E}_2 \to S$ deux familles de courbes elliptiques au-dessus d'une courbe complexe quasi-projective S. On suppose que la surface abélienne $\mathcal{E}_1 \times \mathcal{E}_2$ est non-isotriviale et que $\mathcal{E}_1$ et $\mathcal{E}_2$ ne sont pas géométriquement génériquement isogènes. Alors l'ensemble (avec multiplicités) des points s dans S tel que $\mathcal{E}_{1,s}$ est isogène à $\mathcal{E}_{2,s}$ par une isogénie cyclique de degré N s'équidistribue sur S quand $N \to \infty$ par rapport à la mesure naturelle déduite de la métrique de Petersson sur $X(1)(\mathbb{C}) \times X(1)(\mathbb{C})$ où $X(1)$ est la courbe modulaire sur $\mathbb{Z}$.*

1.1.3 Les variétés de Shimura orthogonales

La stratégie de la preuve du théorème 1.1.9 tire profit de la nature arithmétique du champ de modules classifiant les structures de Hodge de type K3. Plus précisément, notons G_0 le groupe algébrique $\mathrm{SO}(L_{\mathbb{Q}}, Q)$. Alors (G_0, D_L) définit une donnée de Shimura au sens de Deligne [38]. Soit K_0 le sous groupe compact ouvert de $\mathrm{SO}(L \otimes \widehat{\mathbb{Z}})$ constitué des éléments qui agissent trivialement sur $L^\vee/L$ où $L^\vee$ désigne le réseau dual de L défini par

$$L^\vee := \{x \in V,\, \forall y \in L,\, (x.y) \in \mathbb{Z}\}.$$

On a alors une variété [1] de *Shimura orthogonale* M_0 définie sur $\mathbb{Q}$, de points complexes

$$M_0(\mathbb{C}) = G_0(\mathbb{Q})\backslash G_0(\mathbb{A}_f) \times D_L/K_0,$$

et munie d'un fibré en droites amples $\mathcal{L}$, *le fibré de Hodge*. Par les travaux de Griffiths, on dispose d'une application de périodes holomorphe

$$\rho : S \to M_0(\mathbb{C}).$$

Cette application est en fait algébrique par un théorème de Borel [19]. Le tiré-en-arrière du fibré de Hodge $\mathcal{L}$ le long de ρ est égal à $F^2\mathcal{V}$. Pour obtenir le théorème 1.1.9, la première étape est d'obtenir une estimée globale de l'ensemble (compté avec multiplicité)

$$\{s \in S,\, \exists\, \lambda \in \mathbb{V}_{\mathbb{Z},s},\, (\lambda.\lambda) = -2n\}.$$

Ceci est réalisé grâce à un théorème de modularité de Borcherds, qui est une vaste généralisation du théorème de Gross-Kohnen-Zagier [56] et Hirzebruch-Zagier [60] qu'il est utile de rappeler. Pour cela, on introduit une famille de diviseurs spéciaux $Z(\beta, m)(\mathbb{C})$ indexés par $\beta \in L^\vee/L$ et $m \in Q(\beta) + \mathbb{Z}$, $m < 0$ et $Z(0,0)$ est un diviseur dont la classe dans le groupe de Picard de $M_0(\mathbb{C})$ est égale au dual du fibré de Hodge $\mathcal{L}$, voir 3.2.5. Le théorème de Borcherds affirme alors que la série génératrice

$$\sum_{\substack{\gamma \in V^\vee/V \\ n \in -Q(\gamma)+\mathbb{Z}}} [\mathcal{Z}(\gamma, -n)] q^n v_\gamma$$

1. C'est en vérité un champ de Deligne-Mumford, mais ignorons cet aspect pour l'instant.

est le q-développement d'une forme modulaire vectorielle de poids $1 + \frac{b}{2}$. Ici $(v_\beta)_\beta$ désigne une base de $\mathbb{C}[L^\vee/L]$, voir la section 2.2.2 pour plus de détails.

La construction de Borcherds laisse de nombreuses questions ouvertes, la première étant d'étendre les relations obtenues entre les diviseurs spéciaux à des compactifications toroidales de la variété de Shimura M_0. Ceci a fait précisément l'objet des travaux de Peterson dans [93] et cette extension nous est cruciale pour borner supérieurement le cardinal du lieu de Hodge pour les courbes quasi-projectives. Une deuxième question est relative à l'intégralité de la construction de Borcherds. En effet, pour $b \geq 1$, les variétés de Shimura orthogonales ainsi que leurs diviseurs spéciaux admettent des modèles entiers sur $\mathbb{Z}$ et qui seront discutés dans la prochaine section.

1.2 Modèles entiers et théorie d'Arakelov

Dans la seconde partie de ce mémoire, on explore des analogues arithmétiques des résultats de théorie de Hodge obtenus dans la première partie. L'énoncé archétypique qu'on étudie est celui du saut du nombre de Picard géométrique des spécialisations d'une surface K3 définie sur un corps de nombres. Nous nous placerons cependant dans un cadre plus général reposant sur la construction de Kuga-Satake que nous commençons par introduire. On garde les mêmes notations qu'avant, en particulier (L, Q) désigne un réseau pair de signature $(2, b)$ avec $b \geq 3$.

1.2.1 Schéma abélien de Kuga-Satake

Commençons par introduire la *variété de Shimura de type GSpin*, on réfère à [41, 1.3] pour plus de détails. Notons

$$G = \mathrm{GSpin}(L_\mathbb{Q}, Q)$$

le groupe algébrique spinoriel généralisé sur $\mathbb{Q}$: c'est une extension centrale de G_0 par $\mathbb{G}_m$ et c'est un sous-groupe du groupe des unités de l'algèbre de Clifford $C(L_\mathbb{Q})$ de $(L_\mathbb{Q}, Q)$. Notons K le groupe $G(\mathbb{A}_f) \cap C(L_\mathbb{Z})^\times$, d'image égale à K_0 dans $G_0(\mathbb{A}_f)$ par [81, Lemma 2.6]. On a alors une variété[1] de Shimura M de type GSpin définie sur $\mathbb{Q}$ telle que

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) \times D_L / K,$$

et un morphisme fini étale $M \to M_0$.

La variété M a l'avantage d'être de *type Hodge*. Plus précisément, il existe un accouplement alterné

$$\psi : C(L) \times C(L) \to \mathbb{Z}$$

qui induit une forme symplectique non-dégénérée sur $C(L_\mathbb{Q})$. Notons $\mathcal{G}_\psi = \mathrm{GSp}(\psi)$ le groupe des similitudes symplectiques attaché à ψ et $\mathcal{X}_\psi$ l'union des demi-espaces de Siegel attachés à $(C(L), \psi)$. Le couple $(\mathcal{G}_\psi, \mathcal{X}_\psi)$ est une donnée de Shimura de type Siegel, voir [81, 3.5]. On obtient alors un plongement de données de Shimura

$$(G, D_L) \to (\mathcal{G}_\psi, \mathcal{X}_\psi),$$

voir [81, 3.5] pour plus de détails. En notant $\mathcal{K}_\psi \subset \mathcal{G}_\psi(\mathbb{A}_f)$ le stabilisateur de $C(L_{\widehat{\mathbb{Z}}})$, on obtient un morphisme fini et non-ramifié de variétés de Shimura sur $\mathbb{Q}$

$$\iota : M \to Sh_{\mathcal{K}_\psi}(\mathcal{G}_\psi, \mathcal{X}_\psi).$$

Le terme de droite a une interprétation naturelle comme champ de modules de variétés abéliennes polarisées avec structure de niveau additionnelle, voir pour cela [81, 3.9]. En particulier, le tiré-en-arrière par ι du schéma abélien universel sur $Sh_{\mathcal{K}_\psi}(\mathcal{G}_\psi, \mathcal{X}_\psi)$ fournit un schéma abélien sur M, c'est le *schéma abélien de Kuga-Satake*, noté A^{KS}.

Ce schéma abélien ne dépend pas de l'accouplement alterné choisi. Par contre, cela n'est pas vrai de sa polarisation. Les différentes réalisations de la cohomologie du schéma A^{KS} fournissent des fibrés vectoriels filtrés avec connexion intégrable (resp. systèmes locaux ℓ-adiques) sur M, $H_{dR}^{\otimes(1,1)}$ et $\mathcal{V}_{dR}$ (resp. $H_\ell^{\otimes(1,1)}$ et $\mathcal{V}_\ell$) et un plongement $\mathcal{V}_{dR} \hookrightarrow H_{dR}^{\otimes(1,1)}$ (resp. $\mathcal{V}_\ell \hookrightarrow H_\ell^{\otimes(1,1)}$). Le fibré $\mathcal{V}_{dR,\mathbb{C}}$ est en fait le tiré-en-arrière de la variation de structure de Hodge de type K3 universelle sur $M_0(\mathbb{C})$. La construction de Kuga-Satake est précisément ce qui permet de munir $\mathcal{V}_{dR,\mathbb{C}}$ d'une structure de $\mathbb{Q}$-fibré vectoriel filtré avec connexion intégrable, en raison du théorème de Deligne affirmant que les classes de Hodge sur les variétés abéliennes sont absolues, voir [42]. Le sous-fibré $\mathcal{V}_{dR}$ permet de définir la notion d'endomorphisme spécial de A^{KS} : étant donné un M-schéma T, un endomorphisme f dans $\mathrm{End}(A_T^{KS})$ est dit spécial si son image par l'application

$$\mathrm{End}(A_{T^{an}}^{KS,an}) \to H^0(T^{an}, H_{dR,\mathbb{C}}^{\otimes(1,1)})$$

appartient à $H^0(T^{an}, \mathcal{V}_{dR,\mathbb{C}})$. Par [81, Lemma 5.4], cela est équivalent au fait que pour un ℓ (resp. pour tout ℓ), son image sous l'application

$$\mathrm{End}(A_T^{KS}) \to H^0(T, H_\ell^{\otimes(1,1)})$$

appartient à $H^0(T, \mathcal{V}_\ell)$.

On peut donc associer à chaque point $x \in M(K)$, pour K corps de nombres, une représentation galoisienne ℓ-adique

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(\mathcal{V}_{\ell,x}),$$

où $\mathcal{V}_{\ell,x}$ est la fibre du système local ℓ-adique $\mathcal{V}_\ell$ en x.

Pour presque toute place $\mathfrak{P}$, la représentation ρ_ℓ est non-ramifiée en $\mathfrak{P}$ et le Frobenius en $\mathfrak{P}$ a des valeurs propres de norme 1. Une première formulation de la question qu'on cherche à résoudre est la suivante.

Question : Existe-il une place $\mathfrak{P}$ telle que le Frobenius en $\mathfrak{P}$ admette un invariant exceptionnel, i.e qui ne soit pas d'ordre fini sous $\mathrm{Gal}(\overline{K}/K)$ et qui soit d'orbite finie sous l'action du Frobenius en $\mathfrak{P}$?

Remarque 1.2.1. Dans [30, Theorem 1], Charles décrit certaines situations où la représentation ρ_ℓ admet un invariant exceptionnel en toute place $\mathfrak{P}$. Le point de départ est la remarque que la représentation ρ_ℓ préserve une forme quadratique non-dégénérée sur $\mathcal{V}_{\ell,x}$. Combinée à des arguments de parité, cela montre dans certains cas que le Frobenius en toute place $\mathfrak{P}$ admet toujours 1 (ou -1) comme valeur propre exceptionnelle, et par conséquent un invariant exceptionnel.

La question précédente peut être placée dans un cadre conjectural plus général motivé par la conjecture de Lang-Trotter [74] et ses généralisations [88, 2] que nous rappelons brièvement.

Soit E une courbe elliptique définie sur $\mathbb{Q}$. Pour presque tout nombre premier p, la courbe E a bonne réduction en p. La réduction est ordinaire ou supersingulière, selon que la trace de l'action du Frobenius sur $H^1(E_{\text{ét}}, \mathbb{Z}_\ell)$ est non-nulle ou pas. Lang et Trotter conjecturent que si E n'a pas de multiplication complexe, le nombre de réductions supersingulères est infini avec une estimée précise de croissance. Une généralisation de cette conjecture est faite dans [2], qui dans notre cas s'énonce ainsi : supposons qu'on ait à notre disposition un modèle entier canonique $\mathcal{M}$ de la variété Shimura GSpin M, avec une extension $\mathcal{A}^{KS}$ du schéma abélien A^{KS} et des modèles entiers convenables $\mathcal{Z}(\beta, m)$ des diviseurs spéciaux. Si $\mathcal{A}^{KS}_x \to \operatorname{Spec}(\mathbb{Z})$ correspond à un point x dans $\mathcal{M}(\mathbb{Z})$ Mumford-Tate générique, alors la probabilité que $A^{KS}_{x,p}$ soit dans l'un des $\mathcal{Z}(\beta, m)$ est au moins $\frac{1}{p}$, donc en sommant sur $p < X$, on obtient une borne inférieure en $\log\log(X)$ et donc il y a conjecturalement une infinité de premiers p tels que $\mathcal{A}^{KS}_{x,p}$ acquière un endomorphisme spécial. Bien évidemment, cette heuristique n'est pas très convaincante dans la mesure où elle ne tient pas compte de l'équidistribution des diviseurs $\mathcal{Z}(\beta, m)(\mathbb{C})$ dans la fibre générique de $\mathcal{M}$, un résulat dû à [32] et qui devrait avoir en un certain sens un avatar arithmétique.

Signalons enfin qu'Elkies a montré dans [45] que les courbes elliptiques sans multiplication complexe définies sur un corps de nombres K ayant au moins un plongement réel, admettent une infinité de spécialisations elliptiques. Lorsque (L, Q) est un réseau unimodulaire pair de signature $(2, 2)$, la variété de Shimura M est égale à $Y(1) \times Y(1)$ où $Y(1)$ est la courbe modulaire ouverte sur $\mathbb{Z}$. Les composantes irréductibles des diviseurs spéciaux sont données par les courbes modulaires $Y_0(N)$ pour $N \geq 1$ et elles paramètrent les paires de courbes elliptiques liées par une isogénie cyclique de degré N. Dans ce cadre, Charles montre dans [31] qu'étant données deux courbes elliptiques qui ne sont pas géométriquement isogènes sur un corps de nombres K, il existe une infinité de places telles que leurs réductions soient isogènes, ce qui répond à notre de question de départ dans ce cadre. Notre but dans la suite sera d'obtenir des analogues en dimension supérieure de ce résultat.

1.2.2 Constructions de modèles entiers

Notre approche pour répondre à la question posée ci-dessus est celle suivie par Charles dans [31]. Elle se place dans le cadre de la théorie d'intersection arithmétique de Gillet et Soulé [50] et nécessite donc la construction d'un modèle entier distingué de la variété de Shimura M, de ses diviseurs spéciaux ainsi que l'extension des relations de modularité de Borcherds. Il est donc essentiel de discuter dans un premier temps ces constructions et leurs propriétés, en suivant principalement [81, 4, 5, 62]. Pour un survol agréable de ces résultats, on pourra également consulter [11].

On dira que L est presque auto-dual en un nombre premier p si p ne divise pas le discriminant de L, et si $p = 2$, on requiert que le discriminant de L n'est pas divisible par 4 et $L_{\mathbb{Q}}$ de dimension impaire. Le théorème suivant provient de [5, Theorem 4.4.6] en tenant compte de [62, Remark 6.2.1].

Théorème 1.2.2. *Il existe un unique champ algébrique $\mathcal{M}$, normal et plat sur $\mathbb{Z}$ tel que $\mathcal{M}_{\mathbb{Q}} = M$ et qui vérifie de plus :*

1. *le schéma abélien $A^{KS} \to M$ s'étend en un schéma abélien $\mathcal{A}^{KS} \to \mathcal{M}$;*
2. *le fibré en droites $F^1\mathcal{V}_{dR}$ s'étend canoniquement en un fibré en droites ω sur $\mathcal{M}$;*
3. *$\mathcal{M}$ est lisse en les p où L est presque auto-dual ;*

4. $\mathcal{M}_{(p)}$ a la propriété d'extension suivante : si $E/\mathbb{Q}_p$ est une extension finie et $t \in M(E)$ un point tel que $\mathcal{A}_t^{KS}$ a potentiellement bonne réduction sur $\mathcal{O}_E$, alors le morphisme $\mathrm{Spec}(E) \to M$ *s'étend en un morphisme* $\mathrm{Spec}(\mathcal{O}_E) \to \mathcal{M}_{(p)}$.

Soit (G, D_L), (G, D_L),$(\mathcal{G}_\psi, \mathcal{X}_\psi)$, $K \subset G(\mathbb{A}_f)$, $K_0 \subset G_0(\mathbb{A}_f)$ et $\mathcal{K}_\psi \subset \mathcal{G}_\psi(\mathbb{A}_f)$ les données de Shimura et les sous groupes compact-ouverts introduits précédemment.

Pour p nombre premier impair et ne divisant pas le discriminant de L, l'interprétation modulaire de $Sh_{\mathcal{K}_\psi}(\mathcal{G}_\psi, \mathcal{X}_\psi)$ fournit naturellement un modèle entier $\mathcal{S}_{\mathcal{K}_\psi}$ sur $\mathbb{Z}_{(p)}$ et dans ce cas la construction du modèle entier lisse de M est due à Kisin [68], en suivant une stratégie qui remonte à Milne [85, 2.15]. Cette stratégie a été également considérée par Moonen [86] et Vasiu [107]. Plus précisément, $\mathcal{M}_{(p)}$ est définie comme la normalisation de l'adhérence de M dans $\mathcal{S}_{\mathcal{K}_\psi}$. Le tour de force de Kisin a consisté à montrer que les modèles ainsi construits sont bien lisses.

Quand p est impair et divise le discriminant de L, la construction du modèle entier est due à Madapusi-Pera dans [81]. Sa démarche consiste à choisir un réseau auxiliaire (L', Q') de signature $(2, b')$, auto-dual en p et dans lequel (L, Q) se plonge de manière primitive. Madapusi-Pera décrit alors la variété de Shimura $M_{(p)}$ comme intersection de diviseurs spéciaux à l'intérieur de M' et à laquelle les résultats de Kisin s'appliquent. Les diviseurs en question ont une interprétation modulaire comme lieux où le schéma abélien de Kuga-Satake acquiert un endomorphisme spécial et donc admettent un modèle entier sur $\mathbb{Z}_{(p)}$. Le modèle cherché se construit alors comme intersection de ces diviseurs.

Quand $p = 2$, la construction du modèle entier de M, et plus généralement des variétés de Shimura de type abélien, est due aux travaux de Kim et Madapusi-Pera dans [67]. L'ingrédient essentiel est la classification des groupes 2-divisibles sur une large classes d'anneaux en terme de *displays de Dieudonné* due à Lau [75]. Le reste de la stratégie est similaire.

Signalons que le modèle qu'on considère dans le théorème ci-dessus, qui apparaît également dans [5] et [62], n'est pas celui issu de la construction de Madapusi-Pera dans [81]. En effet, cette dernière donne un modèle entier régulier en excluant certains sous-champs singuliers qui apparaissent dans le processus de normalisation décrit plus haut et qui sont supportés en caractéristique p telle que p^2 divise le discriminant de L, on réfère au 6.27 de [81] et [62, Remark 6.2.4] pour plus de détails. En particulier, la construction de Madapusi-Pera assure que $\mathcal{V}_{dR}$ s'étend en un fibré vectoriel avec une connexion intégrable, alors que nous aurons besoin seulement d'étendre le fibré en droites $F^1\mathcal{V}_{dR}$. Par ailleurs, il est nécessaire de tenir compte de ces lieux singuliers pour pouvoir étendre les relations de Borcherds.

Enfin, le schéma abélien de Kuga-Satake permet de construire des modèles entiers canoniques des diviseurs spéciaux $(\mathcal{Z}(\beta, m))_{\beta,m}$. Howard et Madapusi-Pera montrent alors que la construction des produits de Borcherds peut être faite à un niveau entier. En particulier, ils démontrent [62, Thm 8.2.1] que la série génératrice

$$\Phi_L = \sum_{\substack{\beta \in L^\vee/L \\ m \geq 0, m \in -Q(\beta)+\mathbb{Z}}} \mathcal{Z}(\beta, -m).q^m v_\beta$$

est la q-expansion d'une forme modulaire de poids $1 + \frac{b}{2}$ pour une certaine représentation

$$\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[L^\vee/L]),$$

du groupe métaplectique $\mathrm{Mp}_2(\mathbb{Z})$ et où $(v_\beta)_{\beta\in L^\vee/L}$ désigne une base de $\mathbb{C}[L^\vee/L]$, voir sections 3.3.2 et 3.3.3 pour plus de détails.

Les travaux antérieurs de Bruinier [23] montrent que chacun des diviseurs spéciaux $\mathcal{Z}(\beta, m)$ peut être muni d'une fonction de Green $\Phi_{\beta,m}$ construite par un procédé de relèvement thêta de forme harmonique de poids négatif. On définit ainsi pour tout couple (β, m) un diviseur

$$\widehat{\mathcal{Z}}(\beta, m) = (\mathcal{Z}(\beta, m), \Phi_{\beta,m})$$

dans le groupe de Chow arithmétique $\widehat{\mathrm{CH}}^1(\mathcal{M})$. Howard et Madapusi Pera montrent que pour $b \geq 3$, la série génératrice

$$\widehat{\Phi}_L = \sum_{\substack{\beta\in L^\vee/L \\ m\geq 0, m\in -Q(\beta)+\mathbb{Z}}} \widehat{\mathcal{Z}}(\beta, -m).q^m$$

définit également une forme modulaire, voir [62, Theorem 8.3.1].

1.2.3 Invariants exceptionnels et endomorphismes spéciaux

Venons-en maintenant au résultat principal du Chapitre 3 de ce mémoire. Si x est un point de $M(\mathbb{C})$, alors x correspond à un plan défini positif P_x dans $L_\mathbb{R}$ et pour un élément λ dans $L_\mathbb{R}$, on note λ_x la projection orthogonale de λ sur P_x. On dit que x est :

1. Hodge-générique si x n'appartient à aucun diviseur $\mathcal{Z}(\beta, m)(\mathbb{C})$;
2. modérément approché par les diviseurs spéciaux si pour tout $\beta \in L^\vee/L$, on peut trouver $D > 0$ tel que

$$\forall N > 0, \exists m \leq -N, \forall \lambda \in \beta + L : Q(\lambda) = m \Rightarrow Q(\lambda_x) \geq \frac{1}{|m|^D}. \tag{1.1}$$

Cette dernière condition se traduit géométriquement par le fait qu'on peut trouver une suite de diviseurs spéciaux $(\mathcal{Z}(\beta, m)(\mathbb{C}))_m$ dont la distance à x pour une certaine métrique sur $M(\mathbb{C})$ reste bornée inférieurement par une puissance fixée de m. Cela permet de contrôler l'ordre de grandeur de l'évaluation de la fonction de Green $\Phi_{\beta,m}$ en x, on renvoie à la section 3.4.5 pour plus de détails.

Soit K un corps de nombres. Pour $x \in M(K)$, la fibre du schéma abélien $\mathcal{A}^{KS}$ au-dessus de x est une variété abélienne sur K qu'on appelle *variété abélienne de Kuga-Satake*. On dit que x est modérément approché par les diviseurs spéciaux si pour tout plongement $\sigma : K \hookrightarrow \mathbb{C}$, le point $x^\sigma \in M(\mathbb{C})$ est modérément approché par les diviseurs spéciaux. Par ailleurs, on dit que x est Hodge-générique si pour un plongement (de manière équivalente, pour tout plongement) $\sigma : K \hookrightarrow \mathbb{C}$, le point x^σ est Hodge générique.

Le résultat principal qu'on prouve dans le Chapitre 3 est le suivant.

Théorème 1.2.3. *Soit $x \in \mathcal{M}(\mathcal{O}_K)$ et supposons que $x_K \in \mathcal{M}(K)$ est Hodge-générique et qu'il est modérément approché par les diviseurs spéciaux. Pour un nombre premier $\ell > 0$, notons $(\rho_\ell, \mathcal{V}_{\ell,x})$ la représentation Galoisienne associée à x_K construite en 1.2.1. Alors il existe $\ell > 0$ et une place $\mathfrak{P}$ de K de caractéristique résiduelle différente de ℓ telle que ρ_ℓ admette un invariant exceptionnel.*

Il est important de mentionner que la variété $M(\mathbb{C})$ admet une mesure naturelle non-nulle $G(\mathbb{R})$-invariante et pour laquelle le sous-ensemble des éléments qui ne sont pas modérément approchés par les diviseurs spéciaux est de mesure nulle, voir Proposition 3.4.9.

Le résultat du Théorème 1.2.3 est un premier pas vers une généralisation du résultat principal de [31] qui correspond au cas où le réseau (L, Q) est unimodulaire pair de signature $(2, 2)$. Dans cette situation, $\mathcal{M}$ est le produit $Y(1) \times Y(1)$, où $Y(1)$ est la courbe modulaire ouverte sur $\mathbb{Z}$. D'autres cas où la signature du réseau (L, Q) est $(2, 2)$ ont été considérés dans [98]. Dans les deux derniers cas, les auteurs montrent que l'hypothèse d'approximation modérée est superflue. Nous espérons également enlever cette hypothèse dans un futur travail.

Le théorème précédent admet la traduction géométrique suivante, qui repose sur une version de la conjecture de Tate pour les endomorphismes spéciaux démontrée dans [80]. Si $\mathfrak{P}$ est une place de K, on note $\overline{k(\mathfrak{P})}$ une clôture algébrique du corps résiduel $k(\mathfrak{P})$ de $\mathfrak{P}$.

Corollaire 1.2.4. *Soit $x \in \mathcal{M}(\mathcal{O}_K)$ et soit $\mathcal{A}_x^{KS}$ le schéma abélien de Kuga-Satake correspondant. Supposons que $x_K \in \mathcal{M}(K)$ est Hodge-générique et qu'il est modérément approché par les diviseurs spéciaux. Alors il existe une place $\mathfrak{P}$ de K telle $\mathcal{A}^{KS}_{x,\overline{k(\mathfrak{P})}}$ admette un endomorphisme spécial exceptionnel.*

1.2.4 Nombre de Picard des spécialisations des surfaces K3

Nous expliquons maintenant l'application à l'étude du rang du groupe de Picard des spécialisations dans les familles arithmétiques de surfaces K3. Soit X une surface K3 définie sur un corps de nombres K. On peut trouver un entier $N \geq 1$ et une famille $\mathcal{X} \to S = \mathrm{Spec}(\mathcal{O}_K[\frac{1}{N}])$ de surfaces K3, de fibre générique égale à X. Pour toute place $\mathfrak{P}$ de K, on dispose d'une application de spécialisation injective par [64, Chap.17 Prop.2.10]

$$\mathrm{sp}_{\mathfrak{P}} : \mathrm{Pic}(X_{\overline{K}}) \hookrightarrow \mathrm{Pic}(\mathcal{X}_{\overline{k(\mathfrak{P})}}),$$

où $\overline{K}$ désigne une clôture algébrique de K et $\overline{k(\mathfrak{P})}$ est une clôture algébrique du corps résiduel $k(\mathfrak{P})$ de $\mathfrak{P}$.

Dans [100] puis dans [30], Charles et Shioda analysent le comportement de l'ensemble suivant, lieu de saut (**J***umping* **L***ocus*) du nombre de Picard :

$$JL(X) = \{\mathfrak{P} \in S, \rho(\mathcal{X}_{\overline{\mathfrak{P}}}) > \rho(X_{\overline{K}})\},$$

où pour Y une surface K3, $\rho(Y)$ désigne le rang de son groupe de Picard. Par [30, Theorem 1], la valeur générique du nombre de Picard géométrique des spécialisations de X est complètement déterminé par $\rho(X_{\overline{K}})$ et par le corps des endomorphismes de structures de Hodge du réseau transcendant $T(X_{\mathbb{C}})$, pour n'importe quel plongement de K dans $\mathbb{C}$. Par exemple, quand $\rho(X_{\overline{K}})$ est impair et en tenant compte de la conjecture de Tate, l'ensemble précédent est à égal à S tout entier. Supposons donc que $\rho(X_{\overline{K}})$ est pair.

Corollaire 1.2.5. *Soit $\mathcal{X}$ une famille de surfaces K3 sur $\mathcal{O}_K$. Supposons que $X := \mathcal{X}_K$ est modérément approchée par les diviseurs spéciaux de la variété de Shimura associée au réseau transcendant de X, alors il existe une place $\mathfrak{P}$ telle que $\mathfrak{P}$ appartienne à $JL(X)$.*

La preuve de ce corollaire est une application directe du théorème 1.2.3 en travaillant dans la variété de Shimura orthogonale associée au réseau transcendant de $X_{\mathbb{C}}$, pour n'importe quel plongement de K dans $\mathbb{C}$. On conclut en utilisant la conjecture de Tate pour les surfaces K3, qui est maintenant un théorème grâce aux travaux de Charles [29], Maulik [83], Madapusi-Pera [80] et Nygaard-Ogus [89].

1.3 Courbes rationnelles sur les surfaces K3

Le troisième volet de ce mémoire concerne le problème de la construction de courbes rationnelles sur les surfaces K3. Pour une discussion plus détaillée, nous recommandons le Chapitre 13 de [64].

1.3.1 Aperçu des résultats

Soit X une surface K3 sur $\mathbb{C}$. Une courbe rationnelle sur X est une sous-variété intègre dont la normalisation est isomorphe à $\mathbb{P}^1_{\mathbb{C}}$. L'étude des courbes rationnelles sur les surfaces K3 est intimement liée à leur place particulière au sein de la classification des surfaces algébriques sur $\mathbb{C}$, voir [9]. À toute surface algébrique complexe X, on associe son *nombre de Kodaira* $\kappa(X)$ qui prend les valeurs $-1, 0, 1$ ou 2. Les surfaces de nombre de Kodaira -1 sont relativement bien comprises et sont réglées. Celles de dimension de Kodaira strictement positive sont dites de type général. Il existe de nombreuses conjectures à leur sujet, dont les principales sont les conjectures de Green-Griffiths-Lang qui affirment que de telles surfaces ne peuvent contenir qu'un nombre fini de courbes rationnelles, voir [43]. Enfin, les surfaces K3 ont une dimension de Kodaira nulle et présenteraient donc, au moins conjecturalement, un comportement intermédiaire. Signalons que les courbes rationnelles ne viennent jamais en famille sur les surfaces K3 complexes, ce qui n'est pas le cas quand le corps de base est de caractéristique strictement positive, voir [64, Chap.13, Example 0.1]. On a alors la conjecture suivante, voir aussi [64, Chap.13,0.2].

