# DEFINING EXPONENTIAL FUNCTIONS VIA LIMITS

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Abstract. In this short note we prove a few classical properties of the exponential function $e^x$ from the simple definition

$$e^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

This note is a result of the author's curiosity in exploring and possibly teaching $e^x$ from a different point of view.

## 1. The Exponential Function $e^x$

In standard calculus textbooks, the exponential function $e^x$ is defined by

$$e^x := \sum_{n=0}^{\infty}\frac{x^n}{n!} \tag{1}$$

for all $x \in \mathbb{R}$. The continuity of $e^x$ then follows from the uniform convergence of the power series that defines it on closed and bounded intervals. Similarly, one can show that $e^x$ is differentiable everywhere. An alternative way of defining $e^x$ is by considering the following initial value problem

$$\frac{d}{dx}f(x) = f(x),$$
$$f(0) = 1.$$

One defines $e^x$ to be the unique solution $f(x)$ to this problem. One of the advantages of this approach is that one has the differentiability of $e^x$ for free. However, there is a third method when it comes to defining $e^x$. It is well known that the constant $e$ is commonly defined by

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

It is hence natural to define $e^x$ by

$$e^x := \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n. \tag{2}$$

This definition was introduced by L. Euler [2] who actually derived the power series expansion (1) from it (in a somewhat unrigorous way). Using Bernoulli's inequality that $(1+x)^n > 1+nx$ holds for all positive integers $n \geq 2$ and all nonzero $x > -1$, it is not hard to show that for every $x \in \mathbb{R}\setminus\{0\}$, the sequence $\{a_n(x)\}_{n=1}^{\infty}$ defined by

$$a_n(x) := \left(1+\frac{x}{n}\right)^n$$

is strictly increasing for $n > \max(0, -x)$. It is also not hard to prove that the sequence $\{a_n(x)\}_{n=1}^{\infty}$ is bounded for every fixed $x \in \mathbb{R}$. Indeed, let us fix $x \in \mathbb{R}$ and denote by $m$ the least positive integer greater than or equal to $|x|$. Then for sufficiently large $n$, we have

$$|a_n(x)| \le \left(1+\frac{m}{n}\right)^n \le \left(1+\frac{m}{mn}\right)^{mn} = \left(1+\frac{1}{n}\right)^{mn},$$

where we have used the monotonicity of $\{a_n(x)\}_{n=1}^\infty$. Since

$$\left(1+\frac{1}{n}\right)^n = 2+\sum_{k=2}^n \binom{n}{k}\frac{1}{n^k} \le 2+\sum_{k=2}^n \frac{1}{k!} < 2+\sum_{k=2}^n \frac{1}{k(k-1)} = 3-\frac{1}{n},$$

it follows that

$$|a_n(x)| \le \left(3-\frac{1}{n}\right)^m$$

for all sufficiently large $n$. This proves that $\{a_n(x)\}_{n=1}^\infty$ is bounded for every fixed $x \in \mathbb{R}$. Here the treatment for $(1+1/n)^n$ is classical. However, the author discovered a slightly different way to prove that the sequence $\{a_n(x)\}_{n=1}^\infty$ is bounded for every fixed $x \in \mathbb{R}$. The starting point is another simple inequality due to Bernoulli:

$$(1+x)^n \le \frac{1}{1-nx},$$

where $n$ is a positive integer and $x \in (-1, 1/n)$. This inequality can be proved easily by induction. As a consequence of this inequality, we have

$$|a_n(x)| \le \left(1+\frac{m}{n}\right)^n \le \left(1+\frac{2m}{2mn}\right)^{2mn} = \left(1+\frac{1}{2n}\right)^{2mn} \le 4^m$$

for all sufficiently large $n$. This again proves that $\{a_n(x)\}_{n=1}^\infty$ is bounded for every fixed $x \in \mathbb{R}$. Now it follows from the monotone convergence theorem that $\{a_n(x)\}_{n=1}^\infty$ is convergent for every $x \in \mathbb{R}$. This justifies Euler's definition (2). Moreover, the monotonicity of $\{a_n(x)\}_{n=1}^\infty$ implies that for every $x \in \mathbb{R}\setminus\{0\}$, the inequality

$$e^x > \left(1+\frac{x}{n}\right)^n$$

holds for all $n > \max(0, -x)$. In particular, this yields $e^x > \max(0, 1+x)$ for all $x \in \mathbb{R}\setminus\{0\}$, by Bernoulli's inequality. It is also clear from (2) that $e^x$ is increasing on $\mathbb{R}$.