Conjecture 1.3.1. *Soit X une surface K3 sur un corps algébriquement clos. Alors X contient une infinité de courbes rationnelles.*

Venons-en maintenant aux résultats connus à ce sujet. Le premier résultat est attribué par Mori et Mukai à Bogomolov et Mumford, voir [87, Appendix]. Avec le raffinement de [76, Theorem 2.1], il stipule que si X est une surface K3 sur un corps algébriquement clos k et L est un fibré en droites effectif et non-trivial sur X, alors il existe une courbe dans le système linéaire $|L|$ qui est une somme de courbes rationnelles. L'idée de la preuve repose sur des techniques de déformations. On commence d'abord par montrer le résultat pour des surfaces K3 particulières, des surfaces de Kummer par exemple où on construit explicitement des courbes rationnelles. Ensuite, on déforme la courbe rationnelle à des déformations de la surface K3 en sachant que le fibré en droites associé s'y déforme. Ces techniques de déformation seront élaborées et systématisées dans tous les travaux ultérieurs, notamment à travers l'introduction des espaces de modules de courbes stables par Kontsevich, voir [71]. L'avancée majeure est faite dans [15] puis dans [76]. Le théorème suivant récapitule les cas connus de la conjecture.

Théorème 1.3.2. *Soit X une surface K3 sur un corps algébriquement clos k. Alors X contient une infinité de courbes rationnelles dans les cas suivants :*

1. *Si k est de caractéristique nulle :*
   (a) *X n'est pas définie sur $\overline{\mathbb{Q}}$;*
   (b) *ou X admet une fibration elliptique.*
2. *Si k est de caractéristique différente de 2 et $\rho(X)$ est impair.*

Notre contribution à ce sujet est le théorème suivant qui étend le point 1.(b) du Théorème 1.3.2.

Théorème 1.3.3. *Soit X une surface K3 sur un corps k algébriquement clos. Si X admet une fibration elliptique non-isotriviale, alors X contient une infinité de courbes rationnelles.*

Nous espérons dans un prochain travail éliminer l'hypothèse de non-isotrivialité de sorte que le résultat vaut pour toute surface K3 elliptique sur un corps algébriquement clos. Le théorème 1.3.2 implique que les seules surfaces K3 sur $\overline{\mathbb{Q}}$ pour lesquelles la conjecture 1.3.1 n'est pas connue sont celles qui ont un nombre de Picard égal à 2 ou 4. La stratégie de [15] puis celle de [76] repose sur l'existence d'une infinité de spécialisations en lesquels le nombre Picard saute. Dans le cas où le nombre de Picard est impair, ceci résulte de la conjecture de Tate. Cette stratégie peut donc être poursuivie dans les autres cas si on sait que le nombre Picard saute et que la réduction n'est pas supersingulière. Nous espérons explorer ces questions dans le futur.

Chapitre 2

On the equidistribution of some Hodge loci

Résumé. Étant donnée une variation de structure de Hodge entière polarisée de type K3 au dessus d'une courbe complexe quasi-projective, c'est un résultat classique dû à Green-Voisin que le lieu de Hodge correspondant est dénombrable et dense dès lors que la variation est simple et non-triviale. Dans cet exposé, on étudiera l'équirépartition de ce lieu pour la mesure induite par intégration de la classe de Chern du fibré de Hodge. On donnera une estimée asymptotique du nombre de points sur la base ayant une classe de Hodge de carré donné. On discutera ensuite quelques applications à l'étude des fibrations elliptiques dans les familles de surfaces K3 sur une courbe ainsi que la distribution des classes paraboliques dans les familles de variétés hyperkählériennes sur une courbe.

Abstract. We prove the equidistribution of the Hodge locus for certain polarized, non-isotrivial variations of Hodge structure of weight 2 with $h^{2,0} = 1$ over complex, quasi-projective curves. Given some norm condition, we also give an asymptotic on the growth of the Hodge locus. In particular, this implies the equidistribution of elliptic fibrations in quasi-polarized, non-isotrivial families of K3 surfaces.

Contents

2.1 Introduction

Let S be a complex, quasi-projective curve and let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ be an integral, polarized variation of Hodge structure of weight 2 over S with $h^{2,0} = 1$. We assume that the bilinear form associated Q is even of signature $(2, b)$. For $s \in S$, let $\rho(\mathbb{V}_{\mathbb{Z},s})$ be the *Picard number* of $\mathbb{V}_{\mathbb{Z},s}$, that is, the rank of the group of integral $(1,1)$-classes in $\mathbb{V}_{\mathbb{Z},s}$. Let M be the minimum value of the integers $\rho(\mathbb{V}_{\mathbb{Z},s})$ for $s \in S$. It is a classical result of Green [109, Prop. 17.20] and Oguiso [90] that the Noether-Lefschetz locus

$$\mathrm{NL}(\mathbb{V}_{\mathbb{Z}}) = \{s \in S,\ \rho(\mathbb{V}_{\mathbb{Z},s}) > M\}$$

is a countable dense subset of S, when the variation of Hodge structure is non-trivial. A weaker result obtained in [18] assumes the base S to be projective. One might then ask how the set of points where the Picard rank jumps distributes inside S. The goal of this paper is to investigate quantitative statements on the distribution of the Hodge locus.

Say that the polarized variation of Hodge structure $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ is simple if there is no polarized sub-variation of Hodge structure $\{\mathbb{V}'_{\mathbb{Z}},\ \mathcal{F}^{\bullet}\mathcal{V}', Q\}$ such that $\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}}$ is non-zero and torsion free. In fact, starting from an arbitrary polarized variation of Hodge structure $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V},\ Q\}$ with $h^{2,0} = 1$, the minimal sub-variation $\{\mathbb{V}'_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}', Q\}$ for which $\mathbb{V}_{\mathbb{Z}}/\mathbb{V}'_{\mathbb{Z}}$ is torsion free is simple. Its orthogonal with respect to Q is also an integral sub-variation of Hodge structure by Deligne and Schmid's semi-simplicity theorem [39, 95] and which is purely of type $(1,1)$, thus it is trivial, up to taking a finite étale cover of S.

Say also that $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^{\bullet}\mathcal{V}, Q\}$ is non-trivial if the line bundle $\mathcal{F}^2\mathcal{V}$ is not flat. By a result of Griffiths [52, Chapter II] the first Chern class ω of $\mathcal{F}^2\mathcal{V}$ is positive definite, and the integration with respect to ω defines a finite measure μ on S.

Assume that for each $s \in S$, the lattice $(\mathbb{V}_{\mathbb{Z},s}, Q)$ is isomorphic to an even quadratic lattice $(V, (\,.\,))$ of signature $(2, b)$ with $b \geq 3$ and $Q(x) = \frac{(x.x)}{2}$ for all $x \in V$.

Let $\mathbb{V}_{\mathbb{Z}}^{\vee} \subset \mathbb{V}_{\mathbb{Q}}$ be the dual local system to $\mathbb{V}_{\mathbb{Z}}$ with respect to Q, i.e the fiber $\mathbb{V}_{\mathbb{Z},s}^{\vee}$ at each point $s \in S$ is equal to

$$\{x \in \mathbb{V}_{\mathbb{Q},s},\ \forall y \in \mathbb{V}_{\mathbb{Z},s},\ (x.y) \in \mathbb{Z}\}.$$

The fibers of the local system $\mathbb{V}_{\mathbb{Z}}^{\vee}/\mathbb{V}_{\mathbb{Z}}$ are isomorphic to the finite group $V^{\vee}/V$. For $s \in S$ and $\lambda \in \mathbb{V}_{\mathbb{Z},s}^{\vee}$, there exists $\gamma \in V^{\vee}/V$ such that $\lambda \in \gamma + V$ with the previous identification, and therefore $Q(\lambda) \in Q(\gamma) + \mathbb{Z}$. In general we have $Q(\lambda) \in \cup_{\gamma \in V^{\vee}/V}(Q(\gamma) + \mathbb{Z})$.

If H is a subgroup of $V^{\vee}/V$ we let $H^{\perp}$ be the orthogonal of H in $V^{\vee}/V$ with respect to the reduction of the form $(\,.\,)$ valued in $\mathbb{Q}/\mathbb{Z}$. The main result of this paper is the following theorem.

Theorem 2.1.1. *Let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a non-trivial, polarized, simple variation of Hodge structure of weight 2 over a quasi-projective curve S with $h^{2,0} = 1$. Assume that the quadratic lattice $(V, (\ ,\ ))$ is even and that the local system $\mathbb{V}_{\mathbb{Z}}^\vee/\mathbb{V}_{\mathbb{Z}}$ is trivial. Let H be a maximal totally isotropic subgroup of $V^\vee/V$. Let μ be the measure defined by integrating the first Chern class of $\mathcal{F}^2\mathcal{V}$ and let $\gamma \in H^\perp \subset V^\vee/V$. Set $A_\gamma = \{-Q(x+\gamma),\ x \in V\}$. Then*

(i) For $n \in \mathbb{Q}_{>0}$, the number $N(\gamma, n)$ of points $s \in S$ (counted with multiplicity) for which there exists a $(1,1)$-element x in $\mathbb{V}_{\mathbb{Z},s}^\vee$ of class γ in $\mathbb{V}_{\mathbb{Z},s}^\vee/\mathbb{V}_{\mathbb{Z},s}$ and $(x.x) = -2n$ is equal to zero if $n \notin A_\gamma$, otherwise it satisfies

$$N(\gamma, n) \sim \mu(S) \frac{(2\pi)^{1+\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}.\prod_p \mu_p(\gamma, n, V)$$

as n tends to infinity along A_γ, where

$$\prod_{p<\infty} \mu_p(\gamma, n, V) \asymp 1.$$

If S is projective, then the error term is $O_\epsilon(n^{\frac{2+b}{4}+\epsilon})$ for every $\epsilon > 0$.

(ii) The set of such points equidistributes in S with respect to μ.

We refer to Example 2.2.3 for the definition of the factors $\mu_p(\gamma, n, V)$. The product $\prod_{p<\infty} \mu_p(\gamma, n, V)$ is called the *singular series*. The Hodge locus has a schematic structure (see [109, Chapter 17]). The multiplicity of a point evoked in Theorem 2.1.1 is the multiplicity in this schematic sense.

This number can also be seen as the multiplicity of intersection of S with special divisors, the so-called *Heegner divisors*, in the moduli space of Hodge structure of K3 type over V. This moduli space is in fact a Shimura variety of orthogonal type. As a part of their study of the André–Oort conjecture [110], Clozel and Ullmo proved in [32] that the Heegner divisors are equidistributed with respect to the measure induced by the Bergman metric ([59, Chap VIII]). The latter is simply given, up to an absolute constant, by integrating the top power of the first Chern class of the Hodge bundle. What we prove here is the equidistribution of the intersection of the Heegner divisors with any fixed quasi-projective curve as above with respect to the measure given by integration the first Chern class of the Hodge bundle.

Corollary 2.1.2. *Let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a non-trivial, polarized, simple variation of Hodge structure of weight 2 over a quasi-projective curve S and μ the measure on S defined by integrating the first Chern class of $\mathcal{F}^2\mathcal{V}$. Then the set of points $s \in S$ for which there exists a $(1,1)$-class x in $\mathbb{V}_{\mathbb{Z},s}$ such that $(x.x) = -2n$ are equidistributed with respect to μ along positive integers n in the infinite set $\{-Q(x),\ x \in V\}$.*

Indeed, we can always take a finite étale cover $\tilde{S}$ of S for which the pullback of the local system $\mathbb{V}_{\mathbb{Z}}^\vee/\mathbb{V}_{\mathbb{Z}}$ is trivial and apply Theorem 2.1.1 to $\tilde{S}$ which imply the corollary.

In particular, we deduce various equidistribution results for points in some 1-parameter families of complex varieties.

Corollary 2.1.3. *Let Λ_{K_3} be the K3 lattice and $P \subset \Lambda_{K_3}$ a primitive Lorentzian anisotropic sublattice of rank $\rho \leq 4$. Let $\mathcal{X} \xrightarrow{\pi} S$ be a non-isotrivial family of K3 surfaces with generic Picard group equal to P over a quasi-projective curve S and let $\{R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}, \mathcal{F}^\bullet\mathcal{H}, Q\}$ be the induced variation of Hodge structure on S. Let μ be the measure induced by integrating the first Chern class of $\mathcal{F}^2\mathcal{H}$. Fix $H \subset P^\vee/P$ a maximal isotropic group, $\gamma \in H^\perp$ and let $A_\gamma = \{Q(x+\gamma),\, x \in P\}$.*

(i) For $n \in \mathbb{Q}_{>0}$, the number $N(\gamma, n)$ of points $s \in S$ (counted with multiplicity) for which $\mathcal{X}_s$ admits a parabolic line bundle of type (γ, n) is zero if $n \notin A_\gamma$, otherwise it satisfies

$$N(\gamma, n) \sim \mu(S) \frac{(2\pi)^{\frac{22-\rho}{2}}.n^{10-\frac{\rho}{2}}}{\sqrt{|P^\vee/P|}\Gamma(\frac{22-\rho}{2})}. \prod_{p<\infty} \mu_p(n, \gamma, V)$$

as n tends to infinity in A_γ, and where $V = P^\perp$.
If S is projective, then the error term is $O_\epsilon(n^{\frac{2+b}{4}+\epsilon})$ for every $\epsilon > 0$.

(ii) The previous set is equidistributed in S with respect to μ.

(iii) If $P^\vee/P$ has no non-trivial isotropic subgroup, then the set of points $s \in S$ (counted with multiplicity) for which $\mathcal{X}_s$ admits an elliptic fibration of norm less than n is equidistributed with respect to μ as n tends to infinity.

For the definition of a parabolic line bundle of type (γ, n) and the norm of an elliptic fibration, we refer to Definition 2.4.4. If the Lorentzian sublattice P is generated by a single element, the corollary says that the number of elliptic surfaces of norm less than n^2 (or volume less than n) in a generic family of quasi-polarized K3 surfaces "grows like" n^{20}. In the case of twistor families of K3 surfaces, an analogous result was shown by Simion Filip in [48] and an improvement of the error term was given by Bergeron and Matheus in [13]. The main term there grows also like n^{20}, and Filip works with the full K3 lattice Λ_{K3}. His method is different from ours, although it was the starting point of this paper. Notice also the analogy between the coefficient of the main term in our case and in Filip's case. Indeed, due to the Siegel mass formula (see [47]), the product $\prod_{p<\infty} \mu_p(n, \gamma, P)$ can be expressed as a sum of volumes of some homogeneous spaces (compare to Filip's formula 3.1.6 in [48]). There is also a generalization of the previous corollary which concerns families of hyperkähler manifolds over a quasi-projective curve which we discuss of the section 4.3.

There are several arithmetic statements which shed light on the arithmetic analogues of the above results, the curve S being replaced by an open subset of the spectrum of the ring of integers of a number field. A result by Charles [31] shows that the set of primes where the reduction of two elliptic curves defined over a number field are geometrically isogenous is infinite. More recently, Shankar and Tang [98] proved by using similar techniques that, given a simple abelian surface defined over a number field and which has real multiplication, there are infinitely many places where its reduction is not absolutely simple.

2.1.1 Outline of the proof

Let us now sketch the proof of Theorem 2.1.1. Let D_V be the period domain associated to the quadratic lattice (V, Q), namely the complex analytic variety defined by

$$D_V = \{x \in \mathbb{P}(V_\mathbb{C}),\ (x.x) = 0,\ (x.\overline{x}) > 0\}.$$

Let Γ_V be the stable orthogonal group of V defined by

$$\Gamma_V := \mathrm{Ker}\left(\mathrm{O}(V) \to \mathrm{O}(V^\vee/V)\right),$$

where

$$V^\vee := \{x \in V_\mathbb{Q},\ \forall y \in V,\ (x.y) \in \mathbb{Z}\}$$

denotes as before the dual lattice of V. By Baily and Borel [6], the complex analytic quotient $\Gamma_V\backslash D_V$ can be endowed with a natural structure of quasi-projective variety called an *orthogonal modular variety.* It is the structure that we consider in the whole text. Let $\mathcal{L}$ denote the Hodge bundle on $\Gamma_V\backslash D_V$ and let ω be its first Chern class. Recall that $\mathcal{L}$ is an ample line bundle [6].

For $\gamma \in V^\vee/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ with $n > 0$, let $\mathcal{Z}(\gamma, -n)$ denote the associated Heegner divisor in $\Gamma_V\backslash D_V$ which parametrizes Hodge structures on V for which there exists a rational Hodge class $\lambda \in \gamma + V$ with $(\lambda.\lambda) = -2n$ (see Section 2.2.2 for the precise definition). Let $\{S, \mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q)\}$ be given as in Theorem 2.1.1. Since the local system $\mathbb{V}_\mathbb{Z}^\vee/\mathbb{V}_\mathbb{Z}$ is trivial, we have a corresponding holomorphic period map

$$\rho : S \to \Gamma_V\backslash D_V.$$

This map is in fact algebraic by Borel [19] [1]. The pullback of the Hodge bundle $\mathcal{L}$ along ρ is equal to $\mathcal{F}^2\mathcal{V}$. The idea of the proof is to obtain a global estimate of the cardinality (with multiplicity) of the set

$$\{s \in S,\ \exists\, \lambda \in \gamma + \mathbb{V}_{\mathbb{Z},s},\ (\lambda.\lambda) = -2n\}.$$

To this end, following ideas of Maulik [83], we use linear dependence relations between Heegner divisors to get an upper bound. These relations follow from Borcherds' construction in [17] of a modular form on the Picard group of the orthogonal modular variety $\Gamma_V\backslash D_V$. Then we extend those relations to a suitable toroidal compactification of $\Gamma_V\backslash D_V$. It is at this level that we need a restriction on γ being in $H^\perp$ for a maximal isotropic subgroup H of $V^\vee/V$, since for arbitrary γ, we don't know how to control the intersection of S with the boundary divisor of the given toroidal compactification.

To obtain a lower bound, we construct a suitable fibration over every small enough simply connected open subset $\Delta \subset S$. Then following ideas of Green (see [109, Chap.17]), we obtain a map to the homogeneous space $A_0 = \{x \in V_\mathbb{R},\ Q(x) = -1\}$. This map turns out to be, outside a Lebesgue negligible analytic subset, a local diffeomorphism. We use then a result of equidistribution of integral points on A_0 proven by Eskin–Oh in a more general context in [46, Th.1.2]. The proof of the latter relies on results from ergodic theory, namely the ergodicity of unipotent flows, which is also an important ingredient in the proof of the main result of [32].

1. In fact, in [19] the theorem is stated for smooth quotients but see [64, Remark 4.2] for how one can reduce to this case.

2.1.2 Outline of the paper

In section 2 we recall the construction of the Borcherds' modular form and its implications on the linear dependence relations between Heegner divisors following ideas of Maulik in [83]. We then explain how to extend those relations to the toroidal compactification of $\Gamma_V\backslash D_V$ determined by the perfect cone decomposition following the work of Peterson in [93]. This will allow us, under some mild assumptions, to give global estimates on the growth of the Hodge locus in a curve. We conjecture that these estimates still hold without those assumptions. In section 3 we construct a fibration in spheres over the small open subsets of S which, combined with equidistribution results of Eskin and Oh [46], allow to deduce a lower estimate on the cardinality of Hodge locus. In section 4 we explain how one can reduce to the case where the group $V^\vee/V$ has no non-trivial isotropic subgroup and then prove the result in this case. The end of the section is devoted to prove corollary 2.1.3.

2.1.3 Notations

If $f, g : \mathbb{N} \to \mathbb{R}$ are real functions and g does not vanish, then :

1. $f = O(g)$ if there exists an integer $n_0 \in \mathbb{N}$, a positive constant $C_0 > 0$ such that
$$\forall n \geq n_0,\ |f(n)| \leq C_0|g(n)|.$$
2. $f \asymp h$ if and $f = O(h)$ and $h = O(f)$.

2.2 The Weil representation and modular forms

2.2.1 General setting

We recall in this section some results about Weil representations and certain vector-valued modular forms associated to quadratic lattices. Our main references are [16] and [17].

Let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$: the elements of this group consist of pairs (M, ϕ), where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

and ϕ is a holomorphic function on the Poincaré upper half-plane $\mathbb{H}$ such that $\phi(\tau)^2 = c\tau + d$, $\tau \in \mathbb{H}$. The group structure is defined by

$$(M_1, \phi_1).(M_2, \phi_2) = (M_1 M_2, \tau \mapsto \phi_1(M_2.\tau)\phi_2(\tau)),$$

for $(M_1, \phi_1), (M_2, \phi_2) \in \mathrm{Mp}_2(\mathbb{R})$, where $M_2.\tau$ stands for the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ given by fractional linear transformations.

The map $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ given by $(M, \phi) \mapsto M$ is a double cover of $\mathrm{SL}_2(\mathbb{R})$. Denote by $\mathrm{Mp}_2(\mathbb{Z})$ the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under this map. It is well known (see [96, P.78]) that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the elements :

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau \mapsto \sqrt{\tau}\right).$$

Let $\rho : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be a finite-dimensional complex representation of $\mathrm{Mp}_2(\mathbb{Z})$ that factors through a finite quotient of $\mathrm{Mp}_2(\mathbb{Z})$ and let $k \in \frac{1}{2}\mathbb{Z}$. The group $\mathrm{Mp}_2(\mathbb{Z})$ has a right action on the space of functions $f : \mathbb{H} \to V$ given by

$$(f.(M,\phi)_k)(\tau) = \phi(\tau)^{-2k}\rho(M,\phi)^{-1}f(M.\tau). \tag{2.1}$$

Fix an eigenbasis $(v_\gamma)_{\gamma\in I}$ of V with respect to the action of T. A holomorphic function $f : \mathbb{H} \to V$ which is invariant under the action of T has a Fourier expansion

$$f(\tau) = \sum_{\gamma\in I}\sum_{n\in\mathbb{Q}} c(\gamma,n)e^{2i\pi n\tau}v_\gamma. \tag{2.2}$$

For $(\gamma, n) \in I \times \mathbb{Q}$, the coefficient $c(\gamma, n)$ is non-zero only if $e^{2i\pi n}$ is the eigenvalue of T acting on v_γ. The function f is said to be holomorphic at infinity if $c(\gamma, n) = 0$ for all $n < 0$ and $\gamma \in I$.

Definition 2.2.1. A holomorphic function $f : \mathbb{H} \to V$ is a ***modular form of weight*** k ***and type*** ρ, if it satisfies the following conditions :
(i) f is invariant under the action (2.1) of $\mathrm{Mp}_2(\mathbb{Z})$.
(ii) f is holomorphic at infinity.
Moreover, if $c(\gamma, 0) = 0$ for all $\gamma \in I$ in the formula (2.2), we say that f is a cusp form.

Let $M_k(\rho)$ denote the $\mathbb{C}$-vector space of modular forms of weight k and type ρ, and let $S_k(\rho)$ be the subspace of cusp forms. Both $M_k(\rho)$ and $S_k(\rho)$ are finite-dimensional vector spaces over $\mathbb{C}$ (see [17, Section 2]).

Let (V, Q) be an even lattice of signature (b^+, b^-) with the underlying non-degenerate symmetric bilinear form denoted by $(\,.\,)$ and such that $Q(x) = \frac{(x.x)}{2}$ for $x \in V$. Let $V^\vee$ be the dual lattice of V. We can associate to the quadratic lattice (V, Q) a representation ρ_V of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ whose underlying vector space is $\mathbb{C}[V^\vee/V]$. For this, it is enough to specify the action of S and T on a basis $(v_\gamma)_{\gamma\in V^\vee/V}$ of $\mathbb{C}[V^\vee/V]$ as follows :

$$\begin{aligned} \rho_V(T)v_\gamma &= e^{2i\pi Q(\gamma)}v_\gamma, \\ \rho_V(S)v_\gamma &= \frac{i^{\frac{b^- - b^+}{2}}}{\sqrt{|V^\vee/V|}}\sum_{\delta\in V^\vee/V} e^{-2i\pi(\gamma,\delta)}v_\delta, \end{aligned} \tag{2.3}$$

where $\gamma \in V^\vee/V$. We denote by ρ_V^* the dual representation of ρ_V.

Remark 2.2.2. By a result of McGraw [84, Prop. 5.6], the complex vector space $\mathrm{M}_k(\rho_V^*)$ has a rational structure $\mathrm{M}_k(\rho_V^*)_\mathbb{Q}$ given by modular forms with rational coefficients, and similarly for $S_k(\rho_V^*)$.

We present an example of a modular form which will be crucial for our later study.

Example 2.2.3. Assume V has signature $(2, b)$ where $b \geq 3$. There is an Eisenstein series E_V in $\mathrm{M}_k(\rho_V^*)$ whose Fourier expansion is given by (see [26, Prop.4]) :

$$E_V(\tau) = \sum_{\gamma \in V^\vee/V} \sum_{n \in -Q(\gamma)+\mathbb{Z}, n \geq 0} c(\gamma, n) q^n v_\gamma,$$

where $q = e^{2i\pi\tau}$, $\tau \in \mathbb{H}$, and the coefficients $c(\gamma, n)$ are given by :

$$\begin{cases} c(0,0) &= 2 \\ c(\gamma, n) &= -\frac{2^{2+\frac{b}{2}}.\pi^{1+\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}.\prod_p \mu_p(\gamma, n, V) \quad \text{for all}\, n > 0, \end{cases}$$

where the product is ranging over all primes p. The factors $\mu_p(\gamma, n, V)$ are defined as follows. For $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$ such that n is positive and a a positive integer, let

$$N(\gamma, n, L, a) = |\{\alpha \in L/aL,\, Q(\alpha + \gamma) + n \equiv 0 \pmod a\}|.$$

For a prime p, Siegel proves in [101, Hilfssatz 13] that for s sufficiently large, the value of $p^{-(1+b)s} N(\gamma, n, L, p^s)$ is independent of s and we define

$$\mu_p(\gamma, n, V) := \lim_{s \to \infty} p^{-(1+b)s} N(\gamma, n, V, p^s).$$

The infinite product $\prod_p \mu_p(\gamma, n, V)$ converges as long as every factor is different from zero. Since Q is indefinite of rank greater than 5, this is equivalent by Hasse-Minkowski theorem ([96, p.41]) to the equation $Q(\alpha) + n = 0$ having a solution α in $\gamma + V$. In this situation, we say that n satisfy *local congruence conditions* and by [22, Proposition 2] (see also [66, Section 11.5]), we have

$$\prod_p \mu_p(\gamma, n, V) \asymp 1.$$

Hence the estimate

$$c(\gamma, n) \asymp n^{\frac{b}{2}}.$$

The coefficients $c(\gamma, n)$ are rational numbers by [26, Proposition 14] so that $E_V \in \mathrm{M}_k(\rho_V^*)_{\mathbb{Q}}$.

2.2.2 Borcherds' modular form

In this section we introduce the Heegner divisors and state a modularity result of their generating series. This will allow us later to control the growth of their intersection with a curve S supporting a variation of Hodge structure. As before, let (V, Q) be an even quadratic lattice and assume henceforth that it has signature $(2, b)$ with $b \geq 3$. Let $\mathrm{O}(V)$ be the orthogonal group of V and Γ_V the subgroup of elements acting trivially on $V^\vee/V$. We refer to [83] and [64, Chapter 6] for more details on this section.

The period domain

Let D_V be the *period domain* associated to V, that is the complex analytic variety

$$D_V := \{w \in \mathbb{P}(V_{\mathbb{C}}),\, (w, w) = 0,\, (w, \overline{w}) > 0\}.$$

Let D_V^+ be one of the two connected components of D_V. Let G be the connected component of the identity of the real Lie group $\mathrm{O}(V_\mathbb{R})$, where $V_\mathbb{R} := V \otimes_\mathbb{Z} \mathbb{R}$ is endowed with the real extension of Q. The action of discrete subgroup $\Gamma_V^+ := \Gamma_V \cap G$ on D_V^+ is proper and totally discontinuous and the quotient $\Gamma_V^+ \backslash D_V^+$ has the structure of a quasi-projective variety with orbifold singularities by [6].

There is another realization of D_V as a Grassmanian. Let $\mathrm{Gr}(2, V_\mathbb{R})$ be the Grassmanian of planes of $V_\mathbb{R}$ and let $\mathrm{Gr}^+(2, V_\mathbb{R})$ be the open subset of positive definite planes. We have a natural split covering of degree 2

$$\begin{aligned} D_V &\longrightarrow \mathrm{Gr}^+(2, V_\mathbb{R}) \\ \omega = X + iY &\mapsto P = < X, Y >, \end{aligned}$$

where P is the oriented plane generated by X and Y. The restriction of the map above to D_V^+ is a diffeomorphism. Both of the previous descriptions of D_V^+ will be interchangeably used henceforth.

Heegner divisors

In this section, the main result from [17] is used to produce linear dependence relations between certain special divisors that will be defined hereafter.

For any vector $v \in V_\mathbb{R}$ such that $Q(v) < 0$, let $v^\perp$ be the set of planes in D_V^+ orthogonal to v. Let $\gamma \in V^\vee/V$ and $n \in Q(\gamma) + \mathbb{Z}$ with $n < 0$. The union of hyperplanes

$$\bigcup_{v \in \gamma + V,\, Q(v) = n} v^\perp$$

is locally finite, invariant under the action of Γ_V^+, and defines an algebraic divisor on $\Gamma_V^+ \backslash D_V^+$ given as

$$\mathcal{Z}(\gamma, n) := \Gamma_V^+ \backslash \left(\bigcup_{v \in \gamma + V,\, Q(v) = n} v^\perp \right).$$

In terms of Hodge structures, $\mathcal{Z}(\gamma, n)$ parametrizes Hodge structures on V for which there exists a rational Hodge class λ in $\gamma + V$ with $Q(\lambda) = n$.

The restriction of the tautological line bundle $\mathcal{O}(-1)$ to $D_V^+ \subset \mathbb{P}(V_\mathbb{C})$ admits a natural Γ_V^+-equivariant action and defines an algebraic line bundle $\mathcal{L} := \Gamma_V^+ \backslash \mathcal{O}(-1)$ on $\Gamma_V^+ \backslash D_V^+$ called the *Hodge bundle*. We define $\mathcal{Z}(0, 0)$ to be a divisor whose class is equal to the dual of the Hodge bundle.

Finally, we set $\mathcal{Z}(\gamma, n) = 0$ if $n > 0$ or if $n = 0$ and $\gamma \neq 0$. The $\mathcal{Z}(\gamma, n)$ are the *Heegner divisors*. They are Cartier divisors on $\Gamma_V^+ \backslash D_V^+$, and we denote by $[\mathcal{Z}(\gamma, n)]$ their associated class in $\mathrm{Pic}(\Gamma_V^+ \backslash D_V^+)$.

Consider the formal power series

$$\Phi_V(q) = \sum_{\substack{\gamma \in V^\vee/V \\ n \in -Q(\gamma) + \mathbb{Z}}} [\mathcal{Z}(\gamma, -n)] q^n v_\gamma \in \mathrm{Pic}(\Gamma_V^+ \backslash D_V^+)[[q^{\frac{1}{2d}}]] \otimes \mathbb{C}[V^\vee/V].$$

Here d is the order of $V^\vee/V$.

The following result is due to the work of Borcherds ([17]), combined with the refinement of McGraw (see remark 2.2.2) :

Theorem 2.2.4. $\Phi_V(q) \in \mathrm{Pic}(\Gamma_V^+\backslash D_V^+) \otimes \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)_{\mathbb{Q}}$.

We will follow ideas of Maulik in [83, Section 3] with some changes in order to translate the previous theorem in terms of linear dependence relations between the Heegner divisors. This will be achieved by writing Φ_V as a sum of a multiple of an Eisenstein series and a cusp from, then using standard bounds on the growth of coefficients of cusp forms.

By [23, p.27], for each γ in a set of representatives of the quotient of $V^\vee/V$ by the involution $x \mapsto -x$, there exists an Eisenstein series E_γ such that the following decomposition holds

$$\mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)_{\mathbb{Q}} = \bigoplus_{\gamma} \mathbb{C}.E_\gamma \oplus \mathrm{S}_{1+\frac{b}{2}}(\rho_V^*)_{\mathbb{Q}}$$

where $E_0 = E_V$ is the Eisenstein series from Example 2.2.3. Since the only $(\gamma, 0)$-coefficient of Φ_V which is non-zero is the one corresponding to $\gamma = 0$, there exists a finite set $\mathcal{I}$, a family $(\mathcal{Z}(\gamma_i, n_i))_{\in\mathcal{I}}$ of Heegner divisors and a family $(g_i)_{i\in\mathcal{I}}$ of cusp forms such that

$$\Phi_V = \frac{1}{2}[\mathcal{Z}(0,0)] \otimes E_V + \sum_{i\in\mathcal{I}} [\mathcal{Z}(\gamma_i, n_i)] \otimes g_i$$

For $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$, by identifying the (γ, n)-coefficient in the above expression we get

$$[\mathcal{Z}(\gamma, -n)] = \frac{1}{2} c(\gamma, n)[\mathcal{Z}(0,0)] + \sum_{i\in\mathcal{I}} g_i(\gamma, n)[\mathcal{Z}(\gamma_i, n_i)]. \qquad (2.4)$$

Notice that all the coefficients in (2.4) are rational numbers. For a cusp form f, the trivial bounds on the order of growth of its coefficients (see [94, Prop. 1.5.5]) say that

$$|a_{\gamma,n}(f)| \leq C_{\epsilon,f} n^{\frac{2+b}{4}+\epsilon},$$

for all $\epsilon > 0$, some constant $C_{\epsilon,f} > 0$, and for all $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n \geq 0$.

Hence, we can find a constant $C_\epsilon > 0$ such that for all $i \in \mathcal{I}$, n and γ as before, we have

$$|g_i(\gamma, n)| \leq C_\epsilon n^{\frac{2+b}{4}+\epsilon}.$$

Taking into account relation (2.4) and the expression in Example 2.2.3, we get :

Proposition 2.2.5. *For every $\epsilon > 0$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$, the following estimate holds in* $\mathrm{Pic}(\Gamma_V\backslash D_V)_{\mathbb{Q}}$

$$[\mathcal{Z}(\gamma, -n)] = -\frac{(2\pi)^{1+\frac{b}{2}} n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\,\Gamma(1+\frac{b}{2})} \prod_p \mu_p(\gamma, n, V)[\mathcal{Z}(0,0)] + O_\epsilon(n^{\frac{2+b}{4}+\epsilon}).$$

The above proposition is a quantitative version of Lemma 3.7 in [83].

2.2.3 Extension to a toroidal compactification

The goal in this section is to extend the estimate in Proposition 2.2.5 to a well chosen toroidal compactification of $\Gamma_V\backslash D_V$. This will allow us to control their growth in cohomology and the growth of their intersection with any curve as in Theorem 2.1.1. We first start by recalling the construction of the Baily-Borel compactification of $\Gamma_V\backslash D_V$. For a short summary in the case of orthogonal modular varieties, see [55] which we follow closely, or [20, Part III] for the general case.

Baily-Borel compactification

There is a "minimal" compactification of $\Gamma_V\backslash D_V$ constructed by Baily and Borel in [6] and which proceeds by adding rational boundary components and then showing that the resulting space is a projective algebraic variety.

The rational boundary components correspond precisely to maximal rational parabolic subgroups of G, which in turn are the stabilizers of totally isotropic subspaces of $V_{\mathbb{Q}}$. Since Q has signature $(2,b)$, such spaces have dimension 1 or 2. Hence, we obtain the following description :

$$(\Gamma_V^+\backslash D_V^+)^{BB} = \Gamma_V^+\backslash D_V^+ \sqcup \bigsqcup_{\Pi} X_\Pi \sqcup \bigsqcup_{\ell} Q_\ell.$$

where ℓ and Π run through representatives of the finitely many Γ_V^+-orbits of isotropic lines and isotropic planes in $V_{\mathbb{Q}}$. Each X_Π is a modular curve, and Q_ℓ is a point. They are also known as 1-*cusps* and 0-*cusps* respectively.

Extension of the relations between Heegner divisors

The boundary of the Baily-Borel can be singular and the Zariski closure of the Heegner divisors may not be Cartier. To solve this problem, we extend the relation (2.4) to a well-chosen toroidal compactification of $\Gamma_V\backslash D_V$. We work with the toroidal compactification considered in [93, Section 5.2] and which is given by the perfect cone decomposition. We denote it by $\overline{\Gamma_V\backslash D_V}^{tor}$. Above each cusp determined by an isotropic subspace I of V, the boundary divisors are determined by the one dimensional rays in the $\mathrm{Stab}(I)$-invariant decomposition of the positive cone of $I^\perp/I$ and in this situation they lie in its boundary. Hence above every 1-cusp F there is only one irreducible Cartier boundary divisor Δ_F and there are no other boundary divisors. Also the closure $\overline{\mathcal{Z}(\gamma,n)}$ of a Heegner divisor $\mathcal{Z}(\gamma,n)$ is Cartier for all $\gamma\in V^\vee/V$ and $n\in Q(\gamma)+\mathbb{Z}$. For more details, see [93, 5.2.4]. The rest of the section is devoted to bound the coefficients of the boundary divisors in some particular cases. We start first by recalling Peterson's results in our context, especially Theorem 5.3.3 in [93].

Let I be an isotropic primitive plane of V, F the associated 1-cusp. The isomorphism class of the definite lattice $I^\perp/I$ depends only on the cusp F. We denote it by K_F and let Θ_F be the associated theta function, i.e the function defined by

$$\Theta_F(\tau) = \sum_{\gamma\in K_F^\vee/K_F}\ \sum_{x\in\gamma+K_F} q^{-Q(x)} v_\gamma,\ q = e^{2i\pi\tau},\ \tau\in\mathbb{H},$$

where $(v_\gamma)_{\gamma\in K_F^\vee/K_F}$ is the standard basis of $\mathbb{C}[K_F^\vee/K_F]$.

Let $I^{\#} = I_{\mathbb{Q}} \cap V^{\vee}$. Following [21, 4.1], I is said to be *strongly primitive* if $I^{\#} = I$. The cardinality N_F of the finite group $H_I = I^{\#}/I$ depends only on F and is called the imprimitivity of F. Let $H_I^{\perp} := \{x \in L^{\vee}/L, \forall y \in H_I, (x, y) = 0\}$.

Proposition 2.2.6. *Let $I \subset V$ a primitive isotropic plane. Then*
(i) $H_I^{\perp}/H_I \simeq K_F^{\vee}/K_F$ as quadratic finite modules.
(ii) $|V^{\vee}/V| = |K_F^{\vee}/K_F|.N_F^2$.

Proof. Assertion (i) follows from Lemma page 77 in [21]. For (ii), notice that $H_I^{\perp} \simeq \{\ell \in \mathrm{Hom}(V^{\vee}/V, \mathbb{Q}/\mathbb{Z}), \ell_{/H_I} = 0\}$ and that the cardinality of the latter is equal to $\frac{|V^{\vee}/V|}{N_F}$. □

Let $p : H_I^{\perp} \to K_F^{\vee}/K_F$ be the composite of the projection $H_I \to H_I^{\perp}/H_I$ followed by the isomorphism (i) from the last proposition. By construction, it is a morphism of quadratic finite modules. We have an induced map $p^* : \mathbb{C}[K_F^{\vee}/K_F] \to \mathbb{C}[V^{\vee}/V]$ which maps an element v_γ, $\gamma \in K_F^{\vee}/K_F$, to

$$p^* v_\gamma = \sum_{\substack{\delta \in H_I^{\perp} \\ p(\delta) = \gamma}} v_\delta.$$

Using (ii) from the previous proposition, it is straightforward that p^* commutes with the action of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ given by the Weil representation as in Section 2.2.1 Equation (2.3). Hence, for any $k \in \frac{1}{2}\mathbb{Z}$, we have a map

$$p^* : \mathrm{M}_k(\rho_{K_F}^*)_{\mathbb{Q}} \to \mathrm{M}_k(\rho_V^*)_{\mathbb{Q}}.$$

For $\gamma \in V^{\vee}/V$, $n \in -Q(\gamma) + \mathbb{Z}$, let

$$a(\gamma, n, F) = \frac{N_F}{24}(E_2.p^*(\Theta_F))(\gamma, n),$$

where $E_2(\tau) = 1 - 24\sum_{n \geq 1} \sigma_1(n) q^n$, $q = e^{2i\pi\tau}$, $\tau \in \mathbb{H}$, is the weight 2 Eisenstein series.

The following result is an application of Theorem 5.3.3 in [93] to formula (2.4)

Proposition 2.2.7. *Let $\gamma \in V^{\vee}/V$, $n \in -Q(\gamma) + \mathbb{Z}$. Then we have the following linear equivalence relations in* $\mathrm{Pic}(\overline{\Gamma_V \backslash D_V}^{tor})$

$$\begin{aligned}[\overline{\mathcal{Z}(\gamma, -n)}] = \frac{c(\gamma, n)}{2}[\overline{\mathcal{Z}(0,0)}] + \sum_{F \in S_1} u(\gamma, n, F)\Delta_F \\ + \sum_{i \in \mathcal{I}} g_i(\gamma, n)[\overline{\mathcal{Z}(\gamma_i, n_i)}] + \sum_{i \in \mathcal{I}} \sum_{F \in S_1} g_i(\gamma, n) a(\gamma_i, n_i, F)\Delta_F,\end{aligned} \tag{2.5}$$

where

$$u(\gamma, n, F) = \frac{c(\gamma, n)}{2} a(0, 0, F) - a(\gamma, n, F),$$

and the coefficients $c(\gamma, n)$ are defined in 2.2.3.

Taking into account the estimates preceding Proposition 2.2.5, we get

Proposition 2.2.8. *For every $\epsilon > 0$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$, we have :*

$$[\mathcal{Z}(\gamma, -n)] = -\frac{(2\pi)^{1+\frac{b}{2}} n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})} \prod_p \mu_p(\gamma, n, V)[\mathcal{Z}(0,0)] + \sum_{F \in S_1} u(\gamma, n, F)\Delta_F + O_\epsilon(n^{\frac{2+b}{4}+\epsilon}),$$

in $\mathrm{Pic}(\overline{\Gamma_V \backslash D_V}^{tor})_{\mathbb{Q}}$.

Remark 2.2.9. The term $u(\gamma, n, F)$ can a priori be as large as $c(\gamma, n)$. However, when F is strongly primitive, Lemma 2.2.11 shows that $c(\gamma, n)$ cancels because of the term $a(\gamma, n, F)$, hence giving a sharper control on the growth of $u(\gamma, n, F)$.

2.2.4 Some consequences

We turn now to the consequences of the previous proposition on the distribution of Hodge loci in 1-dimensional variation of Hodge structure. Let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a simple, non trivial, polarized variation of Hodge structure over a complex quasi-projective curve S such that the local system $\mathbb{V}_{\mathbb{Z}}^\vee/\mathbb{V}_{\mathbb{Z}}$ is trivial. Let $\rho : S \to \Gamma_V \backslash D_V$ be the corresponding period map. Let $\overline{S}$ be a smooth compactification of S such that the following diagram is commutative

$$\begin{array}{ccc} S & \xrightarrow{\rho} & \Gamma_V \backslash D_V \\ \downarrow & & \downarrow \\ \overline{S} & \xrightarrow{\overline{\rho}} & \overline{\Gamma_V \backslash D_V}^{tor} \end{array}$$

Let $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ such that $n > 0$. Since the variation is assumed to be simple, we can express the degree of the divisor $\overline{\rho}^*\overline{\mathcal{Z}(\gamma, -n)}$ on $\overline{S}$ as follows :

$$\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma, -n)}) = \sum_{s \in \overline{S}} \mathrm{ord}_s(\overline{\rho}^*\overline{\mathcal{Z}(\gamma, -n)}),$$

where $\mathrm{ord}_s(\overline{\rho}^*\overline{\mathcal{Z}(\gamma, -n)})$ is the multiplicity of the intersection of $\overline{S}$ with $\overline{\mathcal{Z}(\gamma, -n)}$ at a point $s \in \overline{S}$.

Notice that $\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(0,0)}) = -\mu(S)$, where μ is the finite measure on S given by integration of the first Chern class of $\mathcal{F}^2\mathcal{V}$. By Proposition 2.2.8 we have :

Corollary 2.2.10. *For every $\epsilon > 0$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$, we have :*

$$\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma, -n)}) = \frac{(2\pi)^{1+\frac{b}{2}} n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})} \prod_p \mu_p(\gamma, n, V)\mu(S) + \sum_{F \in S_1} u(\gamma, n, F) \deg_{\overline{S}}(\overline{\rho}^*\Delta_F) + O_\epsilon(n^{\frac{2+b}{4}+\epsilon}).$$

Assume now that $\deg_{\overline{S}}(\overline{\rho}^*\Delta_F) = 0$ if F corresponds to a totally isotropic plane which is not strongly primitive. The following lemma gives a control on the coefficient $u(\gamma, n, F)$ when F is associated to a strongly primitive totally isotropic plane.

Lemma 2.2.11. *Let $\gamma \in V^\vee/V$, I an isotropic, strongly primitive plane of V and F the associated 1-cusp. Then for all $\epsilon > 0$ we have the following estimate :*

$$|u(\gamma, n, F)| = O_\epsilon(n^{\frac{b}{2}-1+\epsilon})$$

Proof. Let $\mathrm{M}_k^{\leq s}(\rho_V^*)$ be the vector space of vector-valued quasi-modular from of weight k and depth less than s (see [65, Definition 1] and [82, Section 17.1] for definitions and properties of quasi-modular forms). Let D be the derivation operator $q\frac{d}{dq}$. Then we have the following structure theorem

$$\mathrm{M}_{1+\frac{b}{2}}^{\leq 1}(\rho_V^*) = \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*) \oplus D(\mathrm{M}_{\frac{b}{2}-1}(\rho_V^*)).$$

For a proof, we refer to [82, Section 17.1] where it is proven for scalar quasi-modular forms, but the reader may notice that the proof generalizes easily to vector-valued quasi-modular forms.

The product $E_2.p^*(\Theta_F)$ is an element of $\mathrm{M}_{1+\frac{b}{2}}^{\leq 1}(\rho_V^*)$, hence we can write

$$E_2.p^*(\Theta_F) = \sum_i \alpha_i E_L^i + g + D(\tilde{g}), \tag{2.6}$$

where g is a cusp form of weight $1+\frac{b}{2}$, $(E_L^i)_i$ is a basis of Eisenstein series of $\mathrm{M}_{\frac{b}{2}+1}(\rho_V^*)$ with $E_L^0 = E_L$ and $\tilde{g} \in \mathrm{M}_{\frac{b}{2}-1}(\rho_V^*)$. By comparing the constant coefficients, we get $\alpha_0 = \frac{1}{2}$ and $\alpha_i = 0$ for $i \neq 0$, since I is strongly primitive. Hence for $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$ with $n \geq 0$, we have

$$(E_2.p^*(\Theta_F))(\gamma, n) = \frac{c(\gamma, n)}{2} + g(\gamma, n) + n\tilde{g}(\gamma, n).$$

Since $\tilde{g}$ is a modular form of weight $\frac{b}{2} - 1$, we have $\tilde{g}(\gamma, n) = O_\epsilon(n^{\frac{b}{2}-2+\epsilon})$ for all $\epsilon > 0$. Also g is a cusp form and by (see [94, Prop. 1.5.5]) $|a_{\gamma,n}(f)| \leq C_{\epsilon,f} n^{\frac{2+b}{4}+\epsilon}$. Combining these estimates we get the desired result. □

Remark 2.2.12. If I is not strongly primitive, then for $\gamma \notin H_I^\perp$, we have $u(\gamma, n, F) = c(\gamma, n)$, so the estimate in Lemma 2.2.11 fails. Even for $\gamma \in H_I^\perp$, all the Eisenstein series E_δ for $\delta \in H_I$ appear in the decomposition (2.6) with non-zero coefficients, so again Lemma 2.2.11 fails.

In view of the previous lemma, Corollary 2.2.10 rewrites

Corollary 2.2.13. *If $\overline{S}$ only meets the boundary of $\overline{\Gamma_V \backslash D_V}^{tor}$ in divisors Δ_F corresponding to strongly primitive totally isotropic planes, then for every $\epsilon > 0$ we have*

$$\deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma, -n)}) = \mu(S)\frac{(2\pi)^{1+\frac{b}{2}} n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})} \prod_p \mu_p(\gamma, n, V) + O_\epsilon(n^{u+\epsilon}),$$

for $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$ satisfying local congruence conditions of Example 2.2.3 and $u = \max(\frac{b}{2} - 1, \frac{2+b}{4})$. If S is projective, then we can choose $u = \frac{2+b}{4}$

Remark 2.2.14. In the case where the discriminant of (V, Q) is square free, all the primitive isotropic planes are strongly primitive by Proposition 2.2.6(ii), so the estimate 2.2.11 holds for all the coefficients $u(\gamma, n, F)$ for $\gamma \in V^\vee/V$, $n \in -Q(\gamma) + V$. The condition on the curve S in 2.2.13 is then automatically satisfied. Notice that here the control on the error term is sharper than the one in Theorem 2.1.1. This is because we don't know how to bound the intersection of $\overline{S}$ and $\overline{\mathcal{Z}(\gamma, -n)}$ at the boundary points, see Remark 2.2.12. However we conjecture that $|S \cap \mathcal{Z}(\gamma, -n)|_{mult}$ grows as the main term in the corollary.

2.3 Equidistribution in orthogonal modular varieties

The main goal of this section is to prove Proposition 2.3.8 which gives a lower estimate on the growth of the Hodge locus. The results in this section are independent from those in section 2.2.

2.3.1 Construction of a local map

Let U be a connected complex manifold and let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be an integral, polarized variation of Hodge structure of weight 2 over U with $h^{2,0} = 1$. Assume that the fiber of $(\mathbb{V}_{\mathbb{Z}}, Q)$ at a point u_0 (hence at all points of U) is isomorphic to a quadratic even lattice (V, Q) of signature $(2, b)$ as in Section 2.2.2 and assume also that the local system $\mathbb{V}_{\mathbb{Z}}^\vee/\mathbb{V}_{\mathbb{Z}}$ is trivial. It follows that the monodromy representation factors through Γ_V, the stable orthogonal group of (V, Q). Let $\rho : U \to \Gamma_L \backslash D_V$ be the corresponding period map. We will construct in this section a sphere bundle over U that keep track of Hodge classes and a map from the latter to the quadric $A_0 = \{x \in V_{\mathbb{R}},\ Q(x) = -1\}$.

The line bundle $\mathcal{F}^2\mathcal{V}$ is simply the pullback of the Hodge bundle $\mathcal{L}$ via ρ. Let $\mathcal{V}_{\mathbb{R}}$ be the real vector bundle whose sheaf of differentiable sections is equal to $\mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Q}} \mathcal{C}^\infty_{\mathbb{R}}$, where $\mathcal{C}^\infty_{\mathbb{R}}$ is the sheaf of $\mathcal{C}^\infty$ real-valued functions on U. The fiber at a point $u \in U$ of $\mathcal{V}_{\mathbb{R}}$ is isomorphic to $V_{\mathbb{R}}$. This vector bundle contains a sub-vector bundle that we shall note $\mathcal{V}_{\mathbb{R}}^{1,1}$ and whose sheaf of differentiable sections is

$$\mathcal{F}^1\mathcal{V} \otimes \mathcal{C}^\infty_{\mathbb{C}} \cap \mathbb{V}_{\mathbb{Z}} \otimes \mathcal{C}^\infty_{\mathbb{R}}.$$

Let $\mathcal{V}^{1,1} := \mathcal{F}^1\mathcal{V}/\mathcal{F}^2\mathcal{V}$. Then $\mathcal{V}_{\mathbb{R}}^{1,1}$ is the real part of $\mathcal{V}^{1,1}$, i.e the fiber at each point u is equal to $\mathcal{V}_u^{1,1} \cap V_{\mathbb{R}}$.

Assume that U is simply connected. Parallel transport by the Gauss-Manin connection trivializes the vector bundle $\mathcal{V}_{\mathbb{R}}$, hence it is isomorphic to $U \times V_{\mathbb{R}}$ and this isomorphism preserves the intersection form. Thus one has the commutative diagram

$$\begin{array}{ccc} \mathcal{V}_{\mathbb{R}}^{1,1} & \hookrightarrow & U \times V_{\mathbb{R}} \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Projecting forward to $V_{\mathbb{R}}$, we get the parallel transport map :

$$\Xi : \mathcal{V}_{\mathbb{R}}^{1,1} \to V_{\mathbb{R}}.$$

The locus where this map is not submersive were studied in [109, 17.3.4] and goes back to Griffiths and Green. Let us recall the setting and the main result. By Griffiths' transversality, the integrable connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega_U^1$$

induces a $\mathcal{O}_U$-linear map :

$$\overline{\nabla} : \mathcal{F}^1\mathcal{V}/\mathcal{F}^2\mathcal{V} \to \mathcal{F}^0\mathcal{V}/\mathcal{F}^1\mathcal{V} \otimes \Omega_U^1$$

Let $u \in U$, then taking the fibers at u induce a $\mathbb{C}$-linear map

$$\overline{\nabla}_u : \mathcal{V}_u^{1,1} \to \mathcal{V}_u^{0,2} \otimes \Omega_{U,u}^1$$

Then we have the following lemma, due to Green (see Lemma 17.21 from [109]).

Lemma 2.3.1. *Let $u \in U$, $\lambda \in \mathcal{V}_u^{1,1}$. If the map*

$$\overline{\nabla}_u(\lambda) : T_uU \to \mathcal{V}_u^{0,2}$$

is surjective then Ξ is submersive at (u, λ).

Consider the fibration over U defined by

$$\mathcal{S}_U = \{(u, \lambda),\, u \in U,\, \lambda \in \mathcal{V}_{u,\mathbb{R}}^{1,1},\, Q(\lambda) = -1\} \to U.$$

For every $u \in U$, the restriction of the quadratic form Q to $\mathcal{V}_{u,\mathbb{R}}^{1,1}$ is negative definite, and the fiber $\mathcal{S}_{U,u}$ is thus a $(b-1)$-dimensional sphere. By restriction of Ξ, we get a map :

$$\phi : \mathcal{S}_U \to A_0,$$

where $A_0 = \{x \in V_{\mathbb{R}},\, Q(x) = -1\}$.

Lemma 2.3.2. *Let $u \in U$, $\lambda \in \mathcal{V}_u^{1,1}$ such that $Q(\lambda) = -1$. If the map*

$$\overline{\nabla}_u(\lambda) : T_uU \to \mathcal{V}_u^{0,2}$$

is surjective then ϕ is submersive at (u, λ).

Proof. Let u and λ be as in the statement. The following diagram is commutative

$$\begin{array}{ccccc}
T_{(u,\lambda)}\mathcal{S}_U & \longrightarrow & T_{(u,\lambda)}\mathcal{V}_{\mathbb{R}}^{1,1} & \xrightarrow{d_{(u,\lambda)}(Q \circ \Xi)} & \mathbb{R} \\
\downarrow {\scriptstyle d_{(u,\lambda)}\phi} & & \downarrow {\scriptstyle d_{(u,\lambda)}\Xi} & & \downarrow \\
T_\lambda A_0 & \longrightarrow & T_\lambda V_{\mathbb{R}} & \xrightarrow{d_\lambda Q} & \mathbb{R}
\end{array}$$

The rows are exact by construction. By Lemma 2.3.1, $d_{(u,\lambda)}\Xi$ is surjective . Hence the map $d_{(u,\lambda)}\phi$ is surjective which proves the lemma. □

If $\rho(U)$ is not a point, then for $u \in U$ outside the locus where the differential of ρ is identically zero, there exists $\lambda \in \mathcal{S}_{U,u}$ which satisfies the condition of 2.3.2. Hence, the image $\mathrm{Im}(\phi)$ is open around λ. In particular, the set of points of U for which $\mathcal{V}_u^{1,1}$ contain an extra rational Hodge class x with $Q(x) = -1$ is dense (see [109, Proposition 17.20] and [90, Theorem 1.1] for a proof without the norm condition imposed by Q).

Lemma 2.3.2 shows that on order to study the distribution of the Hodge locus in U, one can first study the distribution of points λ in A_0 for which there exists $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{N}$ such that $\sqrt{n}\lambda \in \gamma + V$, since the locus where ϕ is not submersive is a proper real analytic subset of $\mathcal{S}_U$. Hence it is negligible from a measure-theoretic perspective. This will be explained in the following section.

2.3.2 Eskin-Oh's equidistribution result

The study of the distribution of Hodge locus in U amounts via the map ϕ constructed above to the study of radial projections of integral points of $V_\mathbb{R}$ on A_0. We will need thus to understand the distribution in A_0 of the set $\{\lambda \in A_0,\ \sqrt{n}\lambda \in \gamma+V, \}$ for $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + V$ with $n > 0$. This a is a well studied problem and can be dealt with using Hardy-Littlewood's circle method (see [108]). The results we present here follow [46] and [91] to which we refer for more details. Recall that $G = \mathrm{O}(V_\mathbb{R})^+$ is the connected component of the identity of the real Lie group $\mathrm{O}(V_\mathbb{R})$.

Let μ_∞ be the G-invariant measure on A_0 defined in the following way : take W an open subset of $V_\mathbb{R}$ and let

$$\mu_\infty(W \cap A_0) = \lim_{\epsilon \to 0} \frac{\mathrm{Leb}\left(\{x \in W,\ |Q(x) + 1| < \epsilon\}\right)}{2\epsilon}.$$

Here Leb is the Lebesgue measure on $V_\mathbb{R}$ for which the lattice V is of covolume 1. We can now state the main result of this section which is an application of Theorem 1.2 in [46] (see also [91, Section 5]) and the Siegel mass formula [46, (1.6)].

Proposition 2.3.3. *Let Ω be a compact subset of A_0 with zero measure boundary, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$. Then*

$$|\{\lambda \in \gamma + V,\ \frac{1}{\sqrt{n}}\lambda \in \Omega\}| \sim \mu_\infty(\Omega).n^{\frac{b}{2}}.\prod_p \mu_p(\gamma, n, V),$$

as $n \to +\infty$.

Proof. To see how Theorem 1.2 from [46] can be applied to our situation, we refer to the proof of Theorem 6.1 in *loc. cit.*. The only difference is that here we don't restrict to fundamental discriminants so we need to check that condition (1.3) in [46] holds. In other words, we need to know that for each n_0, there is only finitely many n such that

$$\frac{1}{\sqrt{n}}(\gamma + V) \cap A_0 = \frac{1}{\sqrt{n_0}}(\gamma + V) \cap A_0. \tag{2.7}$$

For a given n, notice that if (2.7) holds, then

$$\mathscr{Z}(\gamma, -n) = \mathscr{Z}(\gamma, -n_0),$$

so by Corollary 2.2.10 this can be true only for finitely many n. $\square$

The G-invariant measure μ_∞ can be recovered as integration of a G-invariant volume form on A_0. Indeed, the group G acts transitively on A_0 and the choice of an element ξ in A_0 determines a surjective map

$$\begin{aligned} \pi_\xi : G &\longrightarrow A_0 \\ g &\mapsto g.\xi. \end{aligned} \tag{2.8}$$

Let H be the stabilizer of ξ. The induced map $G/H \to A_0$ is a diffeomorphism giving A_0 the structure of a symmetric space. Let $\mathfrak{g}_0$ and $\mathfrak{h}_0$ be the Lie algebras of G and H respectively. Then $\mathfrak{g}_0/\mathfrak{h}_0$ is isomorphic to the tangent space of A_0 at ξ via the differential of π_ξ at the identity of G. The space of G-invariant volume forms on A_0 is then identified with $\bigwedge^{b+1}(\mathfrak{g}_0/\mathfrak{h}_0)^\vee$.