## 2. The Continuity and Differentiability of $e^x$

Now we derive from (2) the fact that $e^x$ is continuous everywhere. Fixing $x_0 \in \mathbb{R}$ and $\epsilon > 0$, we have

$$\begin{aligned} e^x - e^{x_0} &= \lim_{n\to\infty}\left[\left(1+\frac{x}{n}\right)^n - \left(1+\frac{x_0}{n}\right)^n\right] \\ &= (x-x_0)\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(1+\frac{x}{n}\right)^{n-1-k}\left(1+\frac{x_0}{n}\right)^k. \end{aligned}$$

If $|x-x_0| < \epsilon$, then

$$\left(1+\frac{x_0-\epsilon}{n}\right)^n \le \frac{1}{n}\sum_{k=0}^{n-1}\left(1+\frac{x}{n}\right)^{n-1-k}\left(1+\frac{x_0}{n}\right)^k \le \left(1+\frac{x_0+\epsilon}{n}\right)^n$$

for sufficiently large $n$. It follows that

$$e^{x_0-\epsilon}|x-x_0| \leq |e^x - e^{x_0}| \leq e^{x_0+\epsilon}|x-x_0| \tag{3}$$

for all $x \in (x_0-\epsilon, x_0+\epsilon)$. This proves that $e^x$ is continuous at $x_0$. Since $x_0 \in \mathbb{R}$ is arbitrary, we conclude that $e^x$ is continuous on $\mathbb{R}$.

Now that we have established the continuity of $e^x$, it follows immediately from (3) and the monotonicity of $e^x$ that $e^x$ is differentiable with

$$\frac{d}{dx}(e^x) = e^x > 0$$

for all $x \in \mathbb{R}$. Thus $e^x$ is strictly increasing and $e^x \in C^\infty(\mathbb{R})$. From this the power series expansion (1) of $e^x$ follows naturally from Taylor's theorem. If $f(x)$ is a differentiable function satisfying $f'(x) = f(x)$, then

$$\frac{d}{dx}\left(\frac{f(x)}{e^x}\right) = \frac{f'(x)-f(x)}{e^x} = 0$$

for all $x \in \mathbb{R}$. This implies $f(x) = Ce^x$ for all $x \in \mathbb{R}$, where $C \in \mathbb{R}$ is a constant. Hence $e^x$ is the unique solution to the initial value problem

$$\begin{aligned}\frac{d}{dx}f(x) &= f(x),\\ f(0) &= 1.\end{aligned}$$

We have thus shown that the definition (2) implies both the power series definition and the differential equation definition of $e^x$. One advantage of this approach is that $e^x$ defined by (2) provides an explicit and elementary solution to the initial value problem in consideration, from which the uniqueness follows naturally as we saw above.

## 3. The Addition Law for $e^x$

The addition law for $e^x$ states that $e^{x+y} = e^x e^y$ for all $x, y \in \mathbb{R}$. It is not immediately clear how the addition law follows from (1), but it can be easily verified once we know $e^x$ is differentiable with derivative $e^x$. Indeed, we have by the chain rule that

$$\frac{d}{dx}(e^{c-x}e^x) = \frac{d}{dx}(e^{c-x})e^x + e^{c-x}\frac{d}{dx}(e^x) = -e^{c-x}e^x + e^{c-x}e^x = 0$$

for all $x \in \mathbb{R}$, where $c \in \mathbb{R}$ is a constant. Thus $e^{c-x}e^x$ is constant. Since its value at $x = 0$ is $e^c$, we find that $e^{c-x}e^x = e^c$ for all $x, c \in \mathbb{R}$. Taking $c = x + y$ yields the addition law. Without doubt, this classical argument [1, §3.1] is neat and elegant.

On the other hand, the author found a way to derive the addition law without using the differentiability of $e^x$. Note first that

$$e^x e^{-x} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\left(1-\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n.$$

By Bernoulli's inequality we have

$$1 - \frac{x^2}{n} \leq \left(1-\frac{x^2}{n^2}\right)^n \leq 1$$

for sufficiently large $n$. It follows that

$$\lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n = 1.$$

Hence $e^x e^{-x} = 1$. More generally, suppose that $x, y \in \mathbb{R}$ are arbitrary. Then

$$e^x e^y = \lim_{n\to\infty}\left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n.$$

For any $\epsilon > 0$ we have

$$\left|\frac{xy}{n^2}\right| < \frac{\epsilon}{n}$$

for all sufficiently large $n$. It follows that

$$\left(1+\frac{x+y-\epsilon}{n}\right)^n < \left(1+\frac{x+y}{n}+\frac{xy}{n^2}\right)^n < \left(1+\frac{x+y+\epsilon}{n}\right)^n.$$

Thus we have

$$e^{x+y-\epsilon} \le e^x e^y \le e^{x+y+\epsilon}.$$

Since $e^x$ is continuous and $\epsilon > 0$ is arbitrary, we have $e^x e^y = e^{x+y}$.