Let $(e_1, e_2, \xi_1, \dots, \xi_b)$ be an orthogonal basis of $V_\mathbb{R}$ such that for $i = 1, 2$ and $j = 1, \dots, b$ we have $Q(e_i) = -Q(\xi_j) = 1$. Let ω_{A_0} be the unique G-invariant volume form on A_0 such that

$$\omega_{A_0,\xi_1} = de_1 \wedge de_2 \wedge d\xi_2 \wedge \cdots \wedge d\xi_b \tag{2.9}$$

in $\bigwedge^{b+1}(T_{\xi_1}A_0)^\vee$. Let μ_{A_0} be the G-invariant measure on A_0 given by integration of ω_{A_0}. We have then the following proposition :

Proposition 2.3.4. *For every open subset W of A_0, we have*

$$\mu_\infty(W) = \frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}\mu_{A_0}(W)$$

Proof. It is enough to prove the equality for W open subset of A_0 containing ξ_1. There exists an open subset U^{b+1} in $\mathbb{R}^{b+1}$ containing 0 such that the map

$$\begin{aligned} U^{b+1} &\to A_0 \\ (x_1, x_2, y_2, \dots, y_b) &\mapsto (x_1, x_2, \sqrt{x_1^2 + x_2^2 - y_2^2 - \cdots - y_b^2 + 1}, y_2, \dots, y_b). \end{aligned}$$

is a local chart around ξ_1. Let W be its image. For $\epsilon > 0$, the image of the map

$$\begin{aligned} U^{b+1} \times]-\epsilon, \epsilon[&\to A_0 \\ (x_1, x_2, y_2, \dots, y_b, r) &\mapsto (x_1, x_2, \sqrt{x_1^2 + x_2^2 - y_2^2 - \cdots - y_b^2 + 1 + r}, y_2, \dots, y_b). \end{aligned}$$

defines a tubular neighborhood W_ϵ of W in $\mathbb{R}^{b+2}$ and one can check that

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{W_\epsilon} \omega = \frac{1}{2}\mu_{A_0}(W),$$

where $\omega = de_1 \wedge de_2 \wedge d\xi_1 \wedge \cdots \wedge d\xi_b$ and $A_\epsilon = \{x \in V_{\mathbb{R}},\ |Q(x)+1| < \epsilon\}$. By change of variable, we have

$$\mathrm{Leb}(\{x \in W_\epsilon, |Q(x)+1| < \epsilon\}) = \frac{2^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}} \int_{W_\epsilon} \omega.$$

Hence

$$\begin{aligned}\mu_\infty(W) &= \lim_{\epsilon \to 0} \frac{\mathrm{Leb}(\{x \in W, |Q(x)+1| < \epsilon\})}{2\epsilon} \\ &= \frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}} \mu_{A_0}(W)\end{aligned}$$

which proves the lemma. □

Corollary 2.3.5. *Let Ω be a compact subset of A_0 with zero measure boundary, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$. Then*

$$|\{\lambda \in \gamma + V,\ \frac{1}{\sqrt{n}}\lambda \in \Omega\}| \sim \mu_{A_0}(\Omega).\frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.n^{\frac{b}{2}}.\prod_p \mu_p(\gamma, n, V),$$

as $n \to +\infty$.

2.3.3 Quantitative study of the Hodge locus

The goal of this section is to put together results from the previous sections in order to prove Proposition 2.3.8 which gives a lower bound on the cardinality of the Hodge locus. Let $\{\mathbb{V}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ be a non-trivial, polarized, simple variation of Hodge structure over a complex quasi-projective curve S and let $\rho : S \to \Gamma_V^+\backslash D_V^+$ be the associated period map.

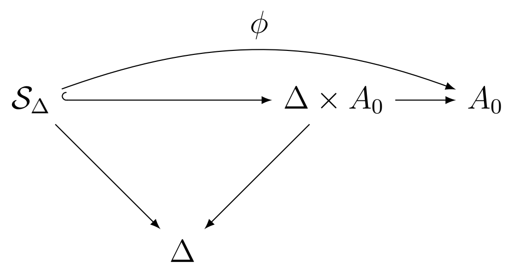
Recall that the Chern class ω of the Hodge bundle $\mathcal{F}^2\mathcal{V}$ defines a volume form on S . For any open subset $\Delta \subset S$, we note $\mu(\Delta) = \int_\Delta \omega$. Let Δ be an open simply connected subset of S. The restriction of ρ to Δ lifts to D_V^+. Let $0 \in \Delta$ be a point in Δ and P_0 the positive definite plane associated to $\rho(0)$. Then P_0 defines a maximal compact subroup $K := \mathrm{SO}(P_0) \times \mathrm{SO}(P_0^\perp)$ of G and a diffeomorphism

$$\begin{aligned}\pi :\ G/K &\to D_V^+ \\ g &\mapsto g.P_0\end{aligned}$$

We constructed in the previous paragraph a map

$$\phi :\ \mathcal{S}_\Delta \to A_0$$

where $\mathcal{S}_\Delta$ is a sphere bundle over Δ that fits into the following commutative diagram



For any $U \subset S$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ with $n > 0$, let

$$|U \cap \mathcal{Z}(\gamma, -n)|_{mult} = \sum_{s \in U \cap \mathcal{Z}(\gamma,-n)} m(s, \gamma, n),$$

where $m(s, \gamma, n) = |\{\lambda \in \mathcal{S}_{\Delta,s},\ \sqrt{n}\lambda \in \gamma + V\}|$.

Lemma 2.3.6. *Let $s \in S$, $\gamma \in V^\vee/V$ and $n \in -Q(\gamma) + \mathbb{Z}$ such that $n > 0$. Then*

$$m(s, \gamma, n) \leq \operatorname{ord}_s(\rho^* \mathcal{Z}(\gamma, n)).$$

Proof. Let γ, n and s as in the statement of the proposition. Assume that

$$\{\lambda \in \mathcal{S}_{\Delta,s},\ \sqrt{n}\lambda \in \gamma + V\} = \{\lambda_1, \dots, \lambda_k\},$$

where $k = m(s, \gamma, n)$. There exists a finite index congruence subgroup Γ of Γ_V^+ such that the orbits $\Gamma.\lambda_1, \dots, \Gamma.\lambda_k$ are pairwise disjoints. In particular, the divisor

$$\mathcal{Z}' := \Gamma \backslash \left(\bigcup_{\substack{\lambda \in \gamma + V \\ Q(\lambda) = -n}} \lambda^\perp \right) \subset \Gamma \backslash D_V^+$$

has at least k irreducible components. The kernel of the morphism $\pi_1(S) \to \Gamma_V/\Gamma$ defines a finite étale cover $S' \xrightarrow{\iota'} S$ and we have a commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\rho'} & \Gamma \backslash D_V^+ \\ \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota} \\ S & \xrightarrow{\rho} & \Gamma_V^+ \backslash D_V^+ \end{array}$$

Remark that $\iota^* \mathcal{Z}(\gamma, n)$ is equal to $\mathcal{Z}'$. Let $s' \in S'$ such that $\iota'(s') = s$. Then $\operatorname{ord}_{s'}(\rho'^* \iota^* \mathcal{Z}(\gamma, n)) \geq k$, since $\mathcal{Z}'$ has at least k irreducible components. By commutativity of the diagram above,

$$\rho'^* \iota^* \mathcal{Z}(\gamma, n) = \iota'^* \rho^* \mathcal{Z}(\gamma, n).$$

Since ι' is étale, we have

$$\operatorname{ord}_{s'}(\iota'^* \rho^* \mathcal{Z}(\gamma, n)) = \operatorname{ord}_s(\rho^* \mathcal{Z}(\gamma, n)),$$

which yields the desired result. □

Remark 2.3.7. We only have an inequality here because the curve S' may have intersection multiplicity strictly greater than one with a given irreducible component of the Heegner divisor Z'. In fact Theorem 2.1.1 implies that this does not happen when n is sufficiently large.

Proposition 2.3.8. *Let $\gamma \in V^\vee/V$. For all $s \in S$, there exists a simply connected open neighborhood $\Delta \subset S$ of s such that*

$$\liminf_n \frac{|\Delta \cap \mathcal{Z}(\gamma, -n)|_{mult}}{n^{\frac{b}{2}} \prod_{p<\infty} \mu_p(\gamma, n, V)} \geq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1 + \frac{b}{2})} \mu(\Delta),$$

where $n > 0$ ranges over numbers in $-Q(\gamma) + \mathbb{Z}$ represented by $-Q$ in $\gamma + V$.

Proof. The map $\pi : G \to D_V^+ \simeq G/K$ is submersive. Hence there exists U a simply connected open subset of D_V^+ around P_0 such that the following diagram is commutative :

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\sim} & U \times K \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

Assume that $\rho(\Delta)$ is contained in U. We have a holomorphic map $\Upsilon : \Delta \to G$ given by the composition

$$\Delta \xrightarrow{\rho} U \to U \times K \xrightarrow{\sim} \pi^{-1}(U)$$
$$t \mapsto \rho(t) \mapsto (\rho(t), 1_K) \mapsto \Upsilon_t$$

Hence, we have a local trivialization of the fibration $\mathcal{S}_\Delta$ given by

$$\begin{array}{ccc} \Delta \times \mathbb{S}^{b-1} & \xrightarrow{\sim} & \mathcal{S}_\Delta \\ (t, \lambda) & \longrightarrow & (t, \Upsilon_t.\lambda) \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

where $\mathbb{S}^{b-1} = \{x \in P_0^\perp,\, Q(x) = -1\}$.

The map

$$\phi : \mathcal{S}_\Delta \to A_0$$

is, by lemma 2.3.2, submersive at (t, λ) if the $\mathbb{C}$-linear map

$$\overline{\nabla}_u(\lambda) : T_uU \to \mathcal{V}_u^{0,2}$$

is surjective, or equivalently not-identically zero, since T_uU is of dimension 1 over $\mathbb{C}$. Let $\mathcal{S}_\Delta^{sing}$ the locus where ϕ is not submersive. Then $\mathcal{S}_\Delta^{sing}$ is a proper real analytic closed subset of $\mathcal{S}_\Delta$ negligible for the Lebesgue measure. Outside $\mathcal{S}_\Delta^{sing}$, ϕ is submersive and in fact a local diffeomorphism by equality of dimensions.

Let ψ be the composite map

$$\psi : \Delta \times \mathbb{S}^{b-1} \to \mathcal{S}_\Delta \xrightarrow{\phi} A_0.$$

The pullback $\phi^*\omega_{A_0}$ is a volume form on $\mathcal{S}_\Delta$ and so is $\psi^*\omega_0$ on $\Delta \times \mathbb{S}^{b-1}$.

Let $\epsilon > 0$, and let $\mathcal{S}_\Delta^{sing,\epsilon}$ be an open subset containing $\mathcal{S}_\Delta^{sing}$ such that

$$\int_{\mathcal{S}_\Delta^{sing,\epsilon}} \phi^*\omega_{A_0} \leq \epsilon.$$

Up to shrinking Δ, we can find a finite open cover W_i of $\mathcal{S}_\Delta \backslash \mathcal{S}_\Delta^{sing,\epsilon}$ such the restriction ϕ_i of ϕ to W_i is a diffeomorphism.

By Corollary 2.3.5, we get

$$\begin{aligned}|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult} &\geq \sum_i |\{\lambda \in \mathrm{Im}(\phi_i),\ \sqrt{n}\lambda \in \gamma + V\}| \\ &= \sum_i \int_{W_i} \phi^*\omega_{A_0}.\frac{2^{\frac{b}{2}}.n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.\prod_p \mu_p(\gamma,n,V) + o(n^{\frac{b}{2}}) \\ &\geq \frac{2^{\frac{b}{2}}.\left(\int_{\mathcal{S}_\Delta}\phi^*\omega_{A_0} - \epsilon\right).n^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.\prod_p \mu_p(\gamma,n,V) + o(n^{\frac{b}{2}})\end{aligned}$$

Here we used that $\int_{\mathcal{S}_\Delta}\phi^*\omega_{A_0} = \mu(\Delta).\frac{2.\pi^{1+\frac{b}{2}}}{\Gamma(1+\frac{b}{2})}$, a result we prove in Lemma 2.3.9 below. Hence we have

$$\liminf_n \frac{|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} \geq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta) - \frac{2^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}}.\epsilon$$

By letting $\epsilon \to 0$, we get the desired result. □

Lemma 2.3.9. *We have :*

$$\int_{\mathcal{S}_\Delta}\phi^*\omega_{A_0} = \mu(\Delta).\frac{2.\pi^{1+\frac{b}{2}}}{\Gamma(1+\frac{b}{2})}.$$

Proof. The differential of the map $\pi : G \to D_L^+$ at the identity of G induces an isomorphism of $\mathfrak{p}_0$ with the tangent space of D_L^+ at P_0. Since ω is a G-invariant 2-form, it corresponds uniquely to an element Ω of $\bigwedge^2 \mathfrak{p}_0^\vee$. Fix an orthogonal basis $(e_1, e_2, \xi_1, \ldots, \xi_b)$ of $V_\mathbb{R}$ compatible with the decomposition $V_\mathbb{R} = P_0 \oplus P_0^\perp$ and such that $Q(e_i) = -Q(\xi_j) = 1$ for $i = 1, 2$ and $j = 1, \ldots, b$. The Lie algebra $\mathfrak{g}_0$ is then identified with $\mathfrak{so}(2,b)$ and an element $M \in \mathfrak{so}(2,b)$ is written by blocks in the following way

$$\begin{pmatrix} 0 & \theta & U \\ -\theta & 0 & V \\ {}^tU & {}^tV & N \end{pmatrix} \tag{2.10}$$

where $\theta \in \mathbb{R}$, U and V are $1 \times b$-dimensional real matrices, and N is a $b \times b$-dimensional antisymmetric real matrix. For $i, j = 1, \ldots, b+2$, let $E_{i,j}$ be the matrix whose coefficients are zero except the coefficient (i,j) which is equal to 1. For $i = 1, \ldots, b$, define $U_i = E_{1,2+i} + E_{2+i,1}$ and $V_i = E_{2,2+i} + E_{2+i,2}$. The family $(U_i, V_i)_{i=1,\ldots,b}$ is a basis of $\mathfrak{p}_0$.

By [28, 13.1] (see also [51, 5.3]), the curvature Θ of the Hodge bundle is given by

$$\Theta(X,Y) = -\lambda([X,Y])$$

for $X, Y \in \mathfrak{p}$ and where λ is the linear form on $\mathfrak{so}(2,b)$ associating to an element M written as in (2.10) the element $i\theta \in \mathbb{C}$. Notice that λ is the differential of the generator χ of the group of characters of K whose associated automorphic line bundle is the Hodge bundle (see [111]). A computation shows that

$$\Omega = \frac{i}{2\pi}\Theta = \frac{1}{2\pi}\sum_{i=1}^{b} dU_i \wedge dV_i. \tag{2.11}$$

Recall that the Killing form B of $\mathfrak{g}_0$ is negative definite on the Lie algebra $\mathfrak{k}_0$ of K and we have thus an orthogonal decomposition (see [59]) :

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Let $\xi \in P_0^{\perp}$ such that $Q(\xi) = -1$. Then $K' = \mathrm{SO}(P_0) \times \mathrm{SO}((\mathbb{R}\xi \oplus P_0)^{\perp})$ is a maximal compact subgroup of H and we have similarly an orthogonal decomposition :

$$\mathfrak{h}_0 = \mathfrak{k}'_0 \oplus \mathfrak{p}'_0.$$

Let $\mathfrak{s}^{b-1}$ and $\widetilde{\mathfrak{p}}$ the orthogonal complements of $\mathfrak{k}'_0$ and $\mathfrak{p}'_0$ in $\mathfrak{k}_0$ and $\mathfrak{p}_0$ respectively with respect to the Killing form B of $\mathfrak{g}_0$:

$$\mathfrak{k}_0 = \mathfrak{k}'_0 \oplus \mathfrak{s}^{b-1} \quad , \quad \mathfrak{p}_0 = \mathfrak{p}'_0 \oplus \widetilde{\mathfrak{p}}.$$

The quotient $\mathfrak{g}_0/\mathfrak{h}_0$ can then be identified to $\mathfrak{s}^{b-1} \oplus \widetilde{\mathfrak{p}}$. The space $\mathfrak{s}^{b-1}$ can be identified, via the differential at the identity of $\mathrm{SO}(P_0^{\perp})$ of the map π_{ξ_1} introduced in (2.8), with the tangent space at ξ_1 of the sphere

$$\mathbb{S}^{b-1} := \{x \in P_0^{\perp},\, Q(x) = -1\},$$

which explains the notation. Let $\omega_{\mathbb{S}^{b-1}}$ be the unique $\mathrm{SO}(P_0^{\perp})$-invariant volume form on $\mathbb{S}^{b-1}$ such that

$$\omega_{\mathbb{S}^{b-1},\xi_1} = d\xi_2 \wedge \cdots \wedge d\xi_b.$$

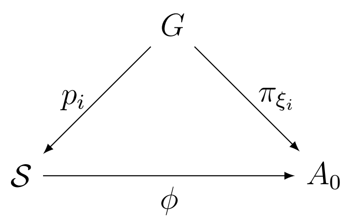
Then $(\mathbb{S}^{b-1}, \omega_{\mathbb{S}^{b-1}})$ is isometric to a sphere of dimension $b-1$ and radius 1 with its standard volume form, hence

$$\int_{\mathbb{S}^{b-1}} \omega_{\mathbb{S}^{b-1}} = \frac{b.\pi^{\frac{b}{2}}}{\Gamma(1+\frac{b}{2})}. \tag{2.12}$$

The group G acts transitively on the left on $\mathcal{S} := \mathcal{S}_{D_L^+}$ via $g.(w,\xi) = (g.w, g.\xi)$ for $g \in G$, $w \in D_V^+$ and $\xi \in \mathcal{S}_u$. The map $\phi : \mathcal{S} \to A_0$ is G-equivariant. For each $i = 1, \ldots, b$, we have thus a surjective map

$$p_i : G \to \mathcal{S}$$
$$g \mapsto (g.P_0, g.\xi_i)$$

that fits into the following commutative diagram



The differential of p_i induces an isomorphism between the tangent space of $\mathcal{S}$ at (P_0, ξ_i) and $\mathfrak{s}^{b-1} \oplus \mathfrak{p}_0$, where $\mathfrak{s}^{b-1}$ is isomorphic to the tangent space of $\mathbb{S}^{b-1}$ at ξ_i. The element $dU_i \wedge dV_i \in \bigwedge^2 \mathfrak{p}_0^\vee$ defines a G-invariant 2-form on $\mathcal{S}$ that we denote by ω_i. Let

$$\begin{aligned} t_i :\ K &\to \mathbb{S}^{b-1} \\ k &\to k.\xi_i. \end{aligned}$$

The pull back of the form $\omega_{\mathbb{S}^{b-1}}$ along t_i is identified to an element $dY_1^i \wedge \dots dY_{b-1}^i$ of $\bigwedge^{b-1} \mathfrak{k}_0^\vee$ for an orthogonal family $(Y_1^i, \dots, Y_{b-1}^i)$ of $\mathfrak{k}_0$. Let $\omega^{(i)}$ be the G-invariant $(b-1)$-from on $\mathcal{S}$ such that $p_i^*\omega^{(i)}$ is equal to $dY_1^i \wedge \dots dY_{b-1}^i$ in $\bigwedge^{b-1} \mathfrak{g}_0^\vee$

For each $i = 1, \dots, r$, we have by (2.9)

$$p_i^*\phi^*\omega_{A_0} = \pi_{\xi_i}^*\omega_{A_0} = dU_i \wedge dV_i \wedge dY_1^i \wedge \dots dY_{b-1}^i.$$

which is equal to $p_i^*\omega_i \wedge p_i^*\omega^{(i)} = p_i^*(\omega_i \wedge \omega^{(i)})$ by the construction itself. Hence, $\phi^*\omega_{A_0} = \omega_i \wedge \omega^{(i)}$ and summing over i yields

$$\phi^*\omega_{A_0} = \frac{1}{b}\sum_{i=1}^{b} \omega_i \wedge \omega^{(i)}$$

Notice now that the restrictions to $\mathbb{S}^{b-1}$ of the forms $\omega^{(i)}$ are all equal to the form $\omega_{\mathbb{S}_{b-1}}$ and that $\sum_{i=1}^{b} \omega_i = 2\pi\omega$ by (2.11). Hence

$$\psi^*\omega_{A_0} = \frac{2\pi}{b}\omega \wedge \omega_{\mathbb{S}^{b-1}}.$$

We have in particular

$$\frac{2\pi}{b}\mu(\Delta). \int_{\mathbb{S}^{b-1}} \omega_{\mathbb{S}^{b-1}} = \int_{\mathcal{S}_\Delta} \phi^*\omega_{A_0}.$$

Combined with (2.12), this yields the desired result. □

2.4 End of the proof and applications

The goal of the section is to prove Theorem 2.1.1. We keep the notations from previous sections, i.e $\{\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ is a simple, non trivial, polarized variation of Hodge structure over a quasi-projective curve S such that the local system $\mathbb{V}_\mathbb{Z}^\vee/\mathbb{V}_\mathbb{Z}$ is trivial and $\rho :\ S \to \Gamma_V\backslash D_V$ is the corresponding period map.

2.4.1 First reduction

Let H be a maximal isotropic subgroup of $V^\vee/V$ with respect to Q and let $\overline{\mathbb{V}}_\mathbb{Z}$ be the inverse image in $\mathbb{V}_\mathbb{Z}$ of $\underline{H}_S$, the trivial local system of fiber H. Then $\{\overline{\mathbb{V}}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ defines a simple, non-trivial, polarized variation of Hodge structure over S. Moreover, the fibers of the local system $\overline{\mathbb{V}}_\mathbb{Z}$ are isomorphic to a lattice $\overline{V}$ which has only strongly primitive totally isotropic planes and $\overline{V}^\vee/\overline{V} \simeq H^\perp/H$.

Proposition 2.4.1. *If Theorem 2.1.1 holds for $\{\overline{\mathbb{V}}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$ then it holds for $\{\mathbb{V}_\mathbb{Z}, \mathcal{F}^\bullet\mathcal{V}, Q\}$.*

Proof. Let Δ be an open simply connected subset of S which satisfies Proposition 2.3.8 and let $\gamma \in H^\perp$ and $n \in -Q(\gamma)+\mathbb{Z}$. Denote by $\overline{\gamma}$ its image in $H^\perp/H \simeq \overline{V}^\vee/\overline{V}$. Then

$$|\Delta \cap \mathcal{Z}_{\overline{V}}(\overline{\gamma}, -n)|_{mult} = \sum_{t\in H} |\Delta \cap \mathcal{Z}(\gamma + t, -n)|_{mult}$$

where $\mathcal{Z}_{\overline{V}}(\overline{\gamma}, -n)$ is the Heegner divisor associated to the lattice $(\overline{V}, Q)$, to $\overline{\gamma}$ and n. By assumption, Theorem 2.1.1 holds for $\{\overline{\mathbb{V}}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{V}, Q\}$:

$$|\Delta \cap \mathcal{Z}_{\overline{V}}(\overline{\gamma}, -n)|_{mult} = -\mu(\Delta)\frac{\overline{c}(\overline{\gamma}, n)}{2} + o(n^{\frac{b}{2}}).$$

The $\overline{c}(\overline{\gamma}, n)$ are the coefficients of the Eisenstein series $E_{\overline{V}}$ constructed out of the lattice $(\overline{V}, Q)$ in a similar fashion to Example 2.2.3. For $t \in H$, we have by Lemma 2.3.8

$$|\Delta \cap \mathcal{Z}(\gamma + t, -n)|_{mult} \geq -\mu(\Delta)\frac{c(\gamma + t, n)}{2} + o(n^{\frac{b}{2}})$$

Thus

$$|\Delta \cap \mathcal{Z}(\gamma, -n)|_{mult} \leq \frac{\mu(\Delta)}{2}.\left(-\overline{c}(\overline{\gamma}, n) + \sum_{t\in H\setminus\{0\}} c(\gamma + t, n)\right) + o(n^{\frac{b}{2}})$$

Using Lemma 2.4.2 below, we have

$$|\Delta \cap \mathcal{Z}(\gamma, -n)|_{mult} \leq -\mu(S)\frac{c(\gamma, n)}{2} + o(n^{\frac{b}{2}})$$

Combined with 2.3.8, we get the desired result. □

Lemma 2.4.2. *Let $\gamma \in H^\perp$, $n \in -Q(\gamma) + \mathbb{Z}$. Then*

$$\sum_{t\in H} c(\gamma + t, n) = \overline{c}(\overline{\gamma}, n) + O_\epsilon(n^{\frac{b+2}{4}+\epsilon})$$

Proof. Let $p : H^\perp \to H^\perp/H \simeq \overline{V}^\vee/\overline{V}$. Then p induces two morphisms

$$p_* : \mathbb{C}[V^\vee/V] \to \mathbb{C}[\overline{V}^\vee/\overline{V}]$$
$$v_\gamma \mapsto v_{p(\gamma)} \,\text{if}\, \gamma \in H^\perp,\, 0\,\text{otherwise.}$$

and

$$p^* : \mathbb{C}[\overline{V}^\vee/\overline{V}] \to \mathbb{C}[V^\vee/V]$$
$$v_\delta \mapsto \sum_{\gamma\in H^\perp,\, p(\gamma)=\delta} v_\gamma$$

which commutes with the Weil representation on both sides. Hence we have two $\mathbb{C}$-linear map : $p_* : \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*) \to \mathrm{M}_{1+\frac{b}{2}}(\rho_{\overline{V}}^*)$ and $p^* : \mathrm{M}_{1+\frac{b}{2}}(\rho_{\overline{V}}^*) \to \mathrm{M}_{1+\frac{b}{2}}(\rho_V^*)$. The modular form $p^*E_{\overline{V}} - p^*p_*E_V$ is then a cuspidal form and Lemma 2.4.2 follows by identifying its coefficients. □

2.4.2 An upper bound

By Theorem 2.4.1, we may assume that $V^\vee/V$ has no non-trivial totally isotropic subgroup in order the prove Theorem 2.1.1. Hence all the primitive isotropic planes of V are strongly primitive (see the definition preceding Proposition 2.2.6). Let $\overline{S}$ be a smooth compactification of S which fits in the following commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\rho} & \Gamma_V\backslash D_V \\ \downarrow & & \downarrow \\ \overline{S} & \xrightarrow{\overline{\rho}} & \overline{\Gamma_V\backslash D_V}^{tor} \end{array}$$

The boundary $\overline{S}\backslash S$ is finite. Let Δ_0 be a finite union of open subsets of $\overline{S}$ around each of those points. Consider Δ an open subset in $S\backslash\Delta_0$ which satisfies lemma 2.3.8. We can find a finite disjoint family of open subsets $(\Delta_i)_{i\in I}$ included in S which satisfy lemma 2.3.8 and such that $\mu(S) = \mu(\Delta) + \mu(\Delta_0) + \sum_{i\in I}\mu(\Delta_i)$. For each $i \in I$, we have :

$$\liminf_n \frac{|\Delta_i \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} \geq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta_i),$$

for $\gamma \in V^\vee/V$ and $n \in -Q(\gamma)+\mathbb{Z}$ satisfying local congruence conditions. Also, by lemma 2.3.6

$$|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult} \leq \deg_{\overline{S}}(\overline{\rho}^*\overline{\mathcal{Z}(\gamma,-n)}) - \sum_{i\in I}|\Delta_i \cap \mathcal{Z}(\gamma,-n)|_{mult}$$

Hence by Corollary 2.2.13

$$\begin{aligned} \limsup_n \frac{|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} &\leq \frac{(2\pi)^{1+\frac{b}{2}}\left(\mu(S)-\sum_i\mu(\Delta_i)\right)}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})} \\ &\leq \frac{(2\pi)^{1+\frac{b}{2}}\left(\mu(\Delta)+\mu(\Delta_0)\right)}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}. \end{aligned}$$

Since the volume of Δ_0 can be chosen arbitrarily small, we deduce that

$$\limsup_n \frac{|\Delta \cap \mathcal{Z}(\gamma,-n)|_{mult}}{n^{\frac{b}{2}}\prod_{p<\infty}\mu_p(\gamma,n,V)} \leq \frac{(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma(1+\frac{b}{2})}\mu(\Delta).$$

Combined with Lemma 2.3.8, this yields the desired equidistribution result.

2.4.3 Elliptic fibrations in families of K3 surfaces

We now derive some equidistribution results in quasi-polarized families of K3 surfaces. We begin by some background on K3 surfaces. The main references are [64] and [7].

Let X be a K3 surface. The second cohomology group with integer coefficients of X endowed with its intersection form $(.,.)$ is an even unimodular lattice of signature $(3,19)$, hence isomorphic abstractly to the *K3 lattice*

$$\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic lattice and E_8 the unique definite positive even unimodular lattice of rank 8, up to isomorphism. Denote by $Q = \frac{(.,.)}{2}$ the associated quadratic form.

Definition 2.4.3. An elliptic K3 surface is a projective K3 surface X together with a surjective morphism $\pi : X \to \mathbb{P}^1$ such the generic fiber is a smooth integral curve of genus one.

Recall that there is an elliptic fibration on X if, and only if there exists a *parabolic line bundle* on X, i.e a non-trivial line bundle L with $(L.L) = 0$. Indeed, if X admits an elliptic fibration $\pi : X \to \mathbb{P}^1$, then the class of a fiber gives a non-trivial element $e \in \mathrm{Pic}(X)$ such that $(e,e) = 0$. Conversely, let L be a non-trivial line bundle with square zero. Either L or L^{-1} is effective by Riemann-Roch. Assume L is effective. In [64, **8**.2.13], it is shown that up to acting on L by the Weyl group of X, we can assume that L is nef of square zero. Then [**2**.3.10, *loc.cit.*] shows that $\pi_V : X \to \mathbb{P}(\mathrm{H}^0(X,L)^\vee)$ factors through $\mathbb{P}^1$ and induces an elliptic fibration whose fiber class is equal to L.

Let $P \subset \Lambda_{K3}$ be a primitive Lorentzian anistropic sublattice of rank $\rho \leq 4$ and let $V = P^\perp$. Then (V,Q) is an even quadratic lattice of signature $(2, 20-\rho)$ and we have an isomorphism of quadratic finite modules $(V^\vee/V, Q) \simeq (P^\vee/P, -Q)$ (see[64, Prop.**14**.0.2]). Recall that a P-K3 surface is a K3 surface X with a fixed primitive embedding $P \to \mathrm{Pic}(X)$ such that the image of P contains a quasi-polarization ℓ. If $L \in \mathrm{Pic}(X)$ is of square 0, we can write $L = L_P + L_V$ where $L_P \in P^\vee$ and $L_V \in V^\vee$. Then $(L.L) = (L_P.L_P) + (L_V.L_V)$ and $(L_V.L_V) \leq 0$ since the restriction of the form $(\, , \,)$ to V is negative definite. Hence $(L_P.L_P) > 0$, unless $L = L_P$, which is excluded since P is assumed to be anisotropic.

Definition 2.4.4. Let X be a P-K3 surface, $\gamma \in P^\vee/P$ and $n \in Q(\gamma) + \mathbb{Z}$. A parabolic line bundle L on X is said to be of type (γ, n) if $L_P \in \gamma + P$ and $(L_P.L_P) = 2n$. An elliptic fibration is said to be of type (γ, n) if a line bundle defining the fibration is so. We call n the norm of the elliptic fibration.

We are now in the setting of Section 2.2.2 and we follow its notations, namely D_V is the period domain associated to the lattice (V,Q) and $\mathcal{Z}(\gamma,n)$ is the Heegner divisors associated to $\gamma \in V^\vee/V$ and $n \in Q(\gamma) + \mathbb{Z}$.

Proposition 2.4.5. *Let X be a P-K3 surface, $\gamma \in P^\vee/P$ and $n \in Q(\gamma) + \mathbb{Z}$. Then X admits a parabolic line bundle of type (γ, n) if and only if there exists $t \in \gamma + P$ such that $(t.t) = 2n$ and the period of X lies on the Heegner divisor $\mathcal{Z}(\gamma, -n)$.*

Proof. Let L be a line bundle on X defining an elliptic fibration of type (γ, n). Write $L = L_P + L_V$ as above. Then the element $L_V \in \gamma + V$ satisfies $(L_V.L_V) = -2n$ and take $t = L_P$. Hence, the period of X lies on the Heegner divisor $\mathcal{Z}(\gamma, -n)$. Conversely, if the period of X lies in $\mathcal{Z}(\gamma, -n)$, then there exists $\lambda \in H^{1,1}(X) \cap (\gamma + V)$ such that $(\lambda, \lambda) = -2n$. By assumption, there exists $t \in \gamma + V$ such that $(t.t) = 2n$. Then $L = \lambda + t \in \mathrm{Pic}(X)$ is of square zero and non-trivial. □

Proposition 2.4.6. *Let X be a P-K3 surface. Then X admits an elliptic fibration of norm less than n if and only if the period of X lies on the union of the Heegner divisors $\mathcal{Z}(\gamma, -s)$ for $\gamma \in P^\vee/P$ and $s \in]0, n]$ represented by Q in $\gamma + P$.*

Proof. The forward direction is clear. For the converse, we can construct a parabolic line bundle L on X of norm less than n in the same way as it was done above : either L or L^{-1} is effective by Riemann-Roch and we can thus assume that L is effective. If L is nef, then L defines an elliptic fibration of degree less than n and we are done. Otherwise, there exists a -2-curve C such that $(L.C) < 0$. Then $s_C(L) := L + (L, C).C$ is a parabolic line bundle with positive intersection with C and of norm less than the norm of L. We repeat the process if $s_C(L)$ is not nef. After a finite number of actions by the Weil group, we get a nef line bundle. □

Proof of corollary 2.1.3. Let $\mathcal{X} \xrightarrow{\pi} S$ be a non-isotrivial family of K3 surfaces with generic Picard group equal to P. The orthogonal to $\underline{P}_S$ in $R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}$ defines a polarized variation of Hodge structure of weight 2 over S with fibers isomorphic to the lattice (V, Q) and to which we can apply Theorem 2.1.1. Using Proposition 2.4.5, this proves (i) and (ii). For (iii), let $\Delta \subset S$ an open subset and $\widetilde{N}(n, \Delta)$ the number of $s \in \Delta$ (counted with multiplicity) for which $\mathcal{X}_s$ admits an elliptic fibration of norm less than n. Then by Proposition 2.4.6 and Theorem 2.1.1 we have

$$\frac{\widetilde{N}(n, \Delta)}{\widetilde{N}(n, S)} = \frac{\sum_{\gamma \in V^\vee/V} \sum_{s \leq n, s \in Q(\gamma+P)} |\Delta \cap Z(\gamma, -s)|_{mult}}{\sum_{\gamma \in V^\vee/V} \sum_{s \leq n, s \in Q(\gamma+P)} |S \cap Z(\gamma, -s)|_{mult}} \underset{n \to \infty}{\longrightarrow} \frac{\mu(\Delta)}{\mu(S)}$$

□

Remark 2.4.7. There is an analogous result which concerns families of hyperkähler manifolds and which we state below. See [63] for definitions. Indeed, given a hyperkähler manifold, the *Beauville-Bogomolov-Fujiki* form, defined in [10], endows its second integral Betti cohomology group with a structure of a lattice of signature $(3, b_2 - 3)$.

Corollary 2.4.8. *Let d be an integer. Let $(\mathcal{X}, \mathcal{L}_{2d}) \to S$ be a non-isotrivial, split quasi-polarized family of hyperkähler manifolds of degree $2d$ over a quasi-projective curve S with generic Picard rank equal to 1 and let $\{R^2\pi_*\underline{\mathbb{Z}}_{\mathcal{X}}, \mathcal{F}^\bullet\mathcal{H}\}$ be the induced variation of Hodge structure over S. Let μ be the measure induced by integrating the first Chern class of $\mathcal{F}^2\mathcal{H}$. Then the set of points $s \in S$ for which $\mathcal{X}_s$ admits a parabolic line bundle L such that $(L.\mathcal{L}_{2d,s}) = 2dn$ becomes equidistributed in S with respect to μ as $n \to +\infty$.*

For the definition of split polarization, we refer to [54, Definition 3.9].

Chapitre 3

Exceptional invariants of some Galois representations

Résumé. Étant donné un corps de nombres K et une représentation galoisienne de type K3 associée à un K-point d'une variété de Shimura orthogonale, on prouve sous une certaine condition d'approximation l'existence d'une place de K qui admet un invariant exceptionnel. Ceci a des applications au problème du saut de nombre de Picard des spécialisations d'une surface K3 définie sur un corps de nombres.

Abstract. Given a number field K and a Galois representation of K3 type associated to a K-point of an orthogonal Shimura variety, we prove under some approximation condition the existence of a place of K with an exceptional invariant. This has application to Picard rank jumps of specializations of a K3 surface defined over a number field.

Contents

3.1 Introduction

Let (L, Q) be an integral quadratic even lattice of signature $(2, b)$ with $b \geq 3$. Assume that L is a maximal lattice in $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ over which Q is $\mathbb{Z}$-valued and let $L^\vee$ be the dual lattice of L. Associated to this data there is a reductive algebraic group $G = \mathrm{GSpin}(V)$ over $\mathbb{Q}$, a certain compact open subgroup $K \subset G(\mathbb{A}_f)$ and a hermitian domain

$$D_L = \{\omega \in \mathbb{P}(V_{\mathbb{C}}),\ (\overline{\omega}.\omega) > 0, (\omega.\omega) = 0\}$$

on which the group $G(\mathbb{R})$ acts, see Section 3.2. There exists a smooth algebraic stack M over $\mathbb{Q}$, the *GSpin Shimura variety* such that $M(\mathbb{C})$ is the b-dimensional complex orbifold

$$G(\mathbb{Q})\backslash D_L \times G(\mathbb{A}_f)/K.$$

It is a Shimura variety of Hodge type and carries a family of abelian varieties $A \to M$, the *Kuga-Satake family*, see section 3.2. Moreover, M admits a canonical integral model $\mathcal{M}$ over $\mathbb{Z}$ which is normal and smooth at primes p at which the lattice (L, Q) is almost self-dual and such that the Kuga-Satake abelian scheme extends to a family $\mathcal{A} \to \mathcal{M}$, see Section 3.2.4.

The different cohomological realizations of the abelian scheme $A \to M$ provides filtered vector bundles with integrable connections (resp. ℓ-adic local systems) over M, $H_{dR}^{\otimes(1,1)}$ and $\mathcal{V}_{dR}$ (resp. $H_\ell^{\otimes(1,1)}$ and $\mathcal{V}_\ell$) such that we have an embedding $\mathcal{V}_{dR} \hookrightarrow H_{dR}^{\otimes(1,1)}$ (resp. $\mathcal{V}_\ell \hookrightarrow H_\ell^{\otimes(1,1)}$). The previous realizations allow to define for every $\beta \in L^\vee/L$ and $m \in Q(\beta) + \mathbb{Z}$ with $m < 0$, a *special divisor*[1] $\mathcal{Z}(\beta, m) \to \mathcal{M}$ parameterizing Kuga-Satake abelian varieties with extra special endomorphisms, see section 3.2.5.

Let $x \in M(\mathbb{C})$. Then x defines a positive definite plane P_x in $L_{\mathbb{R}}$ and for an element $\lambda \in L_{\mathbb{R}}$ we denote by λ_x the orthogonal projection of λ on P_x. We say that x is

1. Hodge-generic if x does not lie on any divisor $\mathcal{Z}(\beta, m)(\mathbb{C})$.
2. moderately approximated by the special divisors if for every $\beta \in L^\vee/L$ there exists $D > 0$ such that

$$\forall N > 0, \exists m \leq -N, \forall \lambda \in \beta + L : Q(\lambda) = m \Rightarrow Q(\lambda_x) \geq \frac{1}{|m|^D}. \qquad (3.1)$$

1. also called Heegner divisor in the literature.

The last condition has geometric meaning : it ensures that we can find infinitely many special divisors $(\mathcal{Z}(\beta, m)(\mathbb{C}))_m$ whose distance to x for a certain metric in $M(\mathbb{C})$ is lower bounded by a power of m. This is essential in order to bound the order of growth of the evaluation of certain Green functions, see 3.4.5. It is satisfied for small values of b, namely $b \leq 2$ as shown in the work of [31] and [98].

Let K be a number field and let $x \in M(K)$. We say that x is moderately approximated by the special divisors if for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, the point $x^\sigma \in M(\mathbb{C})$ is moderately approximated by the special divisors. Say also that x is Hodge-generic if for one embedding (equivalently any) $\sigma : K \hookrightarrow \mathbb{C}$, the point x^σ is Hodge-generic.

Let $\ell > 0$. The fiber of the ℓ-adic local system $\mathcal{V}_\ell$ at x defines a Galois representation

$$\rho_\ell : \mathrm{Gal}(\overline{K}/K) \to \mathcal{V}_{\ell,x}$$

which is unramified at almost every prime $\mathfrak{P}$ of K. These are the *K3 type Galois representations.* At an unramified place $\mathfrak{P}$, we say that ρ_ℓ has an exceptional invariant if there exists an element of $\mathcal{V}_{\ell,x}$ which has a finite orbit under the Frobenius at $\mathfrak{P}$ but not under the whole Galois group $\mathrm{Gal}(\overline{K}/K)$.

The main result we prove in this paper is the following theorem.

Theorem 3.1.1. *Let $x \in \mathcal{M}(\mathcal{O}_K)$ and assume that $x_K \in M(K)$ is Hodge-generic and moderately approximated by the special divisors. For a prime $\ell > 0$, let $(\rho_\ell, \mathcal{V}_{\ell,x})$ be the Galois representation associated to x_K constructed above. Then there exists $\ell > 0$ and a place $\mathfrak{P}$ of K of residual characteristic prime to ℓ such that ρ_ℓ admits an exceptional invariant at $\mathfrak{P}$.*

It is worth noting that $M(\mathbb{C})$ carries a natural nonzero $G(\mathbb{R})$-invariant measure, which pulls back to the measure on D_L induced by the Bergman metric and such that the subset of well-approximated elements by the family $(\mathcal{Z}(\beta, m))_{(\beta,m)}$ has zero measure. In fact this will be true for any finite measure with density.

The result of Theorem 3.1.1 is a generalization of the main result of [31] which was the starting point of our investigation. In the complex setting, the analogous result is well-understood. More precisely, the Hodge locus of a non-trivial polarized variation of Hodge structure of weight 2 over a complex quasi-projective curve is dense for the analytic topology by a well-known result of Green [109, Prop. 17.20]. When this variation is of K3 type, the main result of [105] shows in fact that this locus is equidistributed with respect to a natural measure, which is simply the one induced by the pull-back of the Bergman metric via the period map.

3.1.1 Application : jumps of the Picard rank of K3 surfaces

Let X be a K3 surface over a number field K. There exists $N \geq 1$ and a smooth family of K3 surfaces $\mathcal{X} \to S = \mathrm{Spec}(\mathcal{O}_K[\frac{1}{N}])$ with generic fiber equal to X. For every place $\mathfrak{P}$ of K, we have an injective specialization map by [64, Chap.17 Prop.2.10] :

$$\mathrm{sp}_{\mathfrak{P}} : \mathrm{Pic}(X_{\overline{K}}) \to \mathrm{Pic}(\mathcal{X}_{\overline{\mathfrak{P}}}).$$

In [30], Charles asks about what can be said about the jumping locus defined by :

$$JL(X) = \{\mathfrak{P} \in S, \rho(\mathcal{X}_{\overline{\mathfrak{P}}}) > \rho(X_{\overline{K}})\}.$$

If $\rho(X_{\overline{K}})$ is odd, then the above set is equal to S because of the Tate conjecture, now a theorem for K3 surfaces. Assume thus that $\rho(X_{\overline{K}})$ is even.

Corollary 3.1.2. *Let $\mathcal{X}$ be a family of K3 surfaces over $\mathcal{O}_K$. Assume that the K3 surface $X := \mathcal{X}_K$ is moderately approximated by the special divisors in the Shimura variety associated to the transcendental lattice of X, then there exists a place $\mathfrak{P}$ such that $\mathfrak{P} \in JL(X)$.*

3.1.2 Strategy of the proof

The proof of Theorem 3.1.1 follows the lines of [31] and relies on Arakelov intersection theory on the Shimura variety $\mathcal{M}$. For every pair (β, m) where $\beta \in L^\vee/L$ and $m \in Q(\beta) + \mathbb{Z}$, $m < 0$, the special divisor $\mathcal{Z}(\beta, m)$ is flat over $\mathbb{Z}$ and parametrizes points of $\mathcal{M}$ for which the Kuga-Satake abelian variety admits an extra special $\mathbb{Q}$-endomorphism of type (β, m), see Section 3.2.3. By the work of Bruinier [23], this divisor can be endowed with a Green function $\Phi_{\beta,m}$ which is constructed using theta lift of non-holomorphic Eisenstein series of negative weight and thus yields an arithmetic divisor $\widehat{\mathcal{Z}}(\beta, m) = (\mathcal{Z}(\beta, m), \Phi_{\beta,m})$. The first main input we use is the modularity of the generating series of the special divisors, first proven by Borcherds over the complex fiber in [17], and then extended to the setting of arithmetic divisors by Howard and Madapusi-Pera in [62].

By assumption, we have an abelian scheme $\mathcal{A}_x \to S = \mathrm{Spec}(\mathcal{O}_K)$ with generic fiber isomorphic to A_x and a map $\iota : S \to \mathcal{M}$. Using Arakelov intersection theory, the height of S with respect to the arithmetic divisors $\widehat{\mathcal{Z}}(\beta, m)$ can then be expressed as follows :

$$h_{\widehat{\mathcal{Z}}(\beta,m)}(S) = \sum_{\mathfrak{P}} \sum_{z \in \mathcal{Z}(\beta,m)(\overline{\mathbb{F}}_{\mathfrak{P}})} \frac{(S.\mathcal{Z}(\beta, m))_z}{|\mathrm{Aut}(z)|} \log |\mathcal{O}_K/\mathfrak{P}| + \sum_{\sigma: K \to \mathbb{C}} \frac{\Phi_{\beta,m}(x^\sigma)}{|\mathrm{Aut}(x^\sigma)|}. \quad (3.2)$$

The aforementioned modularity result allows us to control the growth of $h_{\widehat{\mathcal{Z}}(\mu,m)}(S)$ in terms of the coefficient of a certain vector-valued Eisenstein series of weight $1 + \frac{b}{2}$ with respect to the Weil representation associated to the lattice (L, Q). The next step is to control individually the growth of the archimedean term. In order to do so, we use an explicit expression of the Green function $\Phi_{\beta,m}$ given by Bruinier in [23, Section 2.2]. The latter is divided into two parts : one part is estimated using results from [44] on effective equidistribution of projection of lattice points on the quadric $\{x \in L_{\mathbb{R}}, Q(x) = -1\}$. The estimation of the remaining sum is conditional on the moderate approximation hypothesis. More precisely, we prove the following.

Proposition 3.1.3. *If x is moderately approximated by the family $(\mathcal{Z}(\beta, m))_{(\beta,m)}$ then there exists an infinite sequence $(m_i)_i$ such that*

$$h_{\widehat{\mathcal{Z}}(\beta,m_i)}(S) = o(\Phi_{\beta,m_i}(x))$$

as $m_i \to \infty$. In fact

$$|\Phi_{\beta,m_i}(x)| \gg \log(|m_i|)|h_{\widehat{\mathcal{Z}}(\beta,m_i)}(S)|$$

as $m_i \to \infty$.

By the previous proposition, there exists at least one place $\mathfrak{P}$ where the intersection of S with some $\mathcal{Z}(\beta, m_i)$ is not empty for sufficiently large $|m|$, which proves Theorem 3.1.1.

3.1.3 Further discussion

The main result of [31] is stronger than ours. Indeed, Charles makes no hypothesis on the good reduction of the elliptic curves involved nor on the moderate approximation by special divisors. This is due to the fact that the lattice theory appearing there is easier to handle. Moreover, the compactification of the Shimura variety, which is a product of modular curves in this case, is well understood as well as the behavior of the special divisors at the boundary. We are thus tempted to make the following conjecture.

Conjecture 3.1.4. *Let $x \in M(K)$ be a Hodge-generic point. For a prime $\ell > 0$, let $(\rho_\ell, \mathcal{V}_{\ell,x})$ be the Galois representation associated to x. Then there exists $\ell > 0$ such that there exists infinitely many places $\mathfrak{P}$ of K such that ρ_ℓ admits an exceptional invariant at $\mathfrak{P}$.*

As a consequence, we can also make the following conjecture, logically implied by the previous one.

Conjecture 3.1.5. *Let X be a K3 surface over K. The set $JL(X)$ is infinite.*

3.1.4 Organization of the paper

In Section 3.2 we recall the construction of the GSpin Shimura variety associated to the lattice (L, Q) following [4, 5, 62], as well as the construction of its integral model and the construction of the special divisors using the notion of special endomorphisms. In Section 3.3 we recall Brocherds-Howard-Madapusi Pera's modularity result from [62] and derive consequences on the height of the arithmetic curve S with repsect to the Heegner divisors. Section 3.4 is devoted to the proof of the theorem 3.1.1. In Section 3.5 we discuss how the result from [31] fits into our framework and then we discuss the application to K3 surfaces.

3.1.5 Notations

If $f, g : \mathbb{N} \to \mathbb{R}$ are real functions and g does not vanish, then :

1. $f = O(g)$ if there exists an integer $n_0 \in \mathbb{N}$, a positive constant $C_0 > 0$ such that
$$\forall n \geq n_0,\ |f(n)| \leq C_0|g(n)|.$$
2. $f \asymp h$ if $f = O(h)$ and $h = O(f)$.
3. For p a prime number, v_p denotes the p-adic valuation on $\mathbb{Q}$.
4. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ is the real part of s.

3.2 The GSpin Shimura varieties and their special divisors

Let (L, Q) be an integral quadratic even lattice of signature $(2, b)$, $b \geq 1$, with associated bilinear form defined by

$$(x.y) = Q(x + y) - Q(x) - Q(y),$$

for $x, y \in L$. Let $V := L \otimes_{\mathbb{Z}} \mathbb{Q}$ and assume that L is a maximal lattice in V over which Q is $\mathbb{Z}$-valued. We recall in this section the theory of GSpin Shimura varieties associated with (L, Q). Our main references are [4, Section 2], [5, Section 4] and [81, Section 3] [2].

3.2.1 The GSpin Shimura variety

The *Clifford algebra* $C(V)$ of (V, Q) is the $\mathbb{Q}$-algebra defined as the quotient of the tensor algebra $\bigotimes V$ by the ideal generated by $\{(x \otimes x) - Q(x),\ x \in V\}$. It has a $\mathbb{Z}/2\mathbb{Z}$ grading $C(V) = C(V)^+ \oplus C(V)^-$ induced by the grading on $\bigotimes V$. We define similarly the Clifford algebra $C(L)$, which is a $\mathbb{Z}$-algebra and satisfies $C(V) = C(L)_{\mathbb{Q}}$.

Let $G := \mathrm{GSpin}(V)$ be the group of spinor similitudes of V ([81, Section 1]). It is the reductive algebraic group over $\mathbb{Q}$ such that

$$G(R) = \{g \in C^+(V_R),\ gV_R g^{-1} = V_R\}$$

for any $\mathbb{Q}$-algebra R. We denote by $\nu : G \to \mathbb{G}_m$ the spinor similitude factor as defined in [8, Section 3]. The group G acts on V via $g \bullet v = gvg^{-1}$ for $v \in V$ and $g \in G(\mathbb{Q})$. Moreover, there is an exact sequence of algebraic groups

$$1 \to \mathbb{G}_m \to G \xrightarrow{g \mapsto g\bullet} \mathrm{SO}(V) \to 1.$$

Let D_L be the period domain associated to (L, Q) defined by

$$D_L = \{\omega \in \mathbb{P}(V_{\mathbb{C}}),\ (\overline{\omega}.\omega) > 0, (\omega.\omega) = 0\}.$$

It is a hermitian symmetric domain and the group $G(\mathbb{R})$ acts transitively on D_L. Then (G, D_L) defines a Shimura datum as follows : for any class $[z] \in D_L$, choose a representative $z = x + iy$ where $x, y \in V_{\mathbb{R}}$ are orthogonal and $Q(x) = Q(y) = 1$. There is a morphism of algebraic groups over $\mathbb{R}$

$$h_z : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$$

obtained by setting $h_z(i) = xy \in G(\mathbb{R}) \subset C^+(V_{\mathbb{R}})^\times$, hence identifying D_L with a $G(\mathbb{R})$-conjugacy class in $\mathrm{Hom}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m, G_{\mathbb{R}})$. The reflex field of (G, D_L) is equal to $\mathbb{Q}$ by [3, Appendix 1].

Let $K \subset G(\mathbb{A}_f)$ be the compact open subgroup

$$K = \mathrm{G}(\mathbb{A}_f) \cap C(\widehat{L})^\times,$$

where $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. By [81, Lemma 2.6], the image of K in $\mathrm{SO}(\widehat{L})$ is the subgroup of elements acting trivially on $L^\vee/L$, where $L^\vee$ is the dual lattice of L defined by

$$L^\vee := \{x \in V,\ \forall y \in L,\ (x.y) \in \mathbb{Z}\}.$$

By the theory of canonical models, we get a b-dimensional algebraic stack M over $\mathbb{Q}$, *the GSpin Shimura variety associated with* L such that

$$M(\mathbb{C}) = G(\mathbb{Q}) \backslash D_L \times G(\mathbb{A}_f)/K.$$

2. Beware that our convention of sign for the signature of Q is opposite to theirs.

3.2.2 The Kuga-Satake construction

The Kuga-Satake construction was first considered in [72] and later in [39] and [41]. We follow here the exposition of [81, Section 3].

Let $G \to \mathrm{Aut}(N)$ be an algebraic representation of G on a $\mathbb{Q}$-vector space N, and let $N_{\widehat{\mathbb{Z}}} \subset N_{\mathbb{A}_f}$ be a K-stable lattice. Then one can construct a local system $\mathbf{N}_B$ on $M(\mathbb{C})$ whose fiber at a point $[z, g]$ is identified with $N \cap gN_{\widehat{\mathbb{Z}}}$. The corresponding vector bundle $\mathbf{N}_{dR,M(\mathbb{C})} = \mathcal{O}_{M(\mathbb{C})} \otimes \mathbf{N}_B$ is equipped with a holomorphic filtration $\mathcal{F}^\bullet \mathbf{N}_{dR,M(\mathbb{C})}$ which at every point $[z, g]$ equips the fiber with the Hodge structure determined by the cocharacter h_z. Hence we obtain a functor

$$(N, N_{\widehat{\mathbb{Z}}}) \mapsto \left(\mathbf{N}_B, \mathcal{F}^\bullet \mathbf{N}_{dR,M(\mathbb{C}))}\right) \tag{3.3}$$

from the category of algebraic $\mathbb{Q}$-representations of G with a K-stable lattice to variations of $\mathbb{Z}$-Hodge structures over $M(\mathbb{C})$, which are polarized by [41, 1.1.15]. If we apply this construction to the pair $(V, \widehat{L})$, we obtain a canonical polarized variation of $\mathbb{Z}$-Hodge structures $\{\mathbf{V}_B, \mathcal{F}^\bullet \mathbf{V}_{dR,M(\mathbb{C})}\}$ over $M(\mathbb{C})$. For each point $[z, g]$, the induced Hodge decomposition of $V_\mathbb{C}$ is as follows

$$V^{1,-1} = \mathbb{C}z, V^{-1,1} = \mathbb{C}\overline{z}, V^{0,0}_\mathbb{C} = (\mathbb{C}z \oplus \mathbb{C}\overline{z})^\perp.$$

Also, if we denote by H the representation of the group G on $C(V)$ by multiplication on the left and $H_{\widehat{\mathbb{Z}}} = C(L)_{\widehat{\mathbb{Z}}}$, then applying the functor (3.3) to the pair $(H, H_{\widehat{\mathbb{Z}}})$, we get another variation of $\mathbb{Z}$-Hodge structures $(\mathbf{H}_B, \mathbf{H}_{dR,M(\mathbb{C})})$ of type $(-1, 0), (0, -1)$. In fact there is a family of abelian schemes $A \to M$ of relative dimension 2^{n+1}, the Kuga-Satake abelian scheme (see [81, 3.10]), such that the homology of the family $A^{an}(\mathbb{C}) \to M^{an}(\mathbb{C})$ is precisely $(\mathbf{H}_B, \mathbf{H}_{dR,M(\mathbb{C})})$. It is equipped with a right $C(V)$-action and a compatible $\mathbb{Z}/2\mathbb{Z}$-grading : $A = A^+ \times A^-$.

The Kuga-Satake abelian scheme makes it possible to descend over M the vector bundles $\mathbf{N}_{dR,M(\mathbb{C})}$ and $\mathbf{H}_{dR,M(\mathbb{C})}$ to filtered vector bundles with integrable connections $(\mathbf{V}_{dR}, \mathcal{F}^\bullet \mathbf{V}_{dR})$ and $(\mathbf{H}_{dR}, \mathcal{F}^\bullet \mathbf{H}_{dR})$ (see [81, 3.12]). For any prime ℓ, the ℓ-adic sheaves $\mathbb{Z}_\ell \otimes \mathbf{V}_B$ and $\mathbb{Z}_\ell \otimes \mathbf{H}_B$ over $M(\mathbb{C})$ descends also canonically to ℓ-adic étale sheaves $\mathbf{V}_{\ell,\text{ét}}$ and $\mathbf{H}_{\ell,\text{ét}}$ over M. Moreover $\mathbf{H}_\ell$ is canonically isomorphic to the ℓ-adic Tate module of A.

The G-equivariant embedding $V \to \mathrm{End}_{C(V)}(H)$ given by multiplication on the left induces by functoriality the following embeddings

$$\mathbf{V}_B \hookrightarrow \mathrm{End}_{C(L)}(\mathbf{H}_B) \quad \text{and} \quad \mathbf{V}_{dR,M} \hookrightarrow \mathrm{End}_{C(V)}(\mathbf{H}_{dR,M}),$$

the latter being compatible with filtration and integrable connections. Moreover, there is a canonical quadratic form $\mathbf{Q} : \mathbf{V}_{dR,M} \to \mathcal{O}_M$ given on sections by $x \circ x = Q(x).\mathrm{Id}$ where the composition takes places in $\mathrm{End}_{C(V)}(\mathbf{H}_{dR,M})$. Similarly, we have an embedding

$$\mathbf{V}_{\ell,\text{ét}} \hookrightarrow \mathrm{End}_{C(V)}(\mathbf{H}_{\ell,\text{ét}}).$$

There is also a canonical quadratic form on $\mathbf{V}_{\ell,\text{ét}}$ induced by composition on the right-hand side and valued in the constant sheaf $\underline{\mathbb{Z}_\ell}$.

3.2.3 Special divisors

For any vector $\lambda \in L_\mathbb{R}$ such that $Q(\lambda) < 0$, let $\lambda^\perp$ be the set of elements of D_L orthogonal to λ. Let $\beta \in L^\vee/L$ and $m \in Q(\beta) + \mathbb{Z}$ with $m < 0$ and define the complex orbifold

$$Z(\beta,m)(\mathbb{C}) := \bigsqcup_{g\in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K} \Gamma_g\backslash \left(\bigsqcup_{\lambda\in\beta_g+L_g,\, Q(\lambda)=m} \lambda^\perp \right)$$

where $\Gamma_g = G(\mathbb{Q}) \cap gKg^{-1}$, $L_g \subset V$ is the lattice determined by $\widehat{L_g} = g \bullet \widehat{L}$ and $\beta_g = g\bullet\beta \in L_g^\vee/L_g$. Then $Z(\beta,m)(\mathbb{C})$ is the set of complex points of a disjoint union of Shimura varieties associated with orthogonal lattices of signature $(b-1,2)$ and it admits a canonical model over $\mathbb{Q}$ for $b \geq 2$. There is a map $Z(\beta,m)(\mathbb{C}) \to M(\mathbb{C})$ which descends to a closed immersion $Z(\beta,m) \to M$, and in fact $Z(\beta,m)$ is an effective Cartier divisor.

3.2.4 Integral models

We recall the construction of an integral model of M from [5, 4.4], see also [62, 6.2] and the original work of Kisin [68]. Let p be a prime number. We say that L is almost self-dual at p if either L is self-dual at p or $p=2$, $\dim_{\mathbb{Q}}(V)$ is odd and $|L^\vee/L|$ is not divisible by 4. The following proposition is the content of [5, Theorem 4.4.6] and [62, Remark 6.2.1].

Proposition 3.2.1. *There is a unique flat, normal algebraic $\mathbb{Z}$-stack $\mathcal{M}$ such that*

1. *$\mathcal{M}$ is smooth over $\mathbb{Z}_{(p)}$ if L is almost self-dual at p ;*
2. *the Kuga-Satake abelian scheme extends to an abelian scheme $\mathcal{A} \to \mathcal{M}$;*
3. *the line bundle F^1V_{dR} extends canonically to a line bundle $\mathcal{L}$ over $\mathcal{M}$;*
4. *if $L_{(p)}$ is self-dual or p is odd and $p^2 \nmid |L^\vee/L|$, or p is odd and $n \geq 5$, then $\mathcal{M}_{\mathbb{F}_p}$ is a geometrically connected and geometrically normal algebraic stack over $\mathbb{F}_p$.*

Let us give more specific information about the canonical model $\mathcal{M}$ from the above theorem at a prime p at which the lattice L is self-dual over $\mathbb{Z}_{(p)}$. Let $L_{(p)} = L_{\mathbb{Z}_{(p)}}$ and $G_{(p)} = \mathrm{GSpin}(L_{(p)}, Q)$. By the proposition above, $\mathcal{M}_{(p)}$ is smooth over $\mathbb{Z}_{(p)}$ and carries a family of abelian schemes $\mathcal{A}_{(p)} \to \mathcal{M}_{(p)}$. The functor (3.3) extends canonically $N \mapsto N_{dR}$ from algebraic $\mathbb{Z}_{(p)}$ representations N of $G_{(p)}$ to filtered vector bundles with an integrable connection. When $N = H_{(p)}$, the associated filtered vector bundle with connection is equal to the relative first de Rham homology of $\mathcal{A}_{(p)} \to \mathcal{M}_{(p)}$. Moreover, there is a canonical *crystalline realization* functor $N \mapsto \mathbf{N}_{cris}$ from algebraic $\mathbb{Z}_{(p)}$ representations of $G_{(p)}$ to F-crystals over $\mathcal{M}_{(p),\mathbb{F}_p}$. This functor associates to $H_{(p)}$ the relative first crystalline homology of $\mathcal{A}_{(p)}$ over $\mathcal{M}_{(p),\mathbb{F}_p}$ defined by

$$\mathrm{Hom}\left(R^1\pi_{crys,*}\mathcal{O}^{crys}_{\mathcal{A}_{\mathbb{F}_p}/\mathbb{Z}_p}, \mathcal{O}^{crys}_{\mathcal{M}_{\mathbb{F}_p/\mathbb{Z}_p}}\right).$$

When evaluated on $\widehat{\mathcal{M}}_{(p)}$, the formal completion of $\mathcal{M}_{(p)}$ along $\mathcal{M}_{(p),\mathbb{F}_p}$, it is canonically isomorphic to the p-adic completion of $\mathbf{H}_{dR}$ as a vector bundle with integrable connection. The crystalline realization of V admits a canonical embedding

$$\mathbf{V}_{crys} \hookrightarrow \mathrm{End}_{C(L)}(\mathbf{H}_{crys})$$

mapping into a local direct summand of its target and compatible with embedding of de Rham realizations.