## 4. The Natural Logarithm

The natural logarithm, denoted by $\log x$ or $\ln x$, is defined to be the inverse function of $e^x$, namely, $e^{\log x} = x$. It is strictly increasing on its domain $(0, +\infty)$. We now show that $\log x$ is continuous everywhere. Fix $x_0 > 0$ and $0 < \epsilon < x_0$. As in Section 2, we have

$$x - x_0 = (\log x - \log x_0)\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(1+\frac{\log x}{n}\right)^{n-1-k}\left(1+\frac{\log x_0}{n}\right)^k.$$

for all $x > 0$. If $|x - x_0| < \epsilon$, then

$$\left(1+\frac{\log(x_0-\epsilon)}{n}\right)^n \le \frac{1}{n}\sum_{k=0}^{n-1}\left(1+\frac{\log x}{n}\right)^{n-1-k}\left(1+\frac{\log x_0}{n}\right)^k \le \left(1+\frac{\log(x_0+\epsilon)}{n}\right)^n$$

for sufficiently large $n$. Hence

$$|\log x - \log x_0|(x_0-\epsilon) \le |x - x_0| \le |\log x - \log x_0|(x_0+\epsilon).$$

This shows that $\log x$ is continuous at $x_0$. Hence $\log x$ is continuous on $(0, +\infty)$. Moreover, we have

$$\frac{1}{x_0+\epsilon} \le \frac{\log x - \log x_0}{x - x_0} \le \frac{1}{x_0-\epsilon}$$

for all $x > 0$ with $x \neq x_0$ and $|x - x_0| < \epsilon$. This implies that $\log x$ is differentiable at $x_0$ with

$$\left.\frac{d}{dx}(\log x)\right|_{x=x_0} = \frac{1}{x_0}.$$

Hence $\log x$ is differentiable on $(0, +\infty)$ with

$$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

Now one can define $a^x := e^{x \log a}$, where $x \in \mathbb{R}$ and $a > 0$. Then it is easy to see that $a^{x+y} = a^x a^y$ and

$$(a^x)^y = e^{y \log a^x} = e^{xy \log a} = a^{xy}$$

for all $x, y \in \mathbb{R}$ and $a > 0$.

## 5. Extension to the Complex Exponential Function $e^z$

Both the power series expansion (1) and the differential equation approach [1, §3.1] can be extended to define the complex exponential function $e^z$. It is thus tempting to generalize (2) as well by defining

$$e^z := \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n \tag{4}$$

for $z = x + iy \in \mathbb{C}$. Once the existence of the limit on the right-hand side is established for every $z \in \mathbb{C}$, one can show as in Section 2 that $e^z$ defined in this way is continuous on $\mathbb{C}$, but some extra effort is needed to show that $e^z$ is holomorphic everywhere. The author found a way to prove that the sequence $\{a_n(z)\}_{n=1}^{\infty}$ defined by

$$a_n(z) := \left(1 + \frac{z}{n}\right)^n$$

converges uniformly on every bounded subset of $\mathbb{C}$. Let $r > 0$ be a constant and let $E \subseteq \{z \in \mathbb{C}\colon |z| \le r\}$. For any positive integers $n > m \ge 2$ we have

$$|a_n(z) - a_m(z)| \le \sum_{k=2}^{m} \left[\binom{n}{k}\frac{1}{n^k} - \binom{m}{k}\frac{1}{m^k}\right] |z|^k + \sum_{k=m+1}^{n} \binom{n}{k}\frac{|z|^k}{n^k}.$$

It is clear that uniformly for all $z \in E$, we have

$$\sum_{\sqrt[4]{m}<k\le m} \left[\binom{n}{k}\frac{1}{n^k} - \binom{m}{k}\frac{1}{m^k}\right] |z|^k + \sum_{k=m+1}^{n} \binom{n}{k}\frac{|z|^k}{n^k} < \sum_{\sqrt[4]{m}<k\le n} \binom{n}{k}\frac{r^k}{n^k} < \sum_{k>\sqrt[4]{m}} \frac{r^k}{k!} \to 0$$

as $m \to \infty$. For $2 \le k \le \sqrt[4]{m}$, it follows by Bernoulli's inequality that

$$k!\left[\binom{n}{k}\frac{1}{n^k} - \binom{m}{k}\frac{1}{m^k}\right] < 1 - \prod_{l=0}^{k-1}\left(1 - \frac{l}{m}\right) < 1 - \left(1 - \frac{k}{m}\right)^k < \frac{k^2}{m} \le \frac{1}{\sqrt{m}}$$

for sufficiently large $m$. Hence uniformly for all $z \in E$, we have

$$\sum_{2\le k\le \sqrt[4]{m}} \left[\binom{n}{k}\frac{1}{n^k} - \binom{m}{k}\frac{1}{m^k}\right] |z|^k < \frac{1}{\sqrt{m}} \sum_{k=2}^{\infty} \frac{r^k}{k!} \to 0$$