3.2.5 Special endomorphisms

We recall from [5, 4.5] the construction of integral models of the special divisors introduced in 3.2.3 using the notion of special endomorphisms.

Let $S \to \mathcal{M}$ be a scheme and $A_S \to S$ the pull-back of the Kuga-Satake abelian scheme over S. For each $\beta \in L^\vee/L$ and ℓ prime number, let $V_\beta(A_S)$ and $V_{\beta_\ell}(A_S[\ell^\infty])$ denote the space of special endomorphisms of A_S and $A_S[\ell^\infty]$ respectively and which are constructed in [5, 4.5]. By [5, Prop.4.5.4], there is a negative definite quadratic [3] form $Q : V(A_S) \to \mathbb{Z}$ such that for each $x \in V(A_S)$, we have $x \circ x = Q(x).\mathrm{Id}_{A_S}$. If $x \in V_\mu(A_S)$, then $Q(x) \equiv Q(\mu) \pmod{\mathbb{Z}}$.

For $m \in Q(\beta) + \mathbb{Z}$, $m < 0$, the special cycle $\mathcal{Z}(\beta, m)$ is defined as the stack over $\mathcal{M}$ with functor of points

$$\mathcal{Z}(\beta, m)(S) = \{x \in V_\beta(A_S),\, Q(x) = m\} \tag{3.4}$$

The morphism $\mathcal{Z}(\beta, m) \to \mathcal{M}$ is finite and unramified, and so around every geometric point of $\mathcal{M}$, there is an étale neighborhood $U \to \mathcal{M}$ such that $\mathcal{Z}(\beta, m)_{/U}$ defines an effective Cartier divisor on U, and by gluing over an étale cover we obtain an effective Cartier divisor on $\mathcal{M}$, again denoted $\mathcal{Z}(\beta, m)$, see [62, Section 7.1] for more details. If $b \geq 3$, then it is flat over $\mathbb{Z}$ by [62, Proposition 7.1.4].

3.3 Harmonic modular forms and arithmetic cycles

3.3.1 Metrized line bundles

We recall in this section some notions of arithmetic intersection theory on the arithmetic stack $\mathcal{M}$ from proposition 3.2.1. For more details, we refer to [50, 49].

We denote by $F_\infty : \mathcal{M}(\mathbb{C}) \to \mathcal{M}(\mathbb{C})$ the complex conjugation. An *arithmetic divisor* on $\mathcal{M}$ is a pair $(\mathcal{Z}, \Phi)$ consisting of a Cartier divisor $\mathcal{Z}$ on $\mathcal{M}$ and a F_∞-invariant Green function Φ for $\mathcal{Z}(\mathbb{C})$. This means that if $\Psi = 0$ is a local equation for $\mathcal{Z}(\mathbb{C})$, then the function $\Phi + \log|\Psi|^2$ extends smoothly across $\mathcal{Z}(\mathbb{C})$. A *principal arithmetic divisor* is an arithmetic divisor of the form

$$\widehat{\mathrm{div}}(\Psi) = (\mathrm{div}(\Psi), -\log|\Psi|^2),$$

where Ψ is a non-zero rational function on $\mathcal{M}$. The *arithmetic Chow group of Gillet-Soulé*, denoted by $\widehat{\mathrm{CH}}^1(\mathcal{M})$, is the quotient of the group of arithmetic divisors by the subgroup of principal arithmetic divisors.

A *metrized line bundle* $\overline{L} = (L, h)$ on $\mathcal{M}$ is a line bundle L endowed with a smooth F_∞-invariant Hermitian metric h on its complex points. Let $\widehat{\mathrm{Pic}}(\mathcal{M})$ denote the group of isomorphism classes of metrized line bundles. The group law is given by tensor product. There is a map

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \to \widehat{\mathrm{CH}}^1(\mathcal{M})$$

which sends a metrized line bundle $\overline{L} = (L, h)$ to $(\mathrm{div}(\Psi), -\log h(\Psi, \Psi)^2)$ where Ψ is a non-zero rational section of L. By [102, III.4], this map is an isomorphism.

Let K be a number field, $S = \mathrm{Spec}(\mathcal{O}_K)$ and $i : S \to \mathcal{M}$ a morphism. Denote by $x \in M(K)$ the corresponding K-point of $\mathcal{M}$. Let $\widehat{\mathcal{Z}} = (\mathcal{Z}, \Phi)$ be an arithmetic

3. Recall that we work in signature $(2, b)$, while [5] work with signature $(b, 2)$.

divisor on $\mathcal{M}$ such that the image of S is not contained in $\mathcal{Z}$. Then the height $h_{\widehat{\mathcal{Z}}}(S)$ of S with respect to $\widehat{\mathcal{Z}}$ is defined as the image of $\mathcal{Z}$ under the composition

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) \to \widehat{\mathrm{CH}}^1(\mathcal{S}) \xrightarrow{\deg} \mathbb{R},$$

which is expressed, by definition of the arithmetic degree $\widehat{\deg}$ ([5, 6.4]), as :

$$h_{\widehat{\mathcal{Z}}}(S) = \sum_{\mathfrak{P}\subset\mathcal{O}_K} \sum_{z\in\mathcal{Z}(\overline{\mathbb{F}}_{\mathfrak{P}})} \frac{(S.\mathcal{Z})_z}{|\mathrm{Aut}(z)|} \log|\mathcal{O}_K/\mathfrak{P}| + \sum_{\sigma:K\to\mathbb{C}} \frac{\Phi(x^\sigma)}{|\mathrm{Aut}(x^\sigma)|}. \tag{3.5}$$

3.3.2 Arithmetic special divisors

There is a unitary Weil representation

$$\rho_L : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[L^\vee/L]),$$

where $\mathrm{Mp}_2(\mathbb{Z})$ is the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$, see [23, Section 1.1]. Let $k = 1 + \frac{b}{2}$ and let $\mathrm{H}_{2-k}(\rho_L)$ be the $\mathbb{C}$-vector space of vector valued harmonic weak Maass forms of weight k with respect to ρ_L as defined in [27, 3.1]. Then by (3.4a) in *loc.cit*, each $f \in \mathrm{H}_{2-k}(\rho_L)$ has a holomorphic part f^+ :

$$f^+(\tau) = \sum_{\beta\in L^\vee/L} \sum_{\substack{m\in\mathbb{Z}+Q(\beta)\\ m\gg-\infty}} a^+(\beta,m) q^m v_\beta, \tag{3.6}$$

where $(v_\beta)_{\beta\in L^\vee/L}$ is a basis of the $\mathbb{C}$-vector space $\mathbb{C}[L^\vee/L]$, $\tau \in \mathbb{H}$ and $q = e^{2i\pi\tau}$. If $a^+(\beta,m) \in \mathbb{Z}$ for all $\beta \in L^\vee/L$ and $m \in \mathbb{Q}(\beta) + \mathbb{Z}$, $m < 0$, then by [27, (4.7)], one can associate to f an arithmetic divisor $\widehat{\mathcal{Z}}(f)$ in the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M})$, defined by $\widehat{\mathcal{Z}}(f) = (\mathcal{Z}(f), \Phi(f))$, where

$$\mathcal{Z}(f) = \sum_{\beta\in L^\vee/L} \sum_{m\in\mathbb{Z}+Q(\beta), m<0} c^+(\beta,m)\mathcal{Z}(\beta,m),$$

and $\Phi(f)$ is a Green function constructed as the regularized theta lift of f. See also [23, Chap.2] and [24, Section 5].

Let $\beta \in L^\vee/L$, $m \in \mathbb{Z}+Q(\beta), m < 0$. By [23, Def.1.8 and Prop.1.10], there exists a harmonic Maass form, called Hejhal Poincaré series and noted $F_{\beta,m} \in \mathrm{H}_{2-k}(\rho_L)$, such that $\mathcal{Z}(F_{\beta,m}) = \mathcal{Z}(m,\beta)$. In particular $\Phi(\beta,m) := \Phi(F_{\beta,m})$ is a Green function for the divisor $\mathcal{Z}(\beta,m)$. We denote by $\widehat{\mathcal{Z}}(\beta,m) := \widehat{\mathcal{Z}}(F_{\beta,m})$ the arithmetic divisor associated to $F_{\beta,m}$.

Finally, the line bundle $\mathcal{L}$ from proposition 3.2.1 is endowed with the Petersson metric defined as follows : the fiber of $\mathcal{L}$ at a complex point $[z,g] \in M(\mathbb{C})$ is identified with the isotropic line $\mathbb{C}z \subset V_{\mathbb{C}}$, then we set $||z||^2 = \frac{(z.\overline{z})}{4\pi\gamma}$, where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant. Hence we get a metrized line bundle $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{M})$.

There is an another expression for the Green function $\Phi(\beta,m)$ due to Bruinier in [25, Section 4] and which will allow us later to make explicit computations. Let $k = 1 + \frac{b}{2}$, $x \in D_L$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{k}{2}$. Recall that x defines a positive definite plane P_x of $L_{\mathbb{R}}$ and for $\lambda \in L_{\mathbb{R}}$, we denote by λ_x the orthogonal projection of λ on P_x. Let

$$F(s,x) = H\left(s - 1 + \frac{k}{2}, s + 1 - \frac{k}{2}, 2s; x\right),$$

where

$$H(a,b,c;z) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is the Gauss hypergeometric function as in [1, Chapter 15], and $(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}$ for $a, b, c, z \in \mathbb{C}$ and $|z| > 1$. Finally, let

$$\phi_{\beta,m}(x,s) = 2\frac{\Gamma(s-1+\frac{k}{2})}{\Gamma(2s)} \sum_{\substack{\lambda\in\beta+L \\ Q(\lambda)=m}} \left(\frac{m}{m-Q(\lambda_x)}\right)^{s-1+\frac{k}{2}} F\left(s, \frac{m}{m-Q(\lambda_x)}\right). \quad (3.7)$$

In [23, Section 2.2], $\phi_{\beta,m}(x,s)$ is defined as the regularized theta lift of a non-holomorphic Poincaré series closely related to $F_{\beta,m}$. The function $\phi_{\beta,m}(x,s)$ is meromorphic for $\mathrm{Re}(s) > 0$ with a simple pole at $s = \frac{k}{2}$. Let $\phi_{\beta,m}(x)$ be the constant term at $s = \frac{k}{2}$ of the Laurent expansion of $\phi_{\beta,m}(x,s)$. By [25, Prop.4.2], for $x \in D_L$, we have

$$\phi_{\beta,m}(x) = \lim_{s\to\frac{k}{2}} \left(\phi_{\beta,m}(x,s) - \frac{\varphi_{\beta,m}(\frac{k}{2})}{s-\frac{k}{2}}\right)$$

where $\varphi_{\beta,m}$ is defined by [25, Equation (4.10)]. In order to relate $\phi_{\beta,m}$ to $\Phi_{\beta,m}$, we need to introduce more ingredients.

Let $(\tau,s) \to E_0(\tau,s)$ the Eisentein series defined in [25, Equation (1.4)]. It converges normally on $\mathbb{H}$ for $\mathrm{Re}(s) > 1-\frac{k}{2}$ and defines a $\mathrm{Mp}_2(\mathbb{Z})$-invariant real analytic function. Let $\mathrm{W}_{\nu,\mu}(z)$ the W-Whittaker function from [1, Equation 13.1.33]. As in [25, Equation 3.2], we define for $s \in \mathbb{C}$ and $y \in \mathbb{R}\backslash\{0\}$

$$\mathcal{W}_s(y) = |y|^{-\frac{k}{2}} W_{\frac{yk}{2|y|}, \frac{(1-k)}{2}-s}.$$

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1-\frac{k}{2}$, the Eisenstein series $E_0(.,s)$ has a Fourier expansion of the form

$$E_0(\tau,s) = \sum_{\beta\in L^\vee/L} \sum_{m\in -Q(\beta)+\mathbb{Z}} c_0(\beta,m,s,y) v_\beta(mx)$$

By [25, Proposition 3.2], the coefficients $c_0(\beta,m,s,y)$ can be decomposed, for $m \neq 0$, as

$$c_0(\beta,m,s,y) = C(\beta,m,s)\mathcal{W}_s(4\pi m y), \quad (3.8)$$

where the function $C(\beta,m,s)$ is independent of y (see [25, Equation (3.22)]).

The value at $s = 0$ of $E_0(\tau,s)$ is an element of the $\mathbb{C}$-vector space of vector-valued modular forms $\mathrm{M}_k(\rho_L)$, see [23, Definition 1.2] for the definition. For $\beta \in L^\vee/L$, $m \in -Q(\gamma)+\mathbb{Z}$ with $m \geq 0$, we denote by $c(\beta,m)$ its (β,m)-th Fourier coefficient and we can thus write

$$E_0(\tau) = \sum_{\substack{\beta\in L^\vee/L \\ m\in -Q(\beta)+\mathbb{Z}, m>0}} c(\beta,m) q^m v_\beta.$$

For a geometric interpretation of these coefficients, see [25, Prop.4.8]. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1 - \frac{k}{2}$, define

$$b(s) = -\frac{C\left(\beta, -m, s - \frac{k}{2}\right).\left(s - 1 + \frac{k}{2}\right)}{(2s-1).\Gamma\left(s + 1 - \frac{k}{2}\right)}. \tag{3.9}$$

In the notation of [25], it is $b(0,0,s)$ in Equation (4.12) *loc.cit.*. The comparison with the formula given above is given in [25, Proposition 4.3]. In [23], $b(s)$ is defined as the $(0,0)$-th coefficient in the Fourier expansion of $F_{\beta,m}(.,s)$ for $\beta \in L^\vee/L$ and $m \in Q(\beta) + \mathbb{Z}$. It is proven *loc. cit.* that $b(s)$ is a holomorphic function of s in the region $\mathrm{Re}(s) > 1$, see [23, Theorem 1.9]. By [25, Prop.4.8 and (4.56)], the following relation holds

$$\varphi_{\beta,m}\left(\frac{k}{2}\right) = b\left(\frac{k}{2}\right) = -c(\beta, -m). \tag{3.10}$$

The following proposition, which is [23, Proposition 2.11], relates $\phi_{\beta,m}$ to $\Phi_{\beta,m}$. It shows in particular that $\phi_{\beta,m}$ is a Green function for the arithmetic cycle $\mathcal{Z}(\beta,m)$.

Proposition 3.3.1. *For $x \in D_L$, we have :*

$$\Phi(\beta,m)(x) = \phi_{\beta,m}(x) - b'\left(\frac{k}{2}\right).$$

3.3.3 Borcherds-Howard-Madapusi Pera's modularity theorem

Let $M_{1+\frac{b}{2}}(\rho_L)$ denote as before the $\mathbb{C}$-vector space of vector-valued modular forms of weight $1 + \frac{b}{2}$ and type ρ_ℓ. Recall that for $\beta \in L^\vee/L$ and $m \in Q(\beta) + \mathbb{Z}$, we have defined in the previous section an arithmetic divisor $\widehat{\mathcal{Z}}(\beta,m)$. Let also $\widehat{\mathcal{Z}}(0,0)$ be an arithmetic divisor whose class is equal to the metrized line bundle $\overline{\mathcal{L}}^\vee$. The following is the content of [62, Theorem 8.3.1].

Theorem 3.3.2. *Let $b \geq 3$. The formal generating series*

$$\widehat{\Phi}_L = \sum_{\substack{\beta \in L^\vee/L \\ m \geq 0, m \in -Q(\beta)+\mathbb{Z}}} \widehat{\mathcal{Z}}(\beta, -m).q^m v_\beta$$

is modular, in the sense it is an element of $M_{1+\frac{b}{2}}(\rho_L) \otimes \widehat{\mathrm{Pic}}(\mathcal{M})$.

We conclude our discussion by a conjecture on the behavior of of the special divisors on the boundary of a suitable toroidal compactification. This is inspired from the work of Peterson in [93]. Suppose we are given a toroidal compactification $\overline{\mathcal{M}}^{tor}$ in which the closure $\widehat{\widehat{\mathcal{Z}}}(m,\beta)$ of the special arithmetic divisors extend and define elements in the group of metrized lines bundles of $\overline{\mathcal{M}}^{tor}$. Then one can ask how the modularity result above is affected. We make the following conjecture.

Conjecture 3.3.3. *Let $b \geq 3$. There exists a toroidal compactification $\overline{\mathcal{M}}^{tor}$ of $\mathcal{M}$ to which we can extend the arithmetic divisors $\widehat{\mathcal{Z}}(\beta,m)$ to $\widehat{\widehat{\mathcal{Z}}}(\beta,m)$ and such that the formal generating series*

$$\widehat{\Phi}_L = \sum_{\substack{\beta \in L^\vee/L \\ m \geq 0, m \in -Q(\beta)+\mathbb{Z}}} \widehat{\widehat{\mathcal{Z}}}(\beta, -m).q^m v\beta$$

is quasi-modular of weight $1+\frac{b}{2}$ *and depth* 1*, in the sense it is an element of* $M^{\leq 1}_{1+\frac{b}{2}}(\rho_L)\otimes \widehat{\mathrm{Pic}}(\overline{\mathcal{M}}^{tor})$.

For a definition and a discussion on the space of quasi-modular forms, we refer to [65, Definition 1] and [82, Section 17.1]. In [105], we have used Peterson's results in order to bound the multiplicity of intersections with special divisors at the boundary of the Shimura variety $M(\mathbb{C})$.

3.4 Proof of the main theorem

In this section, we prove theorem 3.1.1. We keep the notations from previous sections, namely (L, Q) is an even lattice of signature $(2, b)$ with $b \geq 3$ and $\mathcal{M}$ is the associated GSpin Shimura variety over $\mathbb{Z}$. For $\beta = 0$ and $m \in \mathbb{Z}$, $m < 0$, we denote simply $\widehat{\mathcal{Z}}(m) := \widehat{\mathcal{Z}}(0, m)$ the associated special arithmetic divisor.

Let K be a number field, $S = \mathrm{Spec}(\mathcal{O}_K) \to \mathcal{M}$ an $\mathcal{O}_K$-point of $\mathcal{M}$ such that the corresponding K-point $x \in M(K)$ is Hodge-generic. By formula (3.5), we can write

$$h_{\widehat{\mathcal{Z}}(m)}(S) = \sum_{\mathfrak{P}} \sum_{z\in \mathcal{Z}(m)(\overline{\mathbb{F}}_{\mathfrak{P}})} \frac{(S.\mathcal{Z}(m))_z}{|\mathrm{Aut}(z)|}\log|\mathcal{O}_K/\mathfrak{P}| + \sum_{\sigma:K\to\mathbb{C}} \frac{\Phi_m(x^\sigma)}{|\mathrm{Aut}(x^\sigma)|}. \tag{3.11}$$

For each embedding $\sigma : K \to \mathbb{C}$, we can write furthermore by Proposition 3.10 :

$$\Phi(m)(x^\sigma) = \phi_m(x^\sigma) - b'\left(\frac{k}{2}\right). \tag{3.12}$$

The proof of theorem 3.1.1 proceeds by analyzing the growth of each of the previous terms separately. The first proposition gives an order of growth of the global term. Recall that for $m \in \mathbb{Z}$, the $(0, -m)$-th coefficient of the Eisenstein series E_0 is denoted $c(-m)$ and are described thoroughly in Example 2.2.3.

Proposition 3.4.1. *For every* $\epsilon > 0$*,* $m \in \mathbb{Z}$ *with* $m < 0$*, we have :*

$$h_{\widehat{\mathcal{Z}}(m)}(S) = \frac{-c(-m)}{2}h_{\widehat{\mathcal{L}}}(S) + O_\epsilon(|m|^{\frac{2+b}{4}+\epsilon}).$$

In particular, $h_{\widehat{\mathcal{Z}}(m)}(S) = O(c(-m))$ *as* $|m| \to \infty$*.*

The next two propositions control the growth of the archimedean term and show under the moderate approximation hypothesis that it grows faster than the global intersection term. For the rest of this section, we will fix an embedding $\sigma : K \to \mathbb{C}$ and denote simply $x \in M(\mathbb{C})$ the image of $x \in M(K)$ by σ.

Proposition 3.4.2. *We have :*

$$b'\left(\frac{k}{2}\right) = |c(-m)|\log(|m|) + o\left(c(-m)\log(|m|)\right).$$

as $m \to -\infty$ *in the set* $\{Q(\lambda), \lambda \in L\}$*.*

Proposition 3.4.3. *Let* $m \in \mathbb{Z}$ *with* $m < 0$*.*

(i) *For each* $1 \geq \eta > 0$, *we have a decomposition :*

$$\phi_m(x) = -2 \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1 \\ Q(\lambda_x)\leq \eta}} \log(Q(\lambda_x)) + O(c(-m)).$$

(ii) *If* x *is moderately approximated by the special divisors, then there exists an infinite sequence* (m_i) *such that* $\phi_{m_i} = o(c(-m_i)\log|m|)$.

Now the combination of the above propositions and the fact that the order of the automorphism groups appearing in formula (3.11) are bounded independently from m imply Theorem 3.1.1. The rest of the section is devoted to the proof of the above statements.

Remark 3.4.4. Proposition 3.4.1 shows that the global contribution is negligible in front of the archimedean one, which is completely different from the complex situation in [105].

3.4.1 Global intersection term

In this section we prove proposition 3.4.1. The proof is similar to the one we gave for Proposition 2.2.5.

Proof of proposition 3.4.1. By theorem 3.3.2, the following generating series

$$\sum_{\substack{\beta\in L^\vee/L \\ m\geq 0, m\in -\mathbb{Z}}} h_{\widehat{\mathcal{Z}}(\beta,-m)}(S).q^m v_\beta$$

is the Fourier expansion of an element in $M_{1+\frac{b}{2}}(\rho_L)$. Since $h_{\widehat{\mathcal{Z}}(\beta,0)}(S) = 0$ for $\beta \neq 0$, we can write by [23, p.27] :

$$\sum_{\substack{\beta\in L^\vee/L \\ m\geq 0, m\in -\mathbb{Z}}} h_{\widehat{\mathcal{Z}}(\beta,-m)}(S).q^{-m} = \frac{-h_{\overline{L}}(S)}{2}E_0 + g$$

where $g \in S_{1+\frac{b}{2}}(\rho_L)$ is a cusp form, see [23, Def.1.2] for a definition.

For $m \in \mathbb{Z}$ with $m < 0$, we have thus

$$h_{\widehat{\mathcal{Z}}(m)}(S) = \frac{-c(-m)}{2} h_{\overline{L}}(S) + g(0,-m),$$

where $g(0,-m)$ is the $(0,-m)$-th Fourier coefficient of g. By [94, Prop. 1.5.5], we have the following bound

$$|g(0,-m)| \leq C_{\epsilon,g}|m|^{\frac{2+b}{4}+\epsilon},$$

for all $\epsilon > 0$, some constant $C_{\epsilon,g} > 0$, and for all $m \in \mathbb{Z}$ with $m < 0$. This concludes the proof. □

3.4.2 Growth estimates for Green functions

We prove in this section Proposition 3.4.2.

Proof of proposition 3.4.2. Let $m \in \{Q(x), x \in L\}$. By equation (3.9), we have for $\mathrm{Re}(s) > 1$

$$b(s) = -\frac{C\left(0, -m, s - \frac{k}{2}\right) . \left(s - 1 + \frac{k}{2}\right)}{(2s - 1) . \Gamma\left(s + 1 - \frac{k}{2}\right)}.$$

where $C\left(0, -m, s - \frac{k}{2}\right)$ is the function defined by Equation (5). Taking logarithmic derivatives at $s = \frac{k}{2}$ yields :

$$\frac{b'\left(\frac{k}{2}\right)}{b\left(\frac{k}{2}\right)} = \frac{C'(0, -m, 0)}{C(0, -m, 0)} - \frac{2}{b} + \Gamma'(1)$$

By [25, Theorem 4.10]

$$\frac{C'(0, -m, 0)}{C(0, -m, 0)} = \log(|m|) + \frac{\sigma'_{-m}(k)}{\sigma_{-m}(k)} + O(1)$$

where for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$ the function σ_{-m} is given by :

$$\sigma_{-m}(s) = \begin{cases} \prod\limits_{p \backslash 2m \det(L)} \frac{\mathrm{L}^{(p)}\left(p^{1 - \frac{r}{2} - s}\right)}{1 - \chi_{D_0}(p) p^{-s}}, \text{ if } r = 2 + b \text{ is even,} \\ \prod\limits_{p \backslash 2m \det(L)} \frac{1 - \chi_{D_0}(p) p^{\frac{1}{2} - s}}{1 - p^{1 - 2s}} . \mathrm{L}^{(p)}\left(p^{1 - \frac{r}{2} - s}\right), \text{ if } r \text{ is odd.} \end{cases} \tag{3.13}$$

In the above formula, χ_{D_0} is the quadratic character associated to a fundamental discriminant D_0 of the number field $\mathbb{Q}(\sqrt{D})$ where D is defined by

$$(-1)^{\frac{r}{2}} \det(L), \text{ if } r \text{ is even.}$$
$$2(-1)^{\frac{r+1}{2}} m \det(L), \text{ otherwise.}$$

The polynomial $\mathrm{L}^{(p)}(X)$ is defined by

$$\mathrm{L}^{(p)}(X) = N_m(p^{w_p}) X^{w_p} + (1 - p^{r-1} X) \sum_{v=0}^{w_p - 1} N_m(p^v) X^v \in \mathbb{Z}[X],$$

where

$$N_m(a) = \{x \in L/aL; \, Q(x) - m \equiv 0 \pmod{a}\},$$

and $w_p = 1 + v_p(2m)$. Since $b\left(\frac{k}{2}\right) = |c(-m)|$, it is enough to show that

$$\frac{\sigma'_{-m}(k)}{\sigma_{-m}(k)} = o(\log(|m|)).$$

Taking the logarithmic derivative in (3.13) at $s = k$, we get for r even

$$\frac{\sigma'_{-m}(k)}{\sigma_{-m}(k)} = \sum_p -\frac{p^{1-r} \mathrm{L}^{(p)'}\left(p^{1-r}\right)}{\mathrm{L}^{(p)}\left(p^{1-r}\right)} \log(p) - \frac{\chi_{D_0}(p) \log(p)}{p^k - \chi_{D_0}(p)},$$

and for r odd

$$\frac{\sigma'_{-m}(k)}{\sigma_{-m}(k)} = -\sum_p \left(\frac{p^{1-r} \mathrm{L}^{(p)'}\left(p^{1-r}\right)}{\mathrm{L}^{(p)}\left(p^{1-r}\right)} - \frac{\chi_{D_0}(p)}{p^{k-\frac{1}{2}} - \chi_{D_0}(p)} + \frac{1}{p^{2k-1}} \right) \log(p),$$

where the sums are taken over the primes p dividing $\backslash 2m \det(L)$.

Since $k = 1 + \frac{b}{2} \geq \frac{3}{2}$, we have

$$\left| \sum_{p \backslash 2m \det(L)} \frac{\chi_{D_0}(p) \log(p)}{p^k - \chi_{D_0}(p)} \right| \leq \sum_p \frac{\log(p)}{p^2 - 1} < +\infty,$$

and

$$\left| \sum_{p \backslash 2m \det(L)} \frac{\chi_{D_0}(p) \log(p)}{p^{k-\frac{1}{2}} - \chi_{D_0}(p)} \right| \leq \sum_p \frac{\log(p)}{p^2 - 1} < +\infty.$$

Also

$$\left| \sum_{p \backslash 2m \det(L)} \frac{\chi_{D_0} \log(p)}{p^{k-\frac{1}{2}} - \chi_{D_0}(p)} \right| \leq \sum_p \frac{\log(p)}{p^2 - 1} < +\infty.$$

We have $\mathrm{L}_m^{(p)}(p^{1-r}) = N_m(p^{w_p}) p^{(1-r)w_p}$ and

$$\mathrm{L}_m^{(p)'}(p^{1-r}) = w_p N_n(p^{w_p}) p^{(1-r)(w_p - 1)} - \sum_{v=0}^{w_p - 1} N_m(p^v) p^{(v-1)(1-r)}.$$

Hence

$$\begin{aligned} \frac{p^{1-r} \mathrm{L}^{(p)'}\left(p^{1-r}\right)}{\mathrm{L}^{(p)}\left(p^{1-r}\right)} &= w_p - \sum_{v=0}^{w_p-1} \frac{N_m(p^v)}{N_m(p^{w_p})} p^{(v-w_p)(1-r)} \\ &= w_p - \sum_{v=0}^{w_p-1} \frac{\alpha_m(v)}{\alpha_m(w_p)} \end{aligned}$$

where $\alpha_m(v) = \frac{N_m(p^v)}{p^{v(r-1)}}$.

By Lemma 3.4.5 in the next section, there exists $C > 0$ such that for every m and p, we have

$$\left| \omega_p - \sum_{v=0}^{w_p-1} \frac{\alpha_m(v)}{\alpha_m(w_p)} \right| \leq \frac{C}{p}$$

Thus we have

$$\begin{aligned} \left| \sum_{p \backslash 2m \det(L)} \frac{p^{1-r} \mathrm{L}^{(p)'}\left(p^{1-r}\right)}{\mathrm{L}^{(p)}\left(p^{1-r}\right)} \right| &\leq \sum_{p \backslash 2m \det(L)} \frac{\log(p)}{p} \\ &= O(\log \log(|m|)). \end{aligned}$$

Here we use the fact that for $N \geq 2$,

$$\sum_{p \backslash N} \frac{\log(p)}{p} = O(\log\log(N)).$$

Indeed, let $X = \log(N)$ and use Mertens' first theorem to write

$$\begin{aligned}\sum_{p \backslash N} \frac{\log(p)}{p} &= \sum_{p \backslash N, p<X} \frac{\log(p)}{p} + \sum_{p \backslash N, p \geq X} \frac{\log(p)}{p} \\ &\leq \log(X) + \frac{\log(N)}{X} + O(1) \\ &\leq \log(\log(N)) + O(1).\end{aligned}$$

This concludes the proof of the proposition. □

3.4.3 Counting representations by quadratic forms

The goal of this section is to prove the following lemma.

Lemma 3.4.5. *There exists a constant $C > 0$ such that for every m and p, we have*

$$\left| \omega_p - \sum_{v=0}^{w_p - 1} \frac{\alpha_m(v)}{\alpha_m(w_p)} \right| \leq \frac{C}{p}$$

We first recall some facts on representations by quadratic forms. Let p a prime number and $L_p =: L \otimes \mathbb{Z}_p$. By [69, Section 5.3], the quadratic lattice (L_p, Q) admits an orthogonal decomposition $(L_p, Q) = \bigoplus_j (L_j, p^{\nu_j} Q_j)$ such that each (L_j, Q_j) is unimodular of dimension at most 2 and $\nu_j \geq 0$. Thus we can write

$$\forall x = (x_j)_j \in L_p : \ Q(x) = \sum_j p^{\nu_j} Q_j(x_j).$$

If $p \neq 2$, then every L_j is of dimension 1. Let $m \in \mathbb{Z}$. Let $v \geq 0$ an integer. Set

$$\mathcal{N}_m(p^v) = \{x \in L/p^v L, \ Q(x) - m \equiv \pmod{p^v}\}.$$

Thus $N_m(p^v) = |\mathcal{N}_m(p^v)|$. Let $x \in \mathcal{N}_m(p^v)$. Following [58, Definition 3.1], we will say that x is of

1. zero type if $x \equiv 0 \pmod{p}$,
2. good type if there exists j such that $x_j \not\equiv 0 \pmod{p}$ and $\nu_j = 0$,
3. bad type otherwise.

Let $\mathcal{N}_m^{\text{good}}(p^v)$, $\mathcal{N}_m^{\text{bad}}(p^v)$ and $\mathcal{N}_m^{\text{zero}}(p^v)$ be the set of good type, bad type and zero type solutions respectively. The cardinality of the previous sets will be noted $N_m^{\text{good}}(p^v)$, $N_m^{\text{bad}}(p^v)$ and $N_m^{\text{zero}}(p^v)$ respectively.

Proof of Lemma 3.4.5. First case : Assume that p is odd and coprime with the discriminant of L. In this situation there are no bad type solutions. We will therefore write simply $N_m(p^v)$ and $\alpha_m(p^v)$. By [40, Théorème 8.1], there exists a constant C_1 independent of p and m such that

$$|N_m(p) - p^{r-1}| \leq C_1.p^{r-2}, \tag{3.14}$$

where $r = 2 + b$. It follows that we can find $C_2 > 0$ such that for all m and p

$$|\alpha_m^{\text{good}}(p) - 1| \leq \frac{C_2}{p}. \tag{3.15}$$

For $v \geq 2$, we have

$$N_m(p^v) = N_m^{\text{good}}(p^v) + p^r N_{\frac{m}{p^2}}(p^{v-2}). \tag{3.16}$$

Iterating formula (3.16) yields for $v \geq v_p(m) + 1$

$$N_m(p^v) = \sum_{u=0}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} p^{ur} N_{\frac{m}{p^{2u}}}^{\text{good}}(p^{v-2u}).$$

Hence

$$\alpha_m(p^v) = \sum_{u=0}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} \frac{\alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p^{v-2u})}{p^{u(r-2)}}.$$

By [58, Lemma 3.2], we have for $\ell \geq 1$

$$\alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p^\ell) = \alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p).$$

Thus, if $v \geq v_p(m) + 1$, then

$$\alpha_m(p^v) = \alpha_m(p^{w_p}) = \sum_{u=0}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} \frac{\alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p)}{p^{u(r-2)}}.$$

If $1 \leq v \leq v_p(m)$, then

$$\begin{aligned}
\alpha_m(p^v) &= \sum_{u=0}^{\left\lfloor \frac{v}{2} \right\rfloor - 1} \frac{\alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p^{v-2u})}{p^{u(r-2)}} + \frac{\alpha_{\frac{m}{p^{2\lfloor \frac{v}{2} \rfloor}}}(p^{v-2\lfloor \frac{v}{2} \rfloor})}{p^{\lfloor \frac{v}{2} \rfloor (r-2)}} \\
&= \sum_{u=0}^{\left\lfloor \frac{v}{2} \right\rfloor - 1} \frac{\alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p)}{p^{u(r-2)}} + \frac{\alpha_{\frac{m}{p^{2\lfloor \frac{v}{2} \rfloor}}}(p^{v-2\lfloor \frac{v}{2} \rfloor})}{p^{\lfloor \frac{v}{2} \rfloor (r-2)}}.
\end{aligned}$$

If $v \geq 2$, then

$$\begin{aligned}
|\alpha_m(p^{w_p}) - \alpha_m(p^v)| &= \left| \sum_{u=\left\lfloor \frac{v}{2} \right\rfloor}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} \frac{\alpha_{\frac{m}{p^{2u}}}^{\text{good}}(p)}{p^{u(r-2)}} - \frac{\alpha_{\frac{m}{p^{2\lfloor \frac{v}{2} \rfloor}}}(p^{v-2\lfloor \frac{v}{2} \rfloor})}{p^{\lfloor \frac{v}{2} \rfloor (r-2)}} \right| \\
&\leq \sum_{u=\left\lfloor \frac{v}{2} \right\rfloor}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} \frac{C_3}{p^{u(r-2)}} + \frac{C_3}{p^{\lfloor \frac{v}{2} \rfloor (r-2)}} \qquad (3.17) \\
&\leq \frac{C_4}{p^{\frac{(v-1)(r-2)}{2}}}
\end{aligned}$$

for some constants C_3, C_4 independent from p and m. For $v = 0$, $\alpha_m(p^v) = 1$ and by equation (3.15)

$$\begin{aligned}|\alpha_m(p^{w_p}) - 1| &= \left|\alpha_m^{\text{good}}(p) - 1 + \sum_{u=1}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} \frac{\alpha^{\text{good}}_{\frac{m}{p^{2u}}}(p)}{p^{u(r-2)}}\right| \\ &\leq \frac{C_2}{p} + \sum_{u=1}^{\left\lfloor \frac{v_p(m)}{2} \right\rfloor} \frac{C_3}{p^{u(r-2)}} \\ &\leq \frac{C_5}{p},\end{aligned}$$

for some $C_5 > 0$. For $v = 1$, by equation (3.14) we get

$$\begin{aligned}|\alpha_m(p^{w_p}) - \alpha_m(p)| &\leq |\alpha_m(p^{w_p}) - 1| + |\alpha_m(p) - 1| \\ &\leq \frac{C_6}{p}.\end{aligned}$$

Combining the previous inequalities, we get

$$\begin{aligned}\left| w_p - \sum_{v=0}^{w_p-1} \frac{\alpha_m(p^v)}{\alpha_m(p^{w_p})} \right| &\leq \frac{1}{\alpha_m(p^{w_p})} \left[\sum_{v \geq 2} \frac{C_4}{p^{\frac{(v-1)(r-2)}{2}}} + \frac{C_7}{p} + \frac{C_5}{p} \right] \\ &\leq \frac{C_7}{\alpha_m(p^{w_p})p}\end{aligned}$$

By equation (3.14), for $p > 2C_1$, we have $\alpha_m(p^{w_p}) \geq \frac{1}{2}$, and for $p < 2C_1$, since by assumption m is representable by Q,

$$\alpha_m(p) \geq \frac{1}{p^{r-1}} \geq \frac{1}{(2C_1)^{r-1}}.$$

We conclude that

$$\left| w_p - \sum_{v=0}^{w_p-1} \frac{\alpha_m(p^v)}{\alpha_m(p^{w_p})} \right| \leq \frac{C_8}{p},$$

for some constant C_8 independent from m and p.

Second case : assume now that p is a prime factor of $2\det(L)$. Then formula (3.16) has to be modified as bad type solutions enter the picture. There are two type of bad solutions which we distinguish as follows (see also [58, p.360]). First define

$$\mathbb{S}_0 = \{j, \nu_j = 0\}, \quad \mathbb{S}_1 = \{j, \nu_j = 1\}, \quad \mathbb{S}_2 = \{j, \nu_j \geq 2\},$$

and let $s_i = \sum_{j \in \mathbb{S}_j} \dim(Q_j)$. The bad type solutions satisfy $x_{\mathbb{S}_0} \equiv 0 \pmod p$ and $x \not\equiv 0 \pmod p$. Then x is of

1. bad type I if $x_{\mathbb{S}_1} \not\equiv 0 \pmod p$,
2. bad type II if $x_{\mathbb{S}_1} \equiv 0 \pmod p$.

Let Q' be the quadratic form over $\mathbb{Z}_p$ such that $\nu'_j = \nu_j + 1$ if $j \in \mathbb{S}_0$ and $\nu'_j = \nu_j - 1$ otherwise. Also, let Q'' be the quadratic form over $\mathbb{Z}_p$ such that $\nu''_j = \nu_j - 2$ if $j \in \mathbb{S}_2$ and $\nu''_j = \nu_j$ otherwise. Then by [58, p.360],

$$N^{\mathrm{bad,I}}_{m,Q}(p^v) = p^{s_1+s_2} N^{\mathrm{good}}_{\frac{m}{p},Q'}(p^{v-1}),$$

and

$$N^{\mathrm{bad,II}}_{m,Q}(p^v) = p^{2r-s_0-s_1} N^{x_2 \neq 0 \pmod p}_{\frac{m}{p},Q''}(p^{v-1}).$$

Hence for $v \geq 2$, we have the formula

$$N_m(p^v) = N^{\mathrm{good}}_m(p^v) + p^r N_{\frac{m}{p^2}}(p^{v-2}) + p^{s_1+s_2} N^{I}_{\frac{m}{p},Q'}(p^{v-1}) + p^{2r-s_0-s_1} N^{x_2 \neq 0 \pmod p}_{\frac{m}{p},Q''}(p^{v-1})$$

The solutions of bad type II can be analyzed further by the same reduction maps and by [58, Lemma 3.4], after finitely many steps (equal to the depth defined in 3.3 *loc.cit*) we only have good type solutions that appear. Hence there exists $v_1 \geq 3$ independent of m and p such that for $v \geq v_1$, (using [58, Lemma 3.2]) the good type terms cancel when taking the difference $\alpha_m(p^{w_p}) - \alpha_m(p^v)$. So only zero type solutions contribute and we obtain a formula similar to equation (3.17). Hence for $v \geq v_1$

$$|\alpha_m(p^{w_p}) - \alpha_m(p^v)| \leq \frac{C_4}{p^{\frac{(v-1)(r-2)}{2}}}$$

For $v \leq v_1$, by estimating trivially $\alpha_m(p^v)$, and considering

$$C_{10} = \max_{p \backslash 2\det(L)} p \left| \sum_{v=0}^{v_1-1} \alpha_m(p^{w_p}) - \alpha_m(p^v) \right|,$$

we get

$$\left| \sum_{v=0}^{v_1-1} \alpha_m(p^{w_p}) - \alpha_m(p^v) \right| \leq \frac{C_{10}}{p}.$$

Here we conclude similarly to the first case. □

3.4.4 Estimates via effective equidistribution

We turn now to the proof of statement (i) in Proposition 3.4.3.

Proof of Proposition 3.4.3 (i). Let as before $x \in D_L$ and $m \in \mathbb{Z}$ with $m < 0$. Let $U(m)$ denote the union inside D_L of the hyperplanes $\lambda^\perp$ for $\lambda \in L$ and $Q(\lambda) = m$.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, let $g(s) = 2\frac{\Gamma(s-1+\frac{k}{2})}{\Gamma(2s)}$. For $z \in \mathbb{C}$ with $|z| < 1$, we will write henceforth $F(s,z)$ instead of $F\left(s-1+\frac{k}{2}, s+1-\frac{k}{2}, 2s; z\right)$ from equation (3.7) and write also $F(s,z) = 1 + zG(s,z)$. We have accordingly a decomposition of $\phi_m(x,s)$:

$$\phi_m(x,s) = g(s) \sum_{\substack{\lambda \in L \\ Q(\lambda)=m}} \left(\frac{m}{m - Q(\lambda_x)} \right)^{s-1+\frac{k}{2}} + \widetilde{\phi}_m(x,s).$$

Proposition 3.4.6. *For $x \in D_L$ outside $U(m)$, the series defining the function $s \mapsto \widetilde{\phi}_m(x,s)$ converges absolutely at $s = \frac{k}{2}$ and we have*

$$\widetilde{\phi}_m(x, \frac{k}{2}) = \frac{4}{b} \sum_{\substack{\sqrt{-m}\lambda \in L \\ Q(\lambda)=-1}} \left(\frac{1}{1+Q(\lambda_x)}\right)^k G\left(\frac{k}{2}, \frac{1}{1+Q(\lambda_x)}\right).$$

Proof. Let $s \in]\frac{k}{2}, k]$. There exists $C_m > 0$ such that $Q(\lambda_x) \geq C_m$ for every $\lambda \in L$ with $Q(\lambda) = m$. Since the function $(s,z) \mapsto G(s,z)$ is continuous on $[\frac{k}{2}, k] \times [0, \frac{1}{1+C_m}]$, we can a find a constant A_m such that

$$\left|G\left(s, \frac{1}{1+Q(\lambda_x)}\right)\right| \leq A_m,$$

for every $\lambda \in L$ with $Q(\lambda) = m$. Then we write

$$\begin{aligned}
\frac{\widetilde{\phi}_m(x,s)}{g(s)} &= \sum_{\substack{\sqrt{-m}\lambda \in L \\ Q(\lambda)=-1}} \left(\frac{1}{1+Q(\lambda_x)}\right)^{s+\frac{k}{2}} G\left(s, \frac{1}{1+Q(\lambda_x)}\right) \\
&= \sum_{n \geq 0} \sum_{\substack{\sqrt{-m}\lambda \in L \\ Q(\lambda)=-1 \\ n \leq Q(\lambda_v) < n+1}} \left(\frac{1}{1+Q(\lambda_x)}\right)^{s+\frac{k}{2}} G\left(s, \frac{1}{1+Q(\lambda_x)}\right) \\
&\leq A_m \sum_{n \geq 0} \frac{|\{\lambda \in L\,; Q(\lambda) = m,\, Q\left(\frac{\lambda_v}{|m|}\right) \in [n, n+1[\}|}{(1+n)^{s+\frac{k}{2}}}
\end{aligned}$$

Let $L_{\mathbb{R},-1} = \{\lambda \in L_{\mathbb{R}},\, Q(\lambda) = -1\}$. We can define a $SO(Q)$-invariant measure μ_∞ on $L_{\mathbb{R},-1}$ as in [105, Section 3.2]. In order to estimate the number of points of L whose Q-value is m and which projects to the area $\{\lambda \in L_{\mathbb{R},-1}, Q(\lambda_x) \in [n, n+1[\}$, we make the simple observation that any two such points are distant by at least $\frac{1}{\sqrt{m}}$. Hence the total number number is bounded by the volume of $\{\lambda \in L_{\mathbb{R},-1}, Q(\lambda_x) \in [n, n+1[\}$ with respect to μ_∞ and the implied constant depends only on m. By Lemma 5.4 below, we have the estimate

$$\mu_\infty\left(\{\lambda \in L_{\mathbb{R},-1}, Q(\lambda_x) \in [n, n+1[\}\right) \underset{n \to +\infty}{\asymp} n^{\frac{b}{2}-1}.$$

Thus we get

$$\frac{\widetilde{\phi}_m(x,s)}{g(s)} \leq B_m \sum_{n \geq 0} \frac{n^{\frac{b}{2}-1}}{(1+n)^{s+\frac{k}{2}}}$$

which at $s = \frac{k}{2}$ gives the series $\sum_{n \geq 1} \frac{1}{n^2} < +\infty$, hence the result. □

Lemma 3.4.7. *Let $T > 0$, $x \in D_L$ and $U_T =: \{\lambda \in L_{\mathbb{R},-1},\, Q(\lambda_y) < T\}$. Then*

$$\mu_\infty(U_T) = \frac{2(2\pi)^{1+\frac{b}{2}}(1+T)^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma\left(1+\frac{b}{2}\right)}.$$

Proof. For every $\epsilon > 0$, let $U_{T,\epsilon} =: \{x \in L_{\mathbb{R}}, |Q(x)+1| < \epsilon,\ Q(\lambda_x) < T\}$. Then $U_T = U_{T,\epsilon} \cap L_{\mathbb{R},-1}$ and by definition

$$\mu_\infty(U_T) = \lim_{\epsilon \to 0} \frac{\mu_L(U_{T,\epsilon})}{2\epsilon},$$

where μ_L is the Lebesgue measure on $L_{\mathbb{R}}$ for which L has covolume 1. Denote by P_x the definite positive plane associated to x. Let $\mathcal{E}$ be an orthogonal basis of $L_{\mathbb{R}}$ adapted to the decomposition $P_x \oplus P_x^{\perp}$ and in which the bilinear form associated to Q has the following intersection matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -I_b \end{pmatrix}.$$

Let $\mu_{\mathcal{E}}$ be the associated Lebesgue measure for which the $\mathbb{Z}$-span of $\mathcal{E}$ is of covolume 1. By change of variables, we have

$$\begin{aligned}
\mu_L(U_{T,\epsilon}) &= \frac{2^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}} \mu_{\mathcal{E}}(U_{T,\epsilon}) \\
&= \frac{2^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}} \int_{\substack{(x_1,x_2,y_1,\cdots,y_b)\in\mathbb{R}^{b+1} \\ |x_1^2+x_2^2-y_1^2-\cdots-y_b^2+1|<\epsilon \\ x_1^2+x_2^2<T}} dx_1 dx_2 dy_1 \cdots dy_b \\
&= \frac{2^{2+\frac{b}{2}}\pi}{\sqrt{|V^\vee/V|}} \int_0^{\sqrt{T}} \left(\int_{1+r^2-\epsilon<y_1^2+\cdots+y_b^2<1+r^2+\epsilon} dy_1 \cdots dy_b \right) r dr \\
&= \frac{2(2\pi)^{1+\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma\left(1+\frac{b}{2}\right)} \int_0^{\sqrt{T}} \left((1+r^2+\epsilon)^{\frac{b}{2}} - \left(1+r^2-\epsilon\right)^{\frac{b}{2}} \right) r dr \\
&= \frac{4(2\pi)^{1+\frac{b}{2}}\epsilon(1+T)^{\frac{b}{2}}}{\sqrt{|V^\vee/V|}\Gamma\left(1+\frac{b}{2}\right)} + O(\epsilon^2)
\end{aligned}$$

Dividing by 2ϵ and letting ϵ go to zero, we get the desired result. □

We turn now to estimating the first summand.

Proposition 3.4.8. *For $x \in D_L$, let*

$$R_x(s,m) = g(s) \sum_{\substack{\lambda \in L \\ Q(\lambda)=m}} \left(\frac{m}{m - Q(\lambda_x)} \right)^{s-1+\frac{k}{2}} - \frac{\varphi_m(\frac{k}{2})}{s - \frac{k}{2}}.$$

Then $R_x(s,m)$ is holomorphic at $s = \frac{k}{2}$ and satisfies $R_x(\frac{k}{2}, m) = O_x(c(-m))$.

Proof. We only need to prove that $s \mapsto R_x(s,m)$ is bounded as $s \to \frac{k}{2}$. Let $s \in \mathbb{C}$ with $Re(s) > \frac{k}{2}$. Consider the function

$$\begin{aligned}
h : L_{\mathbb{R},-1} &\to \mathbb{C} \\
\lambda &\mapsto \left(\frac{1}{1+Q(\lambda_x)} \right)^{s-1+\frac{k}{2}}.
\end{aligned}$$

By the effective equidistribution result of [44, Thm.1.1], more precisely its translation to $L_{\mathbb{R},-1}$ as in Section 17.2 *loc.cit.* (see also the discussion after [12, Corollary 1.11]), we have

$$\left| \sum_{\substack{\sqrt{|m|}\lambda\in L \\ Q(\lambda)=-1}} h(\lambda) - a(m).\int_{L_{\mathbb{R},-1}} h(\lambda) d\mu_\infty(\lambda) \right| \leq S(h)$$

where $S(h)$ is a L^2-Sobolev norm of h as defined in [44, Equation (3.8)]. The coefficient $a(\gamma)$ can be expressed as a product of local densities as in [105, Proposition 3.3]. In fact, we show in *loc.cit* the relation

$$a(m) = \frac{\varphi_m(\frac{k}{2}).\Gamma(1+\frac{b}{2}).\sqrt{|L^\vee/L|}}{2.(2\pi)^{1+\frac{b}{2}}},$$

which is simply a reformulation of the Siegel Mass formula, see [47].

The value of the integral is expressed similarly to the proof of Lemma 3.4.7 from which we keep the same notation :

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{L_\epsilon} \left(\frac{1}{1+Q(\lambda_x)}\right)^{s-1+\frac{k}{2}} d\mu_\infty(\lambda) \\
&\quad = \lim_{\epsilon\to 0} \frac{2^{\frac{b}{2}}}{\epsilon\sqrt{|L^\vee/L|}} \int_{(x_1,x_2)} \int_{|y_1^2+\cdots+y_b^2-x_1^2-x_2^2-1|<\epsilon} \frac{dx_1dx_2dy_1\cdots dy_b}{(1+x_1^2+x_2^2)^{s-1+\frac{k}{2}}} \\
&\quad = \frac{(2\pi)^{1+\frac{b}{2}}}{\Gamma\left(\frac{b}{2}\right)} \int_0^{+\infty} \left(\frac{1}{1+r^2}\right)^{s+1-\frac{k}{2}} rdr \\
&\quad = \frac{(2\pi)^{1+\frac{b}{2}}}{\Gamma\left(\frac{b}{2}\right)} \frac{1}{s-\frac{k}{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
a(m).\int_{L_{\mathbb{R},-1}} h(\lambda)d\mu_\infty(\lambda) &= \frac{\Gamma(1+\frac{b}{2})}{2.\Gamma(\frac{b}{2})} \frac{\varphi_m(\frac{k}{2})}{s-\frac{k}{2}} \\
&= \frac{b}{4} \frac{\varphi_m(\frac{k}{2})}{s-\frac{k}{2}}
\end{aligned}$$

yielding the inequality

$$R_x(s,m) \leq \frac{b}{4} S(h).$$

The computation of the Sobolev norm $S(h)$ involves computing L^2-norms of h and a finite number of its derivatives. One can check using a similar computation that the result is holomorphic in s as $s \to \frac{k}{2}$ and thus completing the proof. □

Hence we have the expression for the Green function

$$\phi_m(x) = R_x\left(\frac{k}{2}, m\right) + \widetilde{\phi}_m(x, \frac{k}{2}).$$

We need now to estimate the growth of the second term. Notice that we cannot apply the equidistribution mentioned above because the function

$$\lambda \mapsto \left(\frac{1}{1+Q(\lambda_x)}\right)^k G\left(\frac{k}{2}, \frac{1}{1+Q(\lambda_x)}\right)$$

has a logarithmic singularity along $\{\lambda \in L_{\mathbb{R},-1},\ x \in \lambda^{\perp}\}$. We will hence divide the sum into two terms and estimate each term separately. Let $1 \geq \eta > 0$ be a positive real number and let :

$$\widetilde{\phi}_m^{\leq \eta}(x) = \frac{4}{b} \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1, Q(\lambda_x) \leq \eta}} \left(\frac{1}{1+Q(\lambda_x)}\right)^k . G\left(\frac{k}{2}, \frac{1}{1+Q(\lambda_x)}\right) \tag{3.18}$$

and

$$\widetilde{\phi}_m^{>\eta}(x) = \widetilde{\phi}_m(v, \frac{k}{2}) - \widetilde{\phi}_m^{\leq \eta}(x)$$

By equidistribution of the set $\frac{1}{\sqrt{-m}}(L)$ on $L_{\mathbb{R},-1}$, we have

$$\widetilde{\phi}_m^{>\eta}(x) = O(c(-m).$$

For the term (3.18), we have :

$$\widetilde{\phi}_m^{\leq \eta}(x, \frac{k}{2}) = \frac{4}{b} \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1, Q(\lambda_x) < \eta}} \left(\frac{1}{1+Q(\lambda_x)}\right)^k$$
$$.G\left(\frac{k}{2}, \frac{1}{1+Q(\lambda_x)}\right)$$

Notice that :

$$G\left(\frac{k}{2}, z\right) = \sum_{n \geq 1} \frac{\frac{b}{2}}{n+\frac{b}{2}} z^{n-1}.$$

Thus we can find $C > 0$ such that for every $z \in [\frac{1}{2}, 1[$

$$|z^k.G\left(\frac{k}{2}, z\right) + \frac{b}{2}\log(1-z)| \leq C$$

Hence,

$$\widetilde{\phi}_m^{\leq \eta}(x) = -2 \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1, Q(\lambda_x) \leq \eta}} \log\left(1 - \frac{1}{1+Q(\lambda_x)}\right) + O(c(m))$$
$$= -2 \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1 \\ Q(\lambda_x) \leq \eta}} \log(Q(\lambda_x)) + O(c(m))$$

The last equality holds because $\lambda \mapsto \log(1+Q(\lambda_x))$ is bounded for $Q(\lambda_x) \in [0,1]$ and the equidistribution result of the set $\frac{1}{\sqrt{-m}}(L)$ on $L_{\mathbb{R},-1}$ yields that

$$|\{\lambda \in L_{\mathbb{R},-1},\, Q(\lambda_x) < \eta \,\text{and}\, \sqrt{|m|}\lambda \in L\} = O(|c(-m)|).$$

Finally,

$$\phi_m(x) = -2 \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1 \\ Q(\lambda_x)\leq \eta}} \log(Q(\lambda_x)) + O(c(m)),$$

Which finishes the proof. □

3.4.5 Beyond equidistribution

We deal now with statement *(ii)* in Proposition 3.4.3.

Proof of Proposition 3.4.3 (ii). Assume that x is moderately approximated by the special divisors. We can find $D \geq 0$ and an infinite sequence (m_i) such that for every $\lambda \in L$, we have $-\log(Q(\lambda_x)) \geq \frac{1}{m_i^D}$. Let $\epsilon > 0$, then we can find $\eta > 0$ and m_{i_0} such that for $|m_i| > |m_{i_0}|$ we have

$$\left| 2 \sum_{\substack{\sqrt{|m|}\lambda \in L \\ Q(\lambda)=-1 \\ Q(\lambda_x)\leq \eta}} \log(Q(\lambda_x)) \right| = O(|c(-m_i)| \log |m| \epsilon) \tag{3.19}$$

which proves the claim. □

The following proposition shows that almost every point in $M(\mathbb{C})$ is moderately approximated by the special divisors. Recall that the line bundle $\mathcal{L}$ is ample and we denote by ν the probability measure given by integrating over the top exterior power of the first Chern class of $\mathcal{L}$.

Proposition 3.4.9. *For every $v \in M(\mathbb{C})$ outside a zero measure subset, there exists m_0, such that for all $m \geq m_0$ we have $Q(\lambda_x) \geq \frac{1}{m^b}$.*

Proof. For $x \in M(\mathbb{C})$ outside the union $\cup_m Z(m)$, define for $m \in \mathbb{Z}$ the following function :

$$\theta_m(v) = \min_{\lambda \in L} Q(\lambda_x).$$

Let $A_m = \{v \in M(\mathbb{C}), \theta_m(v) < \frac{1}{|m|^b}\}$. Then $\sum_m \nu(A_m) < +\infty$ and we conclude by Borel-Cantelli lemma. □

Getting rid of "almost everywhere" seems to be a difficult problem. Motivated by Charles' treatment in [31, Proposition 3.3] we make the following conjecture.

Conjecture 3.4.10. *For an integer $D \geq 1$, let*

$$S_v^D = \{N \in \mathbb{N}, \exists \lambda \in A,\, \sqrt{N}\lambda \in L,\, |Q(\lambda_x)| < N^{-D}\}.$$

Then there exists $D \geq 1$ such that

$$\lim_{N\to\infty} \frac{1}{N} |S_v^D \cap \{1, \cdots, N\}| = 0.$$

In the case of lattices of signature $(2,2)$, Charles has shown in [31] that the above conjecture holds. We hope to pursue the investigation in a future work.

3.5 Applications : exceptional isogenies and Picard rank jumps

In this section, we explain how Charles' result from [31] fits in this general theory, then we turn to the case of K3 surfaces and we prove Corollary 3.1.2.

3.5.1 Exceptional isogenies of elliptic curves

The Shimura variety

Given two elliptic curves over $\mathbb{C}$, the $\mathbb{Z}$-module $\mathrm{End}(H_1(E_1,\mathbb{Z}),H_1(E_2,\mathbb{Z}))$ endowed with the quadratic form induced by the polarization forms on E_1 and E_2 is a lattice, abstractly isomorphic to the lattice (L,Q) of signature $(2,2)$ with intersection matrix in a $\mathbb{Z}$ basis equal to $\begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}$ where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. There is then an isomorphism of semi-simple algebraic groups over $\mathbb{Z}$: $\mathrm{Spin}(Q) \simeq \mathrm{SL}_2 \times \mathrm{SL}_2$.

Let

$$D_L = \{\omega \in \mathbb{P}(L_{\mathbb{C}}),\, (\omega,\omega) = 0\, (\omega,\overline{\omega}) > 0\}$$

be the hermitian symmetric domain associated to (L,Q). If $\mathbb{H}$ denotes the Poincaré upper half plane, then the holomorphic map

$$\begin{aligned} \psi :\ & \mathbb{H}\times\mathbb{H} \to D_L \\ & (\tau_1,\tau_2) \mapsto (1,\tau_1\tau_2,\tau_1,\tau_2) \end{aligned}$$

is a biholomorphism onto its image, which is one of the two connected components of D_L that we denote by D_L^+. The group $\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z})$ acts on the left on $\mathbb{H}\times\mathbb{H}$ via linear fractional transformation and this action coincides via ψ with the action of the arithmetic group $\Gamma_L = \mathrm{SO}(L,Q)$ on D_L. Hence the quotient $\Gamma_L\backslash D_L$ is isomorphic to $Y(1)(\mathbb{C})\times Y(1)(\mathbb{C})$, where $Y(1)(\mathbb{C}) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Let $X(1)$ be the modular curve over $\mathbb{Z}$. Then $X(1)(\mathbb{C}) = Y(1)(\mathbb{C})\cup\{\infty\}$, and the j-invariant defines an isomorphism $j : X(1)\to\mathbb{P}^1_{\mathbb{Z}}$. Hence the algebraic variety $\Gamma_L\backslash D_L$ admits a smooth integral compactification which is $X(1)\times X(1)$.