as $m \to \infty$. We have thus shown that $a_n(z) - a_m(z) \to 0$ uniformly for all $z \in E$ as $m \to \infty$. By Cauchy's uniform convergence test, we conclude that $\{a_n(z)\}_{n=1}^{\infty}$ is uniformly convergent on $E$. This proves that $\{a_n(z)\}_{n=1}^{\infty}$ is uniformly convergent on every bounded subset of $\mathbb{C}$. By a theorem of Weierstrass [1, Theorem 1, §5], we know that $a_n(z)$ converges to an entire function, which we denote by $e^z$, with derivative

$$\frac{d}{dz}(e^z) = \lim_{n\to\infty} \frac{d}{dz}\left(1 + \frac{z}{n}\right)^n = \lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^{n-1} = e^z.$$

Having proved this, we obtain immediately the addition law $e^{a+b} = e^a e^b$ for all $a, b \in \mathbb{C}$. It is also clear that this approach allows one to define the exponential function in an arbitrary unital Banach algebra.

In the argument above, we used the fact that the series

$$\sum_{n=0}^{\infty} \frac{r^n}{n!}$$

is convergent for every $r \geq 0$. This is almost trivial when $0 \leq r \leq 1$. For $r > 1$, the convergence of this series follows from the fact that $n! > r^{2n}$ for all sufficiently large $n$. It is worth noting that in comparison to the power series definition of $e^z$, the definition (4) does not give clear clues to Euler's formula $e^{ix} = \cos x + i \sin x$, though it does yield at once $\overline{e^z} = e^{\bar{z}}$ and $|e^z| = \sqrt{e^z e^{\bar{z}}} = \sqrt{e^{z+\bar{z}}} = e^x$ for any $z = x + iy \in \mathbb{C}$. Thus $e^{iy} \colon \mathbb{R} \to S^1$ defines a homomorphism from $\mathbb{R}$ to the unit circle $S^1$, where both $\mathbb{R}$ and $S^1$ are considered as topological groups.

Now we give a proof of Euler's formula $e^{ix} = \cos x + i \sin x$ . Let us write

$$e^{ix} = f(x) + ig(x),$$

where

$$f(x) = \lim_{n\to\infty} \sum_{0\leq k\leq n/2} (-1)^k \binom{n}{2k} \frac{x^{2k}}{n^{2k}},$$

$$g(x) = \lim_{n\to\infty} \sum_{0\leq k\leq (n-1)/2} (-1)^k \binom{n}{2k+1} \frac{x^{2k+1}}{n^{2k+1}}.$$

Since

$$\frac{d}{dx}(e^{ix}) = ie^{ix} = -g(x) + if(x),$$

we have $f'(x) = -g(x)$ and $g'(x) = f(x)$. Therefore, $f(x)$ and $g(x)$ are solutions to the following initial value problem

$$\begin{aligned} f'(x) &= -g(x), & g'(x) &= f(x), \\ f(0) &= 1, & g(0) &= 0. \end{aligned}$$

It follows that $f(x) = \cos x$ and $g(x) = \sin x$. Euler's formula makes it reasonable to define, for all $z \in \mathbb{C}$, the complex trigonometric functions

$$\cos z := \frac{e^{iz} + e^{-iz}}{2} = \lim_{n\to\infty} \sum_{0\leq k\leq n/2} (-1)^k \binom{n}{2k} \frac{z^{2k}}{n^{2k}},$$

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i} = \lim_{n\to\infty} \sum_{0\leq k\leq (n-1)/2} (-1)^k \binom{n}{2k+1} \frac{z^{2k+1}}{n^{2k+1}}.$$

## 6. Final Comments

There are other alternative ways of defining $e^x$. For instance, it is well known that the set of all positive continuous solutions to the Cauchy functional equation $f(x+y) = f(x)f(y)$ is $\{a^x : a > 0\}$. Thus we may define $e^x$ to be the unique positive continuous solution to this Cauchy equation with its value at 1 given by $e$. On the other hand, we may first define $\log x$ to be the the unique nonzero real-valued continuous solution to the Cauchy equation $f(xy) = f(x) + f(y)$ on $(0, +\infty)$ with its value at $e$ given by 1, and then define $e^x$ to be its inverse function. Of course, if one is willing to resort to the theory of integration, then $\log x$ can be defined by the definite integral $\int_1^x 1/t\,dt$, though this is not as elementary as the ones suggested above.

## References

[1] L. V. Ahlfors, *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable*, 3rd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, 1979.

[2] L. Euler and J. D. Blanton (transl.), *Introduction to Analysis of the Infinite*, Book I, Springer-Verlag New York, 1988.

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