The special divisors

Let $N\geq 1$ be an integer. The special divisor $\mathcal{Z}(N)$ on D_L corresponds via ψ to the divisor

$$\widetilde{T}_{N,\mathbb{C}} := \{(\tau_1,\tau_2),\, \exists\gamma\in M_2(\mathbb{Z}),\, \det(\gamma) = N \,\text{and}\, \gamma.\tau_1 = \tau_2\},$$

which parameterizes pairs of elliptic curves linked by an isogeny of degree N. Let $T_{N,\mathbb{C}}$ be the Hecke correspondence which parametrizes pairs of elliptic curves linked by a cyclic isogeny of degree N. Then we have the relations

$$\widetilde{T}_{N,\mathbb{C}} = \sum_{d^2\backslash N} T_{\frac{N}{d^2},\mathbb{C}}$$

Let $X_0(N)$ be the Deligne-Rapoport compactification of the coarse moduli scheme which parametrizes cyclic isogenies of elliptic curves of degree N, see [36]. It is a normal arithmetic surface over Spec($\mathbb{Z}$) and admits a map $X_0(N) \to X(1) \times X(1)$ whose image is the compactification of T_N.

In this situation, the modularity result of Borcherds is trivial since the Picard group of $Y(1)$ is trivial. However Conjecture 3.3.3 is a simple consequence of the fact that $\widetilde{T}_N$ is a divisor of bidegree $(\sigma_1(N), \sigma_1(N))$, $\sigma_1(N)$ being the sum of the divisors of N, and that $1 - 24\sum_{n\geq 1}\sigma_1(N)q^N$ is the Eisenstein series E_2 which is a quasi-modular form of weight 2 and depth 1 with respect to $\mathrm{SL}_2(\mathbb{Z})$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts by multiplication on the left on the set of matrices in $\mathrm{M}_2(\mathbb{Z})$ of determinant N and the following set is a set of representatives of this action

$$H_N := \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}), ad = N, 1 \leq a, 0 \leq b \leq d-1, a \wedge b \wedge d = 1\} \tag{3.20}$$

The cardinality e_N of H_N satisfies

$$e_N = N \prod_{\substack{p \backslash N \\ p \,\mathrm{prime}}} (1 + \frac{1}{p}).$$

There exists an unique polynomial $\Phi_N \in \mathbb{Z}[X, Y]$ (see [73, p.54]), the *modular polynomial of order* N, such that :

$$\forall \tau \in \mathbb{H}, \quad \Phi_N(X, j(\tau)) = \prod_{\gamma \in H_N} (X - j(\gamma.\tau)).$$

Let $\widetilde{\Phi}_N$ be the bi-homogeneous polynomial associated to Φ_N seen as section of the line bundle $\mathcal{O}(e_N) \times \mathcal{O}(e_N)$ and $\phi_N = (j \times j)^*\widetilde{\Phi_N}$, then $T_N = \mathrm{div}(\phi_N)$. In particular, the function

$$\psi_N(\tau_1, \tau_2) = -\log\left(\prod_{\gamma \in H_N} |j(\tau_2) - j(\gamma.\tau_1)|\, |\Delta(\tau_1)\Delta(\tau_2)| y_1^6 y_2^6\right)$$

is a Green function for the divisor T_N. Hence $\widehat{T}_N = (T_N, \psi_N)$ is an arithmetic divisor on $X(1) \times X(1)$ and in fact $\widehat{T}_N = e_N \widehat{T}_1$.

Main result of [31]

Let K be a number field and let E_1, E_2 be two elliptic curves over K. Denote by $\mathcal{O}_K$ the ring of integers of K. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the closure in $X(1)_{\mathcal{O}_K}$ of the points corresponding to E_1 and E_2 in $X(1)(K)$. Then we have a family of abelian surfaces $\mathcal{E}_1 \times \mathcal{E}_2 \to S$ and a morphism $S \to X(1)_{\mathcal{O}_K} \times X(1)_{\mathcal{O}_K}$. If $\sigma : K \to \mathbb{C}$, denote by $|\,.\,|_\sigma$ the corresponding absolute value. Then we have

$$h_{\widehat{T}_N}(S) = (S.T_N)_{\mathrm{finite}} + \sum_{\sigma: K \hookrightarrow \mathbb{C}} \psi_N\left(j(E_1)^\sigma, j(E_2)^\sigma\right). \tag{3.21}$$

Let $h(\Phi_N)$ be the logarithmic height of Φ_N defined as the logarithm of the maximum of the absolute values of its coefficients. By the main theorem of [33], $h(\Phi_N)$ satisfies

$$h(\Phi_N) \underset{N \to +\infty}{\sim} 6e_N \log(N).$$

Also by Lemma 1 *loc.cit*, we have

$$h(\Phi_N) \sim_{N\to\infty} \log \prod_{\gamma\in H_N} \max(1, |j(\gamma.E_2)|_\sigma).$$

Hence, if one would like to estimate the growth of the archimidean term in equation (3.21), one needs to have lower bounds on

$$\sum_{\gamma\in H_N, j(\gamma.E_2)\in\widetilde{K}} \log(|j(E_1) - j(\gamma.E_2)|_\sigma)$$

for some fixed compact set $\widetilde{K} \subset \mathbb{C}$ containing $j(E_1)^\sigma$. This amounts to controlling the approximation of (τ_1, τ_2) by the Hecke correspondences T_N. This can be performed in a similar fashion to 3.4.5. The main advantage here is that the lattice we obtain is of rank 2 and the conjecture 3.4.10 is satisfied in this case.

3.5.2 Application to K3 surfaces

We turn now to the proof of Corollary 3.1.2. This will show also how Conjecture 3.1.4 implies Conjecture 3.1.5. For background on K3 surfaces, we refer to [64].

Let X be a K3 surface over a number field K and, up to taking a finite extension of K, assume that $\mathrm{Pic}(X_{\overline{K}}) = \mathrm{Pic}(X)$. For any embedding $\sigma : K \to \mathbb{C}$, the $\mathbb{Z}$-module $H^2(X_\sigma^{an}, \mathbb{Z})$ endowed with the intersection form given by Poincaré duality is an unimodular even lattice of signature $(3, 19)$. The first Chern class map

$$c_1 : \mathrm{Pic}(X) \to H^2(X_\sigma^{an}, \mathbb{Z})$$

is a primitve embedding and by the Hodge index theorem, $\mathrm{Pic}(X)$ has signature $(1, \rho(X) - 1)$, where ρ is the Picard rank of X. Let (L, Q) be the orthogonal lattice to $\mathrm{Pic}(X)$ in $H^2(X_\sigma^{an}, \mathbb{Z})$. Then (L, Q) is an even lattice whose genus is independent of σ as an abstract lattice and has signature $(2, b)$ where $b = 20-\rho(X)$. Let $\mathcal{M}$ be the GSpin Shimura variety associated to (L, Q). Then X defines a K-point $x \in \mathcal{M}(K)$. Let $\mathfrak{P}$ be a place of K, $\ell > 0$ a prime different from the residual characteristic of $\mathfrak{P}$ and let $\mathcal{V}_\ell$ be the ℓ-adic local system constructed in 3.2.4. By the Tate conjecture for K3 surfaces, now a theorem by the work of Charles [29], Maulik [83], Madapusi-Pera [80] and Nygaard-Ogus [89], and the Mumford-Tate conjecture for K3 surfaces, see [103, 104]the orthogonal of the image of the map

$$\mathrm{sp}_{\mathfrak{P}} : \mathrm{Pic}(X_{\overline{K}}) \otimes \mathbb{Q}_\ell \to \mathrm{Pic}(\mathcal{X}_{\overline{\mathfrak{P}}}) \otimes \mathbb{Q}_\ell,$$

is equal to the union of the invariants $\mathcal{V}_{\ell,x}^U$ where $U \subset \mathrm{Gal}(\overline{K}/K)$ is a finite index subgroup. We refer to the proof of the Theorem 1, page 8 in [30] for further details.

Proof of Corollary 2.1.3. We assume that x extends to an $\mathcal{O}_K$-point of $\mathcal{M}$ and which is moderately approximated by the special divisors of $\mathcal{M}$. It is Hodge-generic by construction of the Shimura variety $\mathcal{M}$ and by Lefschetz theorem on $(1, 1)$ classes. By theorem 3.1.1, there exists a place $\mathfrak{P}$ of K, such that the geometric fiber $\mathcal{X}_{\overline{\mathfrak{P}}}$ of the reduction of X at $\mathfrak{P}$ contains a Tate class. Hence by our above discussion the specialization map $\mathrm{sp}_{\mathfrak{P}}$ is not surjective and this proves Corollary 3.1.2 and the fact that 3.1.4 implies 3.1.5.

□

Chapitre 4

Rational curves on elliptic K3 surfaces

Résumé. Étant donnée une surface K3 sur un corps algébriquement clos, on prouve que si la surface admet une fibration elliptique non-isotriviale, alors elle contient une infinité de courbes rationnelles. Ce résultat étend celui de Bogomolov et Tschinkel obtenu quand le corps de base est de caractéristique zéro.

Abstract. We prove that any non-isotrivial elliptic K3 surface over an algebraically closed field of arbitrary characteristic contains infinitely many rational curves. This result was previously known in characteristic zero due to the work of Bogomolov and Tschinkel.

Contents

4.1 Introduction

Let X be a K3 surface over an algebraically closed field k. In [14, Corollary 3.28], Bogomolov and Tschinkel prove that when the characteristic of k is zero and X admits a non-isotrivial elliptic fibration then X contains infinitely many rational curves. In this note, we extend their result to the case where k has positive characteristic by different and simpler arguments.

Theorem 4.1.1. *If X admits a non-isotrivial elliptic fibration, then X contains infinitely many rational curves.*

In characteristic zero, this is the content of [14, Corollary 3.28]. When k has positive characteristic, the main ingredients are a result on the image of ℓ-adic monodromy representations associated to non-isotrivial 1-dimensional families of elliptic curves (Proposition 4.2.5) and an existence result of separable rational multisections (Lemma 4.2.4). The proof is inspired from [14], though we simplify some arguments presented there. This note is split into three parts. In the first section, some background on elliptic K3 surfaces is recalled. Then the main result for K3 surfaces which are not supersingular in the sense of Shioda (see 4.3) is proved in the second section. The last section covers the case of K3 surfaces whose automorphism group is infinite, which includes supersingular K3 surfaces.

4.2 Elliptic K3 surfaces

Let k be an algebraically closed field of positive characteristic and $\mathbb{P}^1_k$ the projective line over k. We recall some facts about elliptic K3 surfaces. For a more comprehensive introduction, see [64, Chapter 11].

An elliptic K3 surface is a K3 surface X which admits an elliptic fibration $X \xrightarrow{\pi} \mathbb{P}^1_k$, see [64, Chap.11, Def.1.1]. If moreover the morphism π admits a section, then X is said to be a Jacobian elliptic K3 surface. The fibration is said to be non-isotrivial if not all the smooth fibers are isomorphic. Equivalently, for Jacobian elliptic K3 surfaces, the condition is that the j-invariant of the generic fiber is not in k.

4.2.1 Tate-Shafarevich group

Let $X \xrightarrow{\pi} \mathbb{P}^1_k$ be an elliptic K3 surface, and let $\mathrm{J}(X) \to \mathbb{P}^1_k$ be the Jacobian elliptic K3 surface associated to X (see [64, Chap.11, Section 4.1] or [35, Chap.5, Section 3]). It has the property that for every smooth fiber X_t, $t \in \mathbb{P}^1$, the fiber $\mathrm{J}(X)_t$ is isomorphic to the Jacobian elliptic curve associated to X_t. Let $\mathrm{J}(X)^{sm} \subset \mathrm{J}(X)$ be the open set of π-smooth points, viewed as a smooth group scheme over $\mathbb{P}^1_k$. Then the open π-smooth locus $X^{sm} \to \mathbb{P}^1_k$ is a $\mathrm{J}(X)^{sm}$-torsor over $\mathbb{P}^1_k$. Hence for an arbitrary Jacobian elliptic K3 surface $Y \to \mathbb{P}^1_k$, define the *Tate-Shafarevich group* $Ш(Y)$ as the set of isomorphism classes of Y^{sm}-torsors over $\mathbb{P}^1_k$. The group structure on $Ш(Y)$ depends on the choice of the section, however the isomorphism class does not.

Proposition 4.2.1 (Chap.11, Section 5.2, 5.5(i), 5.6 [64])**.** *Let $X \to \mathbb{P}^1_k$ be a Jacobian elliptic K3 surface. The Tate-Shafarevich group $Ш(X)$ is isomorphic to the Brauer group of X and we have an injective map*

$$Ш(Y) \hookrightarrow WC(X_\eta),$$

where $WC(X_\eta)$ is the Weil-Châtelet group of the generic fiber of $X \to \mathbb{P}^1_k$.

Recall that for an elliptic curve E over a field K, the Weil-Châtelet group, denoted $WC(E)$, is defined as the set of isomorphism classes of torsors under E over K, see [64, Chapter 11, Section 5.1].

For every positive integer d, let $\mathrm{J}^d(X) \xrightarrow{\pi_d} \mathbb{P}^1_k$ be the elliptic K3 surface constructed in [64, Chap.11, Remark 4.1] (see also [35, Thm. 5.3.1]). It has the property that that for every smooth fiber X_t, $t \in \mathbb{P}^1_k$, $\mathrm{J}^d(X)_t$ is isomorphic to $\mathrm{Pic}^d(X_t)$. Moreover, one has an isomorphism

$$
\begin{array}{ccc}
X & \xrightarrow{\sim} & J^1(X) \\
& {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\
& \mathbb{P}^1_k &
\end{array}
$$

and $\mathrm{J}(\mathrm{J}^d(X)) \simeq \mathrm{J}(X)$. In addition, the class $[\mathrm{J}^d(X)]$ of $\mathrm{J}^d(X)$ in $\mathrm{Br}(\mathrm{J}(X))$ is equal to $d[X]$.

For every integers d, d', we have natural rational maps of algebraic varieties

$$
\begin{array}{ccc}
\mathrm{J}^d(X) \times_{\mathbb{P}^1_k} \mathrm{J}^{d'}(X) & \dashrightarrow & \mathrm{J}^{d+d'}(X) \\
& \searrow \quad \swarrow & \\
& \mathbb{P}^1_k &
\end{array}
$$

For a positive integer m, the diagonal embedding

$$\mathrm{J}^1(X) \to \underbrace{\mathrm{J}^1(X) \times_{\mathbb{P}^1_k} \cdots \times_{\mathbb{P}^1_k} \mathrm{J}^1(X)}_{m\,\text{times}}$$

defines a rational map η_m which fits into the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{J}^1(X) & \overset{\eta_m}{\dashrightarrow} & \mathrm{J}^m(X) \\
& {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi_m} & \\
& \mathbb{P}^1_k &
\end{array}
$$

The map η_m is defined over the smooth locus of π.

4.2.2 Rational curves

Let X be a K3 surface over k. A rational curve on X is an integral closed subscheme C of dimension 1 and of geometric genus 0. Recall the following existence result, attributed to Bogomolov and Mumford, with a refinement of Li and Liedtke ([76, Theorem 2.1]).

Proposition 4.2.2 (Bogomolov-Mumford)**.** *Let L be a non-trivial effective line bundle on a K3 surface X over k. Then L is linearly equivalent to a sum of effective rational curves.*

4.2.3 Relative effective Cartier divisors

Definition 4.2.3. Let $X \to \mathbb{P}^1_k$ be an elliptic K3 surface. A relative effective Cartier divisor on $X/\mathbb{P}^1_k$ is a closed subscheme $\mathcal{M}$ on X such that $\mathcal{M} \to \mathbb{P}^1_k$ is finite flat. If moreover $\mathcal{M}$ is irreducible, it is called a multisection.

Given an elliptic K3 surface X and a multisection $\mathcal{M}$ on X, the map $\mathcal{M} \to \mathbb{P}^1_k$ is finite flat and its degree is by definition the degree of $\mathcal{M}$.

Let X_0 be a smooth fiber of $X \to \mathbb{P}^1_k$ over a point $0 \in \mathbb{P}^1_k$. Then we have a map given by the intersection product

$$\mathrm{Pic}(X) \xrightarrow{(X_0,\,)} \mathbb{Z}.$$

It sends any multisection to its degree. The image of the above map is a non-zero subgroup of $\mathbb{Z}$, of finite index. Denote by d_X its index. It is called the degree of X.

Lemma 4.2.4. *Let $X \to \mathbb{P}^1_k$ be an elliptic K3 surface.*

1. *The order of $[X]$ in $\mathrm{Br}(\mathrm{J}(X))$ is equal to d_X.*
2. *Every multisection of degree $d_{\mathcal{M}} = d_X$ is rational.*
3. *There exists at least one multisection $\mathcal{M}$ such that $d_{\mathcal{M}} = d_X$ and which is moreover generically étale over $\mathbb{P}^1_k$.*

Proof. For (2), let $\mathcal{M}$ be a multisection of degree d_X. By Proposition 4.2.2, $\mathcal{M}$ is linearly equivalent to a sum of rational curves $\sum_i C_i$. Then there exists a unique curve C_i which is horizontal and all the others are vertical. Let η_{C_i} and $\eta_{\mathcal{M}}$ be the generic points of C_i and $\mathcal{M}$ respectively. Then η_{C_i} and $\eta_{\mathcal{M}}$ are linearly equivalent in X_η, the fiber over the generic point η of $\mathbb{P}^1_k$. Since X_η is a smooth proper curve of genus 1, η_{C_i} and $\eta_{\mathcal{M}}$ must be equal and taking their closure yields the result.

For (1), notice that X_η is a torsor under the elliptic curve $\mathrm{J}(X)_\eta$ and that d_X is the index of X_η, i.e is the greatest common divisor of the degrees of residue fields of closed points of X_η (see [77, 1]). Since the order of X_η in $WC(\mathrm{J}(X)_\eta)$ is equal to its index by [77, Theorem 1], it implies that the order of $[X]$ is exactly d_X. By [77, Section 5, Theorem 4] [1], it is also equal to the minimal degree of residue fields of separable closed points. Hence there exists a closed separable point in X_η of degree d_X. Taking its closure yields a separable multisection which is necessarily rational by (2). This proves (3). □

4.2.4 Monodromy

Let $X \xrightarrow{\pi} \mathbb{P}^1_k$ be an elliptic K3 surface. Let U be the largest Zariski open subset of $\mathbb{P}^1$ over which the map π is smooth. Thus $X_U \to U$ is a torsor under the smooth group scheme $\mathrm{J}(X)_U \to U$. For $b \in U$ a closed point and m prime to $p := \mathrm{char}(k)$, the étale fundamental group $\pi_1^{\text{ét}}(U,b)$ of U acts on the group of m-torsion points in $\mathrm{J}(X)_b$ and defines a group morphism

$$\rho : \pi_1^{\text{ét}}(U,b) \to \mathrm{Aut}\left(\varprojlim_{\gcd(m,p)=1} \mathrm{J}(X)_b[m]\right) = \prod_{\gcd(\ell,p)=1} \mathrm{Aut}(\mathrm{T}_\ell \mathrm{J}(X)_b).$$

This action preserves the Weil paring and factors as follows :

$$\rho : \pi_1^{\text{ét}}(U,b) \to \prod_{\ell \wedge p=1} \mathrm{SL}(\mathrm{T}_\ell \mathrm{J}(X)_b).$$

1. More precisely, see the proof given there.

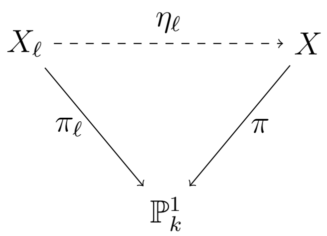
For every prime ℓ, we denote by ρ_{ℓ^∞} the representation of $\pi_1^{\text{ét}}(U, b)$ on the Tate module $\mathrm{T}_\ell \mathrm{J}(X_b)$ and denote by ρ_ℓ its reduction modulo ℓ. Then ρ_{ℓ^∞} is simply the projection on the ℓ-factor in the previous map. The monodromy group Γ is the image of $\pi_1^{\text{ét}}(U, b)$ under ρ. The next result on the image of the monodromy group will be crucial in the proof of Theorem 4.1.1.

Proposition 4.2.5 ([34]). *If the elliptic fibration is not isotrivial, then there exists a constant $c(k)$ depending only on k, such that for every $\ell > c(k)$ the morphism ρ_ℓ is surjective.*

Shafarevitch This is the content of [34, Theorem 1.1] where the surjectivity is proven for the reduction modulo ℓ, then one uses Lemma 2 in [97, IV-23].

4.3 Proof of Theorem 4.1.1

In this part we assume that X is not Shioda supersingular, in the sense that the rank $\rho(X)$ of the Picard group of X satisfies $\rho(X) \leq 20$. For the remaining case, see Corollary 4.4.2 below. The class of X in the Brauer group $\mathrm{Br}(\mathrm{J}(X))$ is a sum of two elements $\alpha_p + \alpha$, where α has torsion prime to p and α_p is torsion of order p^a, for some integer a. Here p is the characteristic of k. We will construct a rational multisection on X of arbitrary large degree which will be enough to prove Theorem 4.1.1. Denote by d_X the degree of X and let ℓ be a prime number with residue 1 (mod p^a) and such that $\ell > \max(d_X, c(k))$, where $c(k)$ is given by Proposition 4.2.5. The prime to p torsion part of $\mathrm{Br}(\mathrm{J}(X))$ is a divisible group by [64, Chap. 18, Example 1.5]. The Kummer exact sequence and the assumption on the supersingularity ensures furthermore that it is not trivial (see formula (1.8) *loc. cit*). We can thus find an elliptic K3 surface $\pi_\ell : X_\ell \to \mathbb{P}^1$ such that $\mathrm{J}(X_\ell) \simeq \mathrm{J}(X)$, $\ell[X_\ell, \pi_\ell] = [X, \pi]$ in $\mathrm{Br}(\mathrm{J}(X))$ and $d_{X_\ell} = \ell d_X$. Take for instance $\alpha_p + \alpha_\ell$, where α_ℓ is a non-trivial element in $\mathrm{Br}(\mathrm{J}(X))$ which satisfies $\ell.\alpha_\ell = \alpha$. Hence $\mathrm{J}^\ell(X_\ell) \simeq X$ and we have a rational map :



By Lemma 4.2.4(2)-(3), X_ℓ contains a rational generically étale multisection $\mathcal{M}_\ell$ of degree $d_{\mathcal{M}_\ell} = d_{X_\ell} = \ell d_X$. If the restriction of η_ℓ to $\mathcal{M}_\ell$ is isomorphic to its images above $\mathbb{P}^1_k$ then $\eta_\ell(\mathcal{M}_\ell)$ is a rational curve on X of degree divisible by ℓ which is the desired result. Otherwise, since the multiplication by ℓ map is étale (by [57, Théorème 2.5]), there exists infinitely many closed points b in the maximal open subset $U \subset \mathbb{P}^1_k$ where π is smooth, $\mathcal{M}_{\ell,U} \to U$ is étale and two distinct points P_1, P_2 in $X_{\ell,b} \cap \mathcal{M}_\ell$ such that $\ell.(P_1 - P_2) = 0$ in $\mathrm{J}(X)_b$. Thus, the point $P_1 - P_2$ is a ℓ-primitive torsion point in $\mathrm{J}(X)_b$. Let $\mathrm{J}(X)_U[\ell] \to U$ be the relative effective Cartier divisor of $\mathrm{J}(X)_U \to U$ of ℓ-torsion points. For ℓ sufficiently large, it has two irreducible components by Proposition 4.2.5 : the zero section and $\mathrm{J}(X)_{U,prim}[\ell]$, the relative effective Cartier divisor of primitive ℓ-torsion points. Since $X_{\ell,U}$ is a $\mathrm{J}(X)_U$-torsor over U, there is an induced map :

$$\mathrm{J}(X)_{U,prim}[\ell] \times \mathcal{M}_{\ell,U} \to X_{\ell,U}$$

The closure of the image in X_ℓ is a curve of X_ℓ which intersects $\mathcal{M}_\ell$ infinitely many times by the non-injectivity of η_ℓ. Hence $\mathcal{M}_\ell$ is isomorphic to an irreducible component of $\mathrm{J}(X)_{U,prim}[\ell] \times \mathcal{M}_{\ell,U}$ and via its first projection, it is surjective over $\mathrm{J}(X)_{U,prim}[\ell]$. Since there are $\ell^2 - 1$ torsion points in each fiber of $\mathrm{J}(X)_{U,prim}[\ell]$ over U, this implies

$$d_{\mathcal{M}_\ell} = \ell d_X \geq \ell^2 - 1.$$

This is a contradiction by our assumption on ℓ.

4.4 Case of infinite automorphism group

We give here a proof of Theorem 4.1.1 in the case where the automorphism group of the given K3 surface is infinite. This includes K3 surfaces which are supersingular in the sense of Shioda. This is a well-known result (see [64, Chap. 13, Remark 1.6]) and we include here an argument for the sake of completeness. By a theorem of Liedtke ([79]), Shioda's conjecture holds, namely supersingular K3 surfaces are unirational. The proof we give here does not use this powerful theorem.

Proposition 4.4.1. *Let X a K3 surface over an algebraically closed field k. Suppose that the automorphism group of X is infinite. Then X contains infinitely many rational curves.*

Proof. Assume the converse. Let $S = \{C_1, \dots, C_r\}$ be the finite set of rational curves on X. Then $\mathrm{Aut}(X)$ acts on S and defines a homomorphism $\rho : \mathrm{Aut}(X) \to \mathsf{S}_r$, where S_r is the symmetric group over r elements. Let g be an element in the kernel of ρ. Then g fixes all the rational curves. By 4.2.2, every non-trivial effective divisor is linearly equivalent to a sum of rational curves. Hence g acts trivially on $\mathrm{Pic}(X)$. It follows that the kernel of ρ is contained in the kernel of the morphism $\mathrm{Aut}(X) \to \mathrm{O}(\mathrm{Pic}(X))$. The latter is finite by [64, Chapter 15, Remark 2.2]. Hence ρ has finite kernel and obviously a finite image and these properties imply that $\mathrm{Aut}(X)$ is finite, which is excluded. □

Corollary 4.4.2. *Let X be a K3 surface with Picard number ≥ 20. Then X has infinitely many rational curves.*

Proof. If the Picard number of X is equal to 22, then by [70, Section 5] the automorphism group of X is infinite and the result follows from Proposition 4.4.1. If the Picard rank of X is equal to 20, then the height of X is finite and by [64, Chapter 9, Proposition 5.8], X admits a projective lift $\mathcal{X} \to \mathrm{Spec}(W(k))$ such that the map

$$\overline{sp} : \mathrm{NS}(\mathcal{X}_{\overline{K}}) \to \mathrm{NS}(X)$$

is an isomorphism and K is the fraction field of $W(k)$. Moreover, since X is not birationally ruled, by [78, Theorem 2.1], we have an injection $\sigma : \mathrm{Aut}(\mathcal{X}_{\overline{K}}) \to \mathrm{Aut}(X)$. By [99], the automorphism group of $X_{\overline{K}}$ is infinite. Hence $\mathrm{Aut}(X)$ is infinite and by proposition 4.4.1 X has infinitely many rational curves. □

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Titre : Sur certains aspects géométriques et arithmétiques des variétés de Shimura orthogonales.

Mots Clefs : Variétés de Shimura, surfaces K3, théorie de Hodge, formes modulaires.

Résumé : Cette thèse a pour objet l'étude de quelques propriétés arithmétiques et géométriques des variétés de Shimura orthogonales. Ces variétés apparaissent naturellement comme espaces de modules de structures de Hodge de type K3. Dans certains cas, elles paramètrent des objets géométriques tels que les surfaces K3 et leurs analogues en dimensions supérieures, les variétés hyperkähleriennes. Ce point de vue modulaire sera notre fil conducteur tout au long de ce mémoire. Ainsi, dans la première partie, on démontre un résultat d'équirépartition du lieu de Hodge dans les variations de structures de Hodge de type K3 au dessus d'une courbe complexe quasi-projective. Dans la deuxième partie, on étudie des analogues arithmétiques du résultat précédent. Un exemple d'énoncés qu'on obtient est est le suivant : étant donnée une surface K3 définie sur un corps de nombres et ayant partout bonne réduction, alors sous certaine hypothèse d'approximation, il existe une spécialisation telle que le nombre de Picard géométrique croît strictement. Dans la troisième partie, on relie les problèmes du saut de nombre de Picard dans les familles de surfaces K3 à la question de construction de courbes rationnelles sur ces surfaces. Enfin, on étend un résultat de Bogomolov et Tschinkel. On montre notamment que toute surface K3 définie sur un corps algébriquement clos de caractéristique quelconque et admettant une fibration elliptique non-isotriviale contient une infinité de courbes rationnelles.

Title : On some geometrical and arithmetical aspects of orthogonal Shimura varieties.

Keys words : Shimura varieties, K3 surfaces, Hodge theory, modular forms.

Abstract : This thesis deals with some arithmetical and geometrical aspects of orthogonal Shimura varieties. These varieties appear naturally as moduli spaces of Hodge structures of K3 type. In some cases, they parametrize geometric objects as K3 surfaces and their analogous in higher dimensions, the hyperkähler varieties. This modular point of view will be our guiding principle throughout this dissertation. In the first part, we prove an equidistribution result of the Hodge locus in variations of Hodge structures of K3 type above complex quasi-projective curves. In the second part, we study analogous results in the arithemtic setting. An example of statements we get is the following : given a K3 surface having everywhere good reduction and satisfying an approximation hypothesis, there exists a specialization with strictly increasing geometric Picard rank. In both cases, our methods take advantage of the rich arithmetic, automorphic and geometric structure of orthogonal Shimura varieties as well as the Kuga-Satake construction that links them to moduli spaces of abelian varieties. Finally, we extend a result of Bogomolov and Tschinkel. In particular, we show that any K3 surface defined over an algebraically closed field of arbitrary characteristic and admitting a non-isotrivial elliptic fibration contains infinitely many rational curves